

Symmetries of Patterns

1 Introduction

- A *pattern* is a subset of the plane.
- A *symmetry* of a pattern is an isometry of the plane that maps the pattern to itself.
- If the pattern does not extend indefinitely in any direction, it is a *Finite Pattern*. Go to Section 2.
- If the pattern has translational symmetry along only one direction (e.g., to the right and left), and there is a basic, smallest, translation symmetry that can be repeated to the right and to the left as many times as desired to generate all possible translation symmetries, and there is a basic design or motif that repeats indefinitely along this direction, then the pattern is a *Border Pattern* (also known as a *Strip Pattern* or *Frieze Pattern*). Go to Section 3.
- If the pattern has translational symmetry along two nonparallel directions, and there are two basic, smallest, translation symmetries in two nonparallel directions that can be repeated in various combinations backwards and forwards as many times as desired to generate all possible translation symmetries, and there is a basic design or motif that repeats indefinitely with these translations, then the pattern is a *Wallpaper Pattern*. Go to Section 4.

2 Finite Patterns

Finite Patterns do not extend indefinitely in any direction.

- **Type Z_1 :** *No* symmetry.
- **Type Z_n :** *No* reflectional symmetry, but has rotational symmetry, and the smallest symmetry rotation is the fraction $1/n$ of a full turn.
- **Type D_n :** Reflectional *and* rotational symmetry, and the smallest symmetry rotation is the fraction $1/n$ of a full turn.
- **Type D_∞ :** Reflectional and rotational symmetry, and can be symmetrically rotated by *any* amount.

3 Border Patterns

Border Patterns have translational symmetry along only one direction (e.g., to the right and left), and there is a basic, smallest, translation symmetry that can be repeated to the right and to the left as many times as desired, and there is a basic design or motif that repeats indefinitely along this direction,

In addition to their translational symmetries (T), these patterns (for convenience in this discussion viewed as extending left and right) might also have one or more of the following symmetries: a reflection across the central horizontal line (H), reflections across various vertical lines (V), rotations by 180° about various central points (R), and glide reflections by various amounts across the central horizontal line (G).

Symmetry Type	Translation	Horizontal Reflection	Vertical Reflection	Half-Turn	Glide Reflection
11 (Hop)	Y	-	-	-	-
1m (Jump)	Y	Y	-	-	Y
m1 (Sidle)	Y	-	Y	-	-
12 (Spin Hop)	Y	-	-	Y	-
1g (Step)	Y	-	-	-	Y
mg (Spin Sidle)	Y	-	Y	Y	Y
mm (Spin Hop)	Y	Y	Y	Y	Y

4 Wallpaper Patterns

Wallpaper Patterns have translational symmetry along two nonparallel directions, and there are two basic, smallest, translation symmetries in two nonparallel directions that can be repeated in various combinations backwards and forwards as many times as desired, and there is a basic design or motif that repeats indefinitely with these translations.

In addition to their translational symmetries, these patterns might also have one or more of the following symmetries: reflections, glide reflections, rotations about various points by multiples of 180° (2-fold rotational centers), rotations about various points by multiples of 90° (4-fold rotational centers), and rotations about various points by multiples of 60° (6-fold rotational centers).

Each of these patterns falls into one of 17 types, and can be classified using the flow chart at this location: http://www.dynamicgeometry.com/documents/recentTalks/islamicArt/Symmetry_Group_Flowchart.pdf.