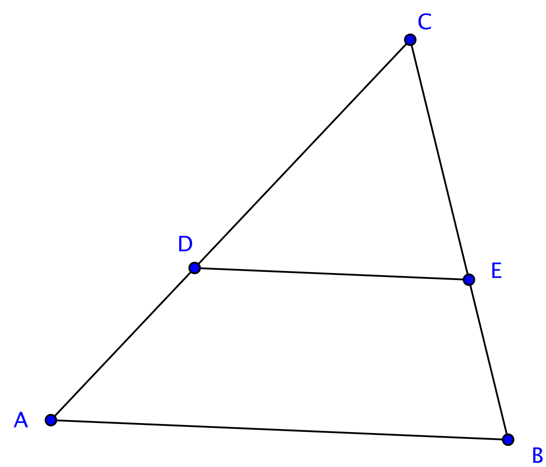
Some Proofs

1. In the figure below segment $\overline{DE}$ cuts across triangle ΔABC, and $CD/CA = CE/CB$. Prove that $\overleftrightarrow{DE}$ is parallel to $\overleftrightarrow{AB}$.



Proof.

Consider the dilation with center C and scaling factor CA/CD.

This dilation fixes C and maps D to A.

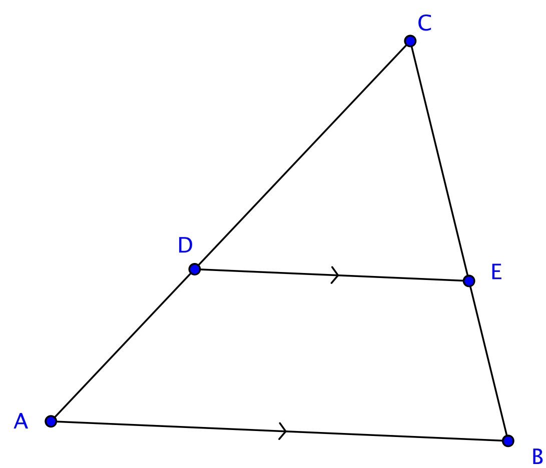
Because we are assuming $CB/CE = CA/CD$, the dilation also maps E to B.

Thus the dilation maps the line $\overleftrightarrow{DE}$ to the line $\overleftrightarrow{AB}$.

But the dilation must map $\overleftrightarrow{DE}$ to a line parallel to $\overleftrightarrow{DE}$ (property of dilations).

Therefore $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$.

2. In the figure below segment $\overline{DE}$ cuts across triangle ΔABC and $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$. Prove that $CD/CA = CE/CB$.



Proof.

Consider the dilation with center C and scaling factor $k = CA/CD$.

This dilation fixes C and maps D to A.

The dilation maps $\overleftrightarrow{DE}$ to a line ℓ through A parallel to $\overleftrightarrow{DE}$.

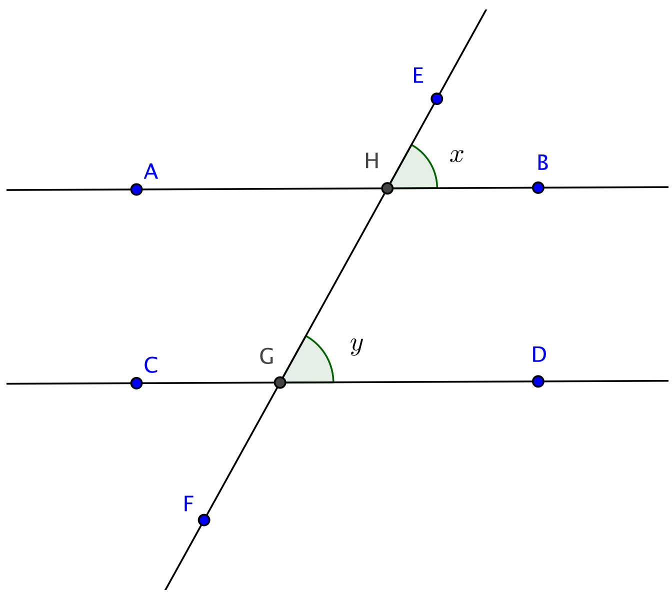
But $\overleftrightarrow{AB}$ passes through A and is parallel to $\overleftrightarrow{DE}$ by assumption.

By the Parallel Postulate, there is at most one line through A parallel to $\overleftrightarrow{DE}$, so lines ℓ and $\overleftrightarrow{AB}$ are in fact the same line.

Thus the dilation must map E to B.

Therefore the scaling factor k must equal CE/CB, and so $CA/CD = CB/CE$.

3. In the diagram below $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ is a transversal. Prove that angles x and y have the same measure.



