# 1 Basic Definitions

**Definition 1.1** Let $x^1, \ldots, x^n \in \mathbf{R}^d$ and $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$. Then $x = \lambda_1 x^1 + \cdots + \lambda_n x^n$ is

1. a *linear combination* of $x^1, \ldots, x^n$;
2. an *affine combination* of $x^1, \ldots, x^n$ if $\lambda_1 + \cdots + \lambda_n = 1$;
3. a *nonnegative combination* of $x^1, \ldots, x^n$ if $\lambda_1, \ldots, \lambda_n \geq 0$;
4. a *convex combination* of $x^1, \ldots, x^n$ if $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_1, \ldots, \lambda_n \geq 0$.

**Remark 1.2** For $x^1, \ldots, x^n \in \mathbf{R}^d$, write

$$A = \begin{bmatrix} x^1 & \cdots & x^n \end{bmatrix}$$

and

$$A' = \begin{bmatrix} x^1 & \cdots & x^n \\ 1 & \cdots & 1 \end{bmatrix}.$$

Define also

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

Then the four cases in the above definition can be expressed, respectively, as

1. $x = A\lambda$.
2. $\begin{bmatrix} x \\ 1 \end{bmatrix} = A'\lambda$.
3. $x = A\lambda$, $\lambda \geq O$.
4. $\begin{bmatrix} x \\ 1 \end{bmatrix} = A'\lambda$, $\lambda \geq O$.

**Definition 1.3** Let $x^1 \ldots, x^n \in \mathbf{R}^d$. Then $\{x^1, \ldots, x^n\}$ is

1. *linearly dependent* if $\exists \lambda_1, \ldots, \lambda_n \in \mathbf{R}$, not all zero, such that $\lambda_1 x^1 + \cdots + \lambda_n x^n = O$. Otherwise $\{x^1, \ldots, x^n\}$ is *linearly independent*.

2. *affinely dependent* if $\exists \lambda_1, \ldots, \lambda_n \in \mathbf{R}$, not all zero, such that $\lambda_1 x^1 + \cdots + \lambda_n x^n = O$ and $\lambda_1 + \cdots + \lambda_n = 0$. Otherwise $\{x^1, \ldots, x^n\}$ is *affinely independent*.

**Theorem 1.4**

1. *$\{x^1, \ldots, x^n\} \subseteq \mathbf{R}^d$ is linearly dependent iff there exists one of the $x^i$ that can be expressed as a linear combination of the others.*

2. *$\{x^1, \ldots, x^n\} \subseteq \mathbf{R}^d$ is affinely dependent iff there exists one of the $x^i$ that can be expressed as an affine combination of the others.*

PROOF. Exercise. □

**Theorem 1.5** *Let $S = \{x^1, \ldots, x^n\} \subseteq \mathbf{R}^d$ and define $A$ and $A'$ as in Remark 1.2.*

1. *$S$ is linearly independent iff* $\operatorname{rank} A = n$ *iff* $\dim\{\lambda \in \mathbf{R}^n : A\lambda = O\} = 0$. *In particular, if $n > d$ then $S$ is linearly dependent.*

2. *$S$ is affinely independent iff* $\operatorname{rank} A' = n$ *iff* $\dim\{\lambda \in \mathbf{R}^n : A'\lambda = O\} = 0$. *In particular, if $n > d + 1$ then $S$ is affinely dependent.*

PROOF. Exercise. □

**Exercise 1.6** $\{x^0, x^1, \ldots, x^n\} \subseteq \mathbf{R}^d$ is affinely independent iff $\{x^1 - x^0, \ldots, x^n - x^0\}$ is linearly independent.

**Definition 1.7** Let $S \subseteq \mathbf{R}^d$.

1. $S$ is a *linear set* or *subspace* if $S \neq \emptyset$ and $\forall x, y \in S$, $\forall \lambda, \mu \in \mathbf{R}$, $\lambda x + \mu y \in S$. I.e., $S$ is nonempty (in particular $S$ contains $O$) and closed under all linear combinations of two elements.

2. $S$ is an *affine set*, *affine space*, or *flat* if $\forall x, y \in S$, $\forall \lambda, \mu \in \mathbf{R}$ such that $\lambda + \mu = 1$, $\lambda x + \mu y \in S$. I.e., $S$ is closed under all affine combinations of two elements.

