MA 515 HOMEWORK #7
Due Monday, November 10

ANNOUNCEMENT: There will be no class on Friday, November 7. Instead, there will be an extra class on Wednesday, November 5, 8:00 am, in CB219.

1. A two-commodity flow problem considers a digraph with distinguished nodes s_1, t_1, s_2, t_2 and arc capacities $u(e)$. A feasible two-commodity flow is an (s_1, t_1)-flow x and an (s_2, t_2)-flow y such that x and y individually satisfy the flow-conservation equations, but jointly must satisfy $x(e) + y(e) \leq u(e)$ for every arc e. The size of x, f_1, is the net flow amount out of s_1, and the size of y, f_2, is the net flow amount out of s_2. The goal of the two-commodity flow problem is to maximize $f_1 + f_2$. Your exercise is to find a two-commodity flow problem such that every capacity $u(e)$ is an integer, but the maximum value of $f_1 + f_2$ is not an integer.

2. Read the statement and proof of Theorem 10.3 on page 206. Note that $\Gamma(X)$ stands for the set of neighboring vertices of X—the set of vertices not in X that are joined to at least one vertex of X by an edge.

   (a) Page 228, problem 2. Note that (pages 206 and 228) $\nu(G)$ is the maximum cardinality of a matching in G; $\tau(G)$ is the minimum cardinality of a set of vertices that covers the edges of G (vertex cover); $\alpha(G)$ is the maximum cardinality of a set of vertices, no two of which are joined by an edge of G (stable set); and $\zeta(G)$ is the minimum cardinality of a set of edges that touch every vertex of G (edge cover). Hint: think of ways of getting stable sets from vertex covers and vv., and ways of getting edge covers from matchings and vv.

   (b) Page 228, problem 3. Hint: First prove that A and B have the same cardinality, where $V = A \cup B$ is the bipartition of the vertices. Then use Theorem 10.3 to prove that G has a perfect matching (one covering every vertex of G).

3. Consider the graph K_3: the complete graph with 3 vertices and three edges e_1, e_2, e_3 (a triangle). Each matching M of K_3 has a corresponding incidence vector x^M:

$$x_i^M = \begin{cases} 1 & \text{if } e_i \in M \\ 0 & \text{if } e_i \notin M \end{cases}$$

   Write down a set of inequalities that precisely describes the convex hull of the incidence vectors of all of the matchings of K_3.