Proof.

Consider the dilation with center F and scaling factor FG/FH.

This dilation maps H to G and maps $\overleftrightarrow{AB}$ to a line ℓ through G and parallel to $\overleftrightarrow{AB}$.

But $\overleftrightarrow{CD}$ is a line through G and parallel to $\overleftrightarrow{AB}$.

By the Parallel Postulate, there is at most one line through G parallel to $\overleftrightarrow{AB}$, so lines ℓ and $\overleftrightarrow{CD}$ are in fact the same line.

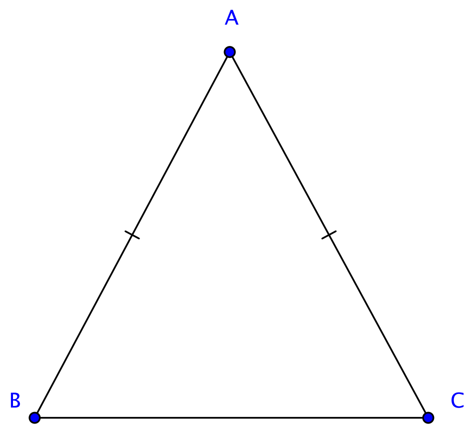
Thus the dilation maps B to a point on $\overleftrightarrow{CD}$.

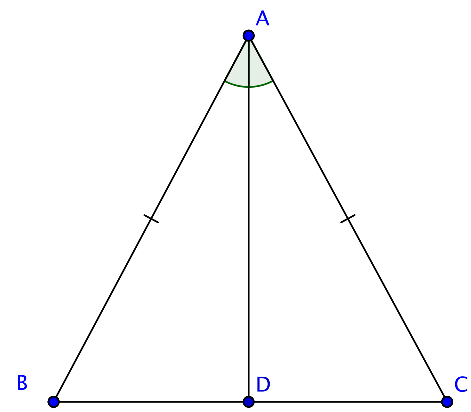
Also, the dilation maps E to a point on the ray $\overrightarrow{GE}$.

So the dilation maps the angle $\angle BHE$ to the angle $\angle DGE$.

But a dilation preserves angle measure, so these two angles must have the same measure.

4. Prove that the base angles of an isosceles triangle are congruent.





Proof.

Assume that $AB = AC$.

Let $\overleftrightarrow{AD}$ be the angle bisector of angle $\angle A$.

Thus $\mathrm{m}\angle BAD = \mathrm{m}\angle CAD$.

Consider the reflection in the line ℓ.

Note that this reflection maps A to A.

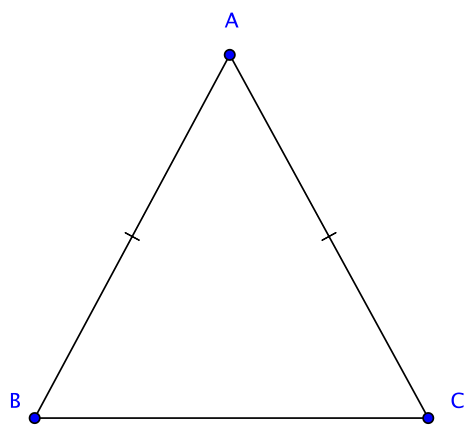
Since reflections preserve angles and distance, B is mapped to C and C is mapped to B.

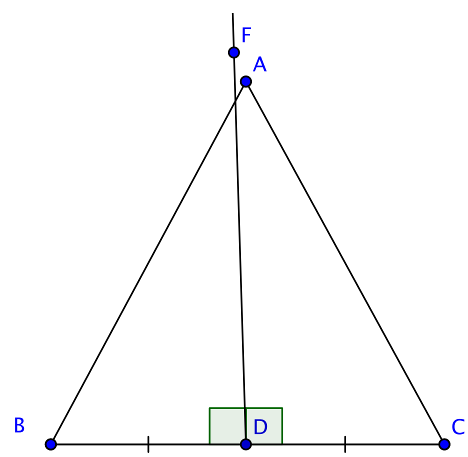
Thus $\angle CBA$ is mapped to $\angle BCA$.

But reflections preserve angle measure, so these two angles have the same measure.

5. Prove that if a triangle has a pair of congruent angles, then it is isosceles.

   Proof.





   Assume that $m\angle ABC = m\angle ACB$.

   Let D be the midpoint of $\overline{BC}$ and ℓ be the perpendicular bisector of $\overline{BC}$.

   Note: We do not yet know that ℓ passes through A.

   Consider the reflection in the line ℓ.

   By construction of ℓ, this reflection maps B to C and C to B.

