Homework #4
Due Monday, July 20

DO THESE PROBLEMS ON YOUR OWN. You may use your notes and your textbook, and you may ask me questions, but otherwise you may not discuss these problems with anyone else or use any other source of information or assistance, whether human or nonhuman.

1. Prove "Sunny's formula" for derangements algebraically:

$$D(n) = nD(n-1) + (-1)^n, \; n \geq 1.$$

2. Problem 4.1.6.

3. Problem 4.2.1.

4. Problem 4.2.9.

5. Let G be a graph with vertex set $\{1, \ldots, n\}$. Assume each edge $\{i, j\}$ of a graph is assigned a positive cost c_{ij}. The cost of a walk is then defined to be the sum of the costs of its edges.

   (a) Prove that if W is a minimum cost walk from i to k, then W must in fact be a path. (Some surgery?)

   (b) Assume that $i \neq k$ and P is a minimum cost path from i to k. Assume that $\{i, j\}$ is the first edge on P. Prove that the portion of P from j to k must be a minimum cost path from j to k. (By contradiction?)

   (c) For convenience let's say $c_{ij} = \infty$ if $\{i, j\}$ is not an edge of the graph, and also $c_{ii} = 0$ for all i. For all i, k define $c_{ik}^{(\ell)}$ to be the cost of a minimum cost path from i to k having no more than ℓ edges. Prove that

$$c_{ik}^{(\ell)} = \min_{j=1}^{n} \{c_{ij} + c_{jk}^{(\ell-1)}\}.$$

   P.S. This problem implies that we can compute the minimum cost of paths between all pairs of vertices by raising the matrix $C = (c_{ij})$ to the power $n-1$, but using "weird matrix multiplication" in which multiplication of entries is replaced by addition, and addition of terms is replaced by taking the minimum.