MA515 Homework #8
Due Friday, November 18

1. Give a simple example of a digraph with edge weights for which Dijktra's algorithm fails when started from a particular vertex of this digraph.

2. In a communications network, represented by a digraph, the probability that the edge from i to j is operative is p_{ij}. The probability that all the edges in any given dipath are operative is the product of the edge probabilities. How can one solve the problem of finding a most reliable dipath from one designated vertex to all of the others? Justify your answer.

3. Suppose a digraph is given with nonnegative capacities c_{ij} assigned to the edges. The capacity of a dipath is defined to be the minimum of the capacities of its edges. How can one solve the problem of finding maximum capacity dipaths between every pair of vertices? Justify your answer.

4. Suppose we are working with matrices with entries in $\mathbf{R} \cup \{\infty\}$. Define a new type of matrix multiplication $\otimes$ as follows: $P = (p_{ij}) = A \otimes B$, where $p_{ij} = \min_k \{a_{ik} + b_{kj}\}$. That is, let ordinary addition take the place of multiplication and minimization take the place of addition.

   (a) Show that $\otimes$ is associative. What is the identity matrix?

   (b) Let $C = (c_{ij})$ be the matrix of edges weights of a complete digraph, setting $c_{ii} = 0$ for all i. Prove that the (i, j)-entry of C^m, where C^m is computed according to the definition of $\otimes$, is the weight of a minimum weight (i, j)-diwalk, subject to the condition that the diwalk contains no more than m edges. This is true whether or not the digraph contains any negative dicycles.

5. A certain project consists of a set of tasks to be performed. Denote the set of tasks by $\{1, \ldots, n\}$. Each task j requires a certain amount of time a_j (assumed to be nonnegative). For each task j there is a certain subset of tasks $S_j \subseteq \{1, \ldots, n\} \setminus \{j\}$ that must be completed some time before task j is begun. For simplicity, assume that task 1 must be completed before all other tasks are begun, and that all other tasks must be completed before task n is begun. (I.e., task 1 is the initial task and task n is the final task.)

   (a) Using unrestricted variables t_j to represent the time that task j is begun, and the objective function $t_n - t_1$, write a linear program to solve the problem of determining the minimum amount of time required to complete the entire project.

(b) Prove that the dual linear program is a max weight dipath problem in a certain digraph. (If it isn't, try another formulation of the primal problem.)

(c) Prove that the shortest completion time equals the weight of a maximum weight dipath between two certain vertices in the digraph.