   It also maps D to D.

   Because reflections preserve angle measures, this reflection maps angle $\angle CBA$ to angle $\angle BCA$ and vice versa, and so maps ray $\overrightarrow{BA}$ to ray $\overrightarrow{CA}$ and vice versa.

   So the intersection of these two rays, namely, A, is fixed by the reflection.

   Thus A lies on ℓ.

   Therefore $AB = AC$ since reflections preserve distance.

6. (CPCTC.) Assume that triangle Δ_1 is congruent to triangle Δ_2. Prove that there is a correspondence between the vertices such that corresponding angles have the same measure and corresponding sides have the same length.

   Proof.

   By the definition of congruence there is a rigid motion that maps Δ_1 to Δ_2.

   This mapping gives a correspondence between the vertices of the two triangles.

   By the definition of rigid motion, distances and angle measures are preserved.

   Therefore corresponding angles have the same measure and corresponding sides have the same length.

7. Assume that triangle Δ_1 is similar to triangle Δ_2. Prove that there is a correspondence between the vertices such that corresponding angles have the same measure and corresponding side lengths are proportional.

   Proof.

   By the definition of similarity there is a similarity transformation that maps Δ_1 to Δ_2.

   This mapping gives a correspondence between the vertices of the two triangles.

   By the definition of similarity transformation, angle measures are preserved, and also there is a positive constant k such that ratios of corresponding side lengths each equals k.

   Therefore corresponding angles have the same measure and corresponding sides are proportional

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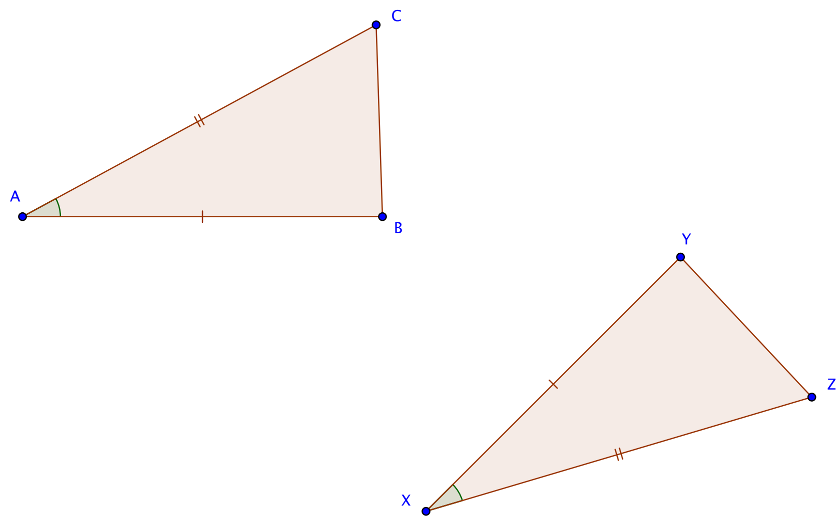
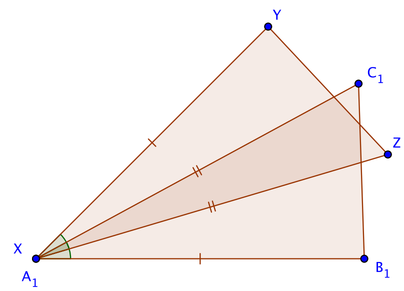
8. (SAS Congruence.) Assume that triangles ΔABC and ΔXYZ are such that $AB = XY$, $AC = XZ$, and $\mathrm{m}\angle A = \mathrm{m}\angle X$. Prove that $\Delta ABC \cong \Delta XYZ$.

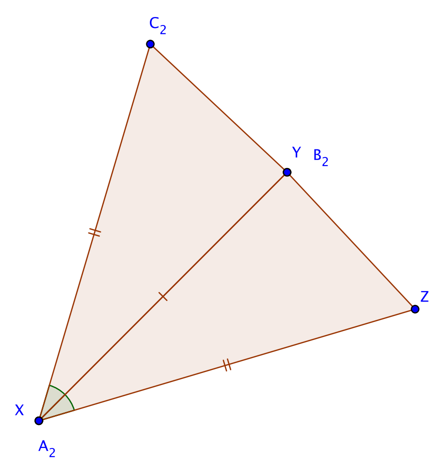


Proof.

First, apply the translation to ΔABC that maps A to X. Call the resulting triangle $\Delta A_1B_1C_1$. So $A_1 = X$. (Note: If it happens that $A = X$, then the translation will just be the identity map fixing all points; no translation is necessary.) See figure below.