3. $S$ is a *(convex) cone* if $S \neq \emptyset$ and $\forall x, y \in S$, $\forall$ nonnegative $\lambda, \mu \in \mathbf{R}$, $\lambda x + \mu y \in S$. I.e., $S$ is nonempty (in particular $S$ contains $O$) and is closed under all nonnegative combinations of two elements.

4. $S$ is a *convex set* if $\forall x, y \in S$, $\forall$ nonnegative $\lambda, \mu \in \mathbf{R}$ such that $\lambda + \mu = 1$, $\lambda x + \mu y \in S$. I.e., $S$ is closed under all convex combinations of two elements.

**Remark 1.8** We often (but not always) refer to elements of linear subspaces and convex cones as *vectors* and elements of affine and convex sets as *points*.

**Theorem 1.9** *Let $\mathcal{L}, \mathcal{A}, \mathcal{K}, \mathcal{C}$ denote the collection of all linear subspaces, affine sets, cones, and convex sets, respectively, of $\mathbf{R}^d$. Then $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{K} \subseteq \mathcal{C}$, and these inclusions are strict. Further, $\mathcal{A} \cap \mathcal{K} = \mathcal{L}$.*

PROOF. Exercise. □

**Exercise 1.10** Classify the following sets.

1. $\{x \in \mathbf{R}^d : \|x\| \leq 1\}$.
2. $\{x \in \mathbf{R}^d : a^T x = \alpha\}$, where $O \neq a \in \mathbf{R}^d$ and $\alpha \in \mathbf{R}$. (Such a set is called a *hyperplane*.)
3. $\{x \in \mathbf{R}^d : Mx = O\}$ where $M$ is an $n \times d$ matrix.
4. $\{x \in \mathbf{R}^d : Mx \leq O\}$ where $M$ is an $n \times d$ matrix.
5. $\{x \in \mathbf{R}^d : Mx = b\}$ where $M$ is an $n \times d$ matrix and $b \in \mathbf{R}^n$.
6. $\{x \in \mathbf{R}^d : Mx \leq b\}$ where $M$ is an $n \times d$ matrix and $b \in \mathbf{R}^n$.

**Theorem 1.11** *Let $S \subseteq \mathbf{R}^d$.*

1. *$S$ is a linear subspace iff $S \neq \emptyset$ and $S$ is closed under linear combinations of finite numbers of vectors in $S$.*
2. *$S$ is an affine set iff $S$ is closed under affine combinations of finite numbers of points in $S$.*
3. *$S$ is a cone iff $S \neq \emptyset$ and $S$ is closed under nonnegative combinations of finite numbers of vectors in $S$.*
4. *$S$ is a convex set iff $S$ is closed under convex combinations of finite numbers of points in $S$.*

PROOF. Exercise. □

**Theorem 1.12** *Let $S$ be a nonempty subset of $\mathbf{R}^d$. Then $S$ is an affine set iff there exists a linear subspace $L$ of $\mathbf{R}^d$ and a point $y$ such that $S = L + y$; i.e., $S = \{x + y : x \in L\}$. In this case $L$ is unique, and $y$ can be chosen to be any particular point of $S$.*

PROOF. Exercise. □

**Theorem 1.13** *Let $S \subseteq \mathbf{R}^d$.*

1. *$S$ is a linear subspace iff $S$ is a set of the form $\{x \in \mathbf{R}^d : Mx = O\}$ for some $n \times d$ matrix $M$.*

2. *$S$ is an affine set iff $S$ is a set of the form $\{x \in \mathbf{R}^d : Mx = b\}$ for some $n \times d$ matrix $M$ and $b \in \mathbf{R}^n$. If $S$ is nonempty, then in this case, $S = L + y$, where $y$ is any particular point in $S$ and $L = \{x \in \mathbf{R}^d : Mx = O\}$.*

PROOF. Exercise. □

**Remark 1.14** Not all cones are of the form $\{x \in \mathbf{R}^d : Mx \leq O\}$. Those that are are called *(convex) polyhedral cones*. Similarly, not all convex sets are of the form $\{x \in \mathbf{R}^d : Mx \leq b\}$. Those that are are called *(convex) polyhedra*.

