MA/STA515 Homework #3

Due Friday, September 17

1. Problem 2A.
2. Problem 2E.
3. For any simple graph G, direct each edge arbitrarily to obtain a digraph G'. Construct a matrix B with rows indexed by the vertices of G', and columns indexed by the edges of G': entry B_{ve} equals -1 if edge v is the tail of e, $+1$ if v is the head of e, and 0 otherwise.
   (a) Prove that a subset of rows of B is dependent (over $\mathbf{R}$) iff the corresponding set of vertices of G contains (but not necessarily equals) all of the vertices in some component of G. Suggestion: Think about a vector in the left nullspace of B as a certain way of labeling the vertices of G with real numbers.
   (b) Prove that the row rank of B equals $|V(G)| - 1$ iff G contains precisely one component.
   (c) Prove that a subset of columns of B is dependent iff the corresponding set of edges of G contains (but not necessarily equals) all of edges in some polygon of G.
   (d) Prove that the column rank of B equals $|E(G)|$ iff G contains no polygons.
   (e) Use the "Rank-Nullity" Theorem of Matrices to prove for a connected graph G that it is a tree iff it has $|V(G)| - 1$ edges.