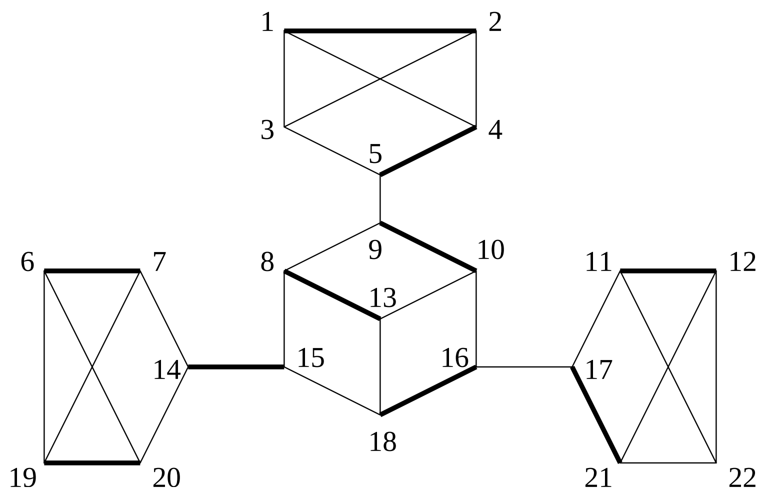
MA 515 HOMEWORK #8
Due Wednesday, December 10

ANNOUNCEMENT: There will be no class or office hours on Friday, December 12, and I will be out of the office all that day, so make arrangements to meet me for questions before that. The final exam is on Monday, December 15, at 8 am, in our regular room.

1. Let G be a graph and M be a maximum matching in G. Grow an alternating forest according to the maximum matching algorithm, and then use the resulting forest to verify that the given matching is of maximum cardinality, and also to find an odd set cover of minimum capacity.



2. Page 305, #2.

3. Page 305, #8, first part.

4. Let $(E, \mathcal{F})$ be a matroid and $r \to \mathbf{Z}_+$ its rank function. Consider the LP:

$$
(P) \quad \begin{aligned} &\max c^T x \\ &\sum_{e \in A} x_e \leq r(A) \text{ for all } A \subseteq E \\ &x \geq O \end{aligned}
$$

Recall that this is the problem solved by the Greedy Algorithm.

   (a) Write the LP (D) that is dual to (P).
   (b) Use some ideas from the proof of Theorem 13.19 and complementary slackness to show how one can obtain an optimal solution to (D) while carrying out the Best-In Greedy Algorithm.

5. Consider the matroid represented by the columns of the following matrix; i.e., independent sets of the matroid correspond to indices of linearly independent subsets of the columns. For convenience, let's assume the columns are indexed in order by the integers from 1 to 5.

$$
\begin{bmatrix} -1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

(a) List the bases of this matroid.

(b) List the circuits of this matroid.

(c) List the closed sets of this matroid.

(d) List the bases of the dual matroid.

(e) Write down a complete set of nonredundant inequalities to describe the associated matroid polytope. An inequality in a system is nonredundant if it cannot be algebraically derived from the other inequalities in the system. Use the fact that if an inequality for the matroid polytope is associated with a subset that is not closed, then it is redundant.

6. Let $M_i = (E, \mathcal{F}_i)$ be a matroid with rank function r_i for $i = 1, 2$. A set $J \subseteq E$ is a *basic subset* if it contains simultaneously a base of M_1 and a base of M_2.

   (a) How would you solve the problem of finding a basic subset with minimum cardinality?

   (b) Prove that if k is the minimum cardinality of a basic subset then

$$k = \max\,\{r_1(E) + r_2(E) - r_1(E - A) - r_2(A) \mid A \subseteq E\}.$$

7. **Extra Credit.** Let $G = (V, E)$ be a connected digraph with no loops, and let B be the matrix whose rows and columns are indexed by nodes, such that the ij entry of B equals the negative of the number of arcs joining i and j in either direction if $i \neq j$, and the number of arcs incident to i if $i = j$.

   (a) Prove $B = AA^T$, where A is the node-arc incidence matrix of G.

   (b) Let B' be the matrix obtained from B by crossing off the last row and the last column of B. Prove that the number of spanning trees of G equals $|\det B'|$. Hint: You may use the Cauchy-Binet Theorem, which states that if X is an $n \times m$ matrix, Y is an $m \times n$ matrix, and $Z = XY$ is an $n \times n$ matrix, then

$$\det Z = \sum_S \det X_S \det Y_S,$$

   where the sum is taken over all subsets S of the index set $\{1, \ldots, m\}$ of cardinality n, X_S is the $n \times n$ submatrix of X consisting of all the rows of X but only the columns of X indexed by S, and Y_S is the $n \times n$ submatrix of Y consisting of all the columns of Y but only the rows of Y indexed by S.