PROOF. Exercise. □

**Theorem 1.15**

1. *Every linear subspace is the intersection of a finite number of hyperplanes containing $O$.*

2. *Every affine set is the intersection of a finite number of hyperplanes.*

PROOF. Exercise. □

**Theorem 1.16** *The intersection of any collection of linear subspaces, affine sets, cones, convex sets is again a linear subspace, affine set, cone, convex set, respectively.*

PROOF. Exercise. □

**Definition 1.17** Let $S \subseteq \mathbf{R}^d$.

1. The *linear span* of $S$, $\operatorname{span} S$, is the intersection of all linear subspaces containing $S$.

2. The *affine span* of $S$, $\operatorname{aff} S$, is the intersection of all affine sets containing $S$.

3. The *cone* of $S$, $\operatorname{pos} S$, is the intersection of all cones containing $S$.

4. The *convex hull* of $S$, $\operatorname{conv} S$, is the intersection of all convex sets containing $S$.

**Definition 1.18** Let $S, T \subseteq \mathbf{R}^d$.

1. If $S = \operatorname{span} T$, then we say $T$ *(linearly) spans* $S$.

2. If $S = \operatorname{aff} T$, then we say $T$ *affinely spans* $S$.

**Theorem 1.19** *Suppose $S$ is a nonempty subset of $\mathbf{R}^d$. Then* $\operatorname{span} S$, $\operatorname{aff} S$, $\operatorname{pos} S$, $\operatorname{conv} S$ *is the set of all linear, affine, nonnegative, convex combinations, respectively, of finite numbers of elements in $S$.*

PROOF. Exercise. □

**Definition 1.20** If $S \subseteq \mathbf{R}^d$ is a finite set, $\operatorname{pos} S$ is called a *finite cone* and $\operatorname{conv} S$ is called a *(convex) polytope*.

**Theorem 1.21** *Let $S \subseteq \mathbf{R}^d$.*

1. *If $S \neq \{O\}$ is a linear subspace, then $\exists x^1, \ldots, x^n \in S$ such that every $x \in S$ can be expressed uniquely as a linear combination of $x^1, \ldots, x^n$. In this case, $n$ is the maximum size of a linearly independent subset of $S$ and the minimum size of a subset of $S$ that spans $S$ linearly.*

2. *If $S \neq \emptyset$ is an affine set, then $\exists x^1, \ldots, x^n \in S$ such that every $x \in S$ can be expressed uniquely as an affine combination of $x^1, \ldots, x^n$. In this case, $n$ is the maximum size of an affinely independent subset of $S$ and the minimum size of a subset of $S$ that spans $S$ affinely.*

PROOF. Exercise. □

**Definition 1.22** In the first case above, $\{x^1, \ldots, x^n\}$ is called a *basis* of $S$ and the *dimension* of $S$, $\dim S$, equals $n$. If $S = \{O\}$, then $\dim S = 0$. In the second case above, $\{x^1, \ldots, x^n\}$ is called an *affine basis* of $S$ and $S$ is the translate $L + y$ of some linear $(n-1)$-dimensional subspace (verify this). We say that the *dimension* of $S$, $\dim S$, equals $n - 1$. An affine set is 0-dimensional iff it is a set consisting of a single point. Affine sets of dimension 1 and 2 are called *lines* and *planes*, respectively.

**Exercise 1.23** Let $S$ be an affine subset of $R^d$. Show that $S$ is a hyperplane iff $\dim S = d - 1$.

**Exercise 1.24** Let $S \subseteq \mathbf{R}^d$.

1. Show that a linear basis for $\operatorname{span} S$ can be chosen from $S$ itself.
2. Show that an affine basis for $\operatorname{aff} S$ can be chosen from $S$ itself.

**Exercise 1.25** Suppose $M$ is an $n \times d$ matrix and $b \in \mathbf{R}^n$.

1. If $S = \{x \in \mathbf{R}^d : Mx = O\}$, then $\dim S = d - \operatorname{rank} M$.
2. If $\emptyset \neq S = \{x \in \mathbf{R}^d : Mx = b\}$, then $\dim S = d - \operatorname{rank} M$.

**Definition 1.26** Suppose $S$ is a nonempty subset of $\mathbf{R}^d$. The *dimension* of $S$, $\dim S$, is defined to be $\dim(\operatorname{aff} S)$. (Note that the definitions of dimension agree if $S$ is an affine set.) The dimension of the empty set is defined to be $-1$.