MA515 EXAM #2

INSTRUCTIONS: This is a take-home exam. You may use my course notes, the course text, and your course notes, but no other source of information or assistance, human or non-human. You may certainly see me if you have questions. Your answers are due Wednesday, November 1, at 4:00 pm — give them directly to me or slide them under my office door.

1. Let $P \subset \mathbf{R}^n$ be a polytope. Suppose $V = \{v^1, \ldots, v^m\}$ is a finite subset of P such that for every $c \in \mathbf{R}^n$ there is at least one point in V at which the function $c^T x$ attains its maximum value. Prove that every point p in P is a convex combination of the points in V by applying a theorem of the alternatives to the following system:

$$\begin{bmatrix} v^1 & \cdots & v^m \\ 1 & \cdots & 1 \end{bmatrix} \lambda = \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\lambda \geq O$$

2. Assume that you have used the simplex method to find an optimal solution to the linear program

$$(P) \quad \begin{array}{c} \max c^T x \\ \text{s.t. } Ax = b \\ x \geq O \end{array}$$

Assume that A has full row rank and that $\overline{y}^T = c_B^T A_B^{-1}$ is associated with the final optimal basis B. Prove that $\overline{y}$ is optimal for the dual (D) of (P):

$$(D) \quad \begin{array}{c} \min y^T b \\ \text{s.t. } y^T A \geq c^T \end{array}$$

Suggestion: Show that $\overline{y}$ is feasible for (D) and satisfies complementary slackness.

3. Consider the linear programs (P) and $(P(u))$:

$$\begin{array}{cc} \max c^T x & \max c^T x \\ \text{s.t. } Ax = b & \text{s.t. } Ax = b + u \\ x \geq O & x \geq O \\ & \\ (P) & (P(u)) \end{array}$$

Assume that (P) has an optimal objective function value z^*. Suppose that there exists a vector y^* and a positive real number ε such that the optimal objective function value

$z^*(u)$ of $(P(u))$ equals $z^* + u^T y^*$ whenever $\|u\| < \varepsilon$. Prove that y^* is an optimal solution to the dual of (P). Suggestion: Let $\overline{y}$ be optimal for the dual of (P) and use a previous homework problem (Exercise 10.7) to (eventually) show $\overline{y} = y^*$.

4. An *affine combination* of points $v^1, \ldots, v^m \in \mathbf{R}^n$ is a linear combination

$$\lambda_1 v^1 + \cdots + \lambda_m v^m$$

for which we also require

$$\lambda_1 + \cdots + \lambda_n = 1.$$

Let $S = \{v^1, \ldots, v^m\}$ be a subset of $\mathbf{R}^n$. Define S to be *affinely independent* if there is no linear combination

$$\lambda_1 v^1 + \cdots + \lambda_m v^m = O$$

for which

$$\lambda_1 + \cdots + \lambda_m = 0$$

other than $\lambda_1 = \cdots = \lambda_m = 0$.

   (a) Prove that the set S is affinely independent if and only if there is no element of S that can be written as an affine combination of the remaining elements of S.

   (b) Prove that the set S is affinely independent if and only if the following subset of $\mathbf{R}^{n+1}$ is linearly independent:

$$\left\{ \begin{bmatrix} v^1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} v^m \\ 1 \end{bmatrix} \right\}$$

   (c) Prove that the set S is affinely independent if and only if the set

$$\{v^1 - v^m, \ldots, v^{m-1} - v^m\} \subset \mathbf{R}^n$$

   is linearly independent.

5. Exercise (Linear over $GF(2) \not\Longrightarrow$ graphic) on the top of page 54 of the text. That is to say, prove that there is no graph G such that the graphic matroid associated with G is isomorphic to the Fano matroid. Suggestion: Try explicitly and systematically to construct a graph G with 7 edges having the same independent sets as the Fano matroid.