MA515 Final Exam
Due Wednesday, December 15, 1:00 pm SHARP

This is a take-home exam. You may consult your text and your notes, and may ask me questions, but otherwise you may not use any other assistance, human or non-human.

1. Problem 7C.

2. Problem 7E(v).

3. Let $G = (V(G), E(G))$ be a directed graph with two distinguished vertices s and t. Assume that st is not an edge. The (s,t)-vertex connectivity of G equals the minimum number of vertices in $V(G) \setminus \{s,t\}$ whose removal destroys all directed simple paths from s to t. Prove that this number equals the maximum number of paths in a collection of internally vertex-disjoint simple directed paths from s to t. Suggestion: First create a new directed graph by splitting each vertex $v \in V(G) \setminus \{s,t\}$ into two vertices v' and v''. Introduce the new directed edge $v'v''$ with capacity 1. For each old edge entering v, have it now enter v', giving it capacity $+\infty$. For each old edge leaving v, have it now leave v'', giving it capacity $+\infty$.

4. We can encode sequences $a_0, a_1, a_2, \ldots$ into generating functions $a_0 + a_1x + a_2x^2 + \cdots$. So, for example, the sequence $1, 1, 1, \ldots$ corresponds to the generating function $1 + x + x^2 + x^3 + \cdots$ which can be expressed in the simpler form $\frac{1}{1-x}$ as a rational function. Also, the generating function for the sequence $a_0, a_1, a_2, \ldots$, where a_i equals 1 if i is a multiple of 2, and otherwise a_i equals 0, is $1 + x^2 + x^4 + \cdots$, which simplifies (by summing a geometric series to the rational function $\frac{1}{1-x^2}$. You can verify this by going to wolframalpha.com and entering "`taylor 1/(1-x^2) at x=0`". Your problem is to find a generating function for the sequence $a_0, a_1, a_2, \ldots$ for which $a_i = 1$ if i is a not a multiple of 2, 3, or 5, and $a_i = 0$ otherwise. So this generating function (unsimplified) starts out $x + x^7 + x^{11} + x^{13} + \ldots$. Simplify your result sufficiently so that it is clear that it is a rational function. You can test your answer with wolframalpha.com, if you wish.

5. Let $a_1, \ldots, a_{30}$ be a sequence of 30 positive integers that sum to 48.

   (a) Prove there is a consecutive subsequence of the form $a_i, a_{i+1}, a_{i+2}, \ldots, a_j$ for which $a_i + a_{i+1} + a_{i+2} + \ldots + a_j = 11$. Suggestion: Let p_i denote the partial sum $a_1 + a_2 + \cdots + a_i$, and consider the sequences $p_1, p_2, \ldots, p_{30}$ and $p_1 + 11, p_2 + 11, \ldots, p_{30} + 11$.

   (b) Prove that that the result is not true if we change "11" to "16".