MA515 Exam #2
Due Wednesday, October 26

Instructions: You may consult me, your course notes, my course notes, and the text by Jon Lee, but no other source of information, whether human or inhuman.

1. Prove that a subset S of $\mathbf{R}^n$ is convex if and only if it is closed under all convex combinations of all *pairs* of elements.

2. Let S be a subset of $\mathbf{R}^n$. Let conv(S) be the set of all convex combinations of elements in S, and let T be the intersection of all convex sets containing S. Prove conv$(S) = T$.

3. Let $S = \{x \in \mathbf{R}^n : Ax = b,\ x \geq O\}$. Define $\overline{x} \in S$ to be a *vertex* of S if there exists an objective function c such that $\overline{x}$ is the unique optimal solution to the LP

$$\begin{array}{c} \max c^T x \\ \text{s.t. } Ax = b \\ x \geq O \end{array}$$

   Prove that $\overline{x}$ is a vertex of S if and only if $\overline{x}$ is a basic feasible solution of S.

4. Suppose $V = \{v^1, \ldots, v^p\}$ is a finite subset of $\mathbf{R}^n$ and $\overline{x} \in \mathbf{R}^n$. Recall that $\overline{x} \in$conv(V) if and only if the following system has a solution:

$$\begin{bmatrix} v^1 & \cdots & v^p \\ 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{bmatrix} = \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix}$$
$$\lambda_1, \ldots, \lambda_p \geq 0$$

   Use this to prove that if S is any subset of $\mathbf{R}^n$ (whether finite or infinite) and $\overline{x} \in$conv(S), then $\overline{x}$ can be expressed a convex combination of at most $n+1$ elements of S.

5. Let G be an undirected bipartite graph. Define $X \subseteq V(G)$ to be a *vertex packing* if no two vertices of X are joined by an edge. Let $\mathcal{S}$ denote the collection of all vertex packings of G and $P_{\mathcal{S}}$ denote the convex hull of all of the characteristic vectors of $\mathcal{S}$. Describe $P_{\mathcal{S}}$ as an H-polytope, carefully justifying your answer.