Second, consider the rotation with center A_1 that maps B_1 to Y. Such a rotation exists because we are given that $AB = XY$, and so $A_1B_1 = XY$. Apply this rotation to $\Delta A_1B_1C_1$ to get triangle $\Delta A_2B_2C_2$. (Note: If it happens that $B_1 = Y$, then the rotation will just be the identity map fixing all points; no rotation is necessary.) See figure below.

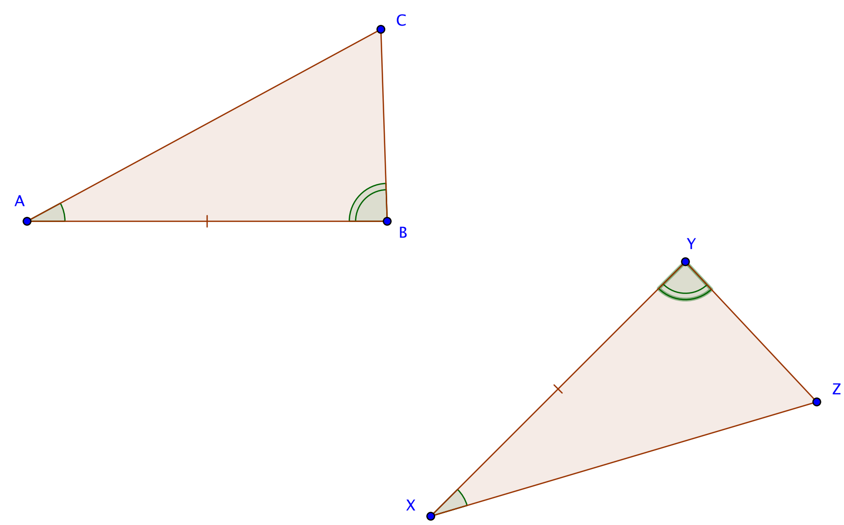


Case I: If $\Delta A_2B_2C_2$ is on the opposite side of $\overleftrightarrow{AB}$ than ΔXYZ (as in the figure above), apply the reflection across $\overleftrightarrow{A_2B_2}$ to $\Delta A_2B_2C_2$ to get $\Delta A_3B_3C_3$. Observe that since $AC = XZ$, then $A_2C_2 = XZ$, and also since $m\angle A = m\angle X$, then $m\angle A_2 = m\angle X$. Thus the reflection will map C_2 onto Z. This means that $\Delta A_3B_3C_3 = \Delta XYZ$.

Case II. Suppose $\Delta A_2B_2C_2$ is on the same side of $\overleftrightarrow{AB}$. Then since $AC = XZ$, then $A_2C_2 = XZ$, and also since $m\angle A = m\angle X$, then $m\angle A_2 = m\angle X$. Thus C_2 must equal the point Z. This means that $\Delta A_2B_2C_2 = \Delta XYZ$

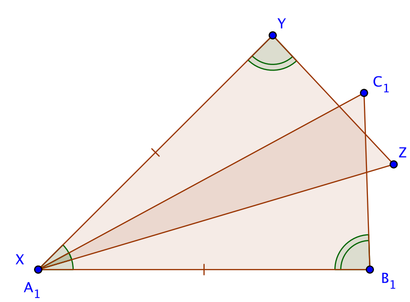
In either case we have found a sequence of rigid motions mapping ΔABC to ΔXYZ, so these two triangles are congruent.

9. (ASA Congruence.) Assume that triangles ΔABC and ΔXYZ are such that $AB = XY$, $m\angle A = m\angle X$, and $m\angle B = m\angle Y$. Prove that $\Delta ABC \cong \Delta XYZ$.

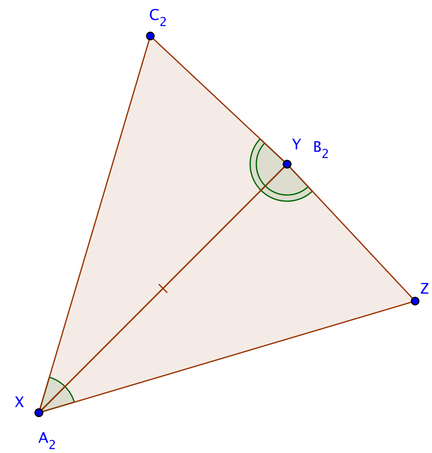


Proof.

First, apply the translation to ΔABC that maps A to X. Call the resulting triangle $\Delta A_1B_1C_1$. So $A_1 = X$. (Note: If it happens that $A = X$, then the translation will just be the identity map fixing all points; no translation is necessary.) See figure below.



Second, consider the rotation with center A_1 that maps B_1 to Y. Such a rotation exists because we are given that $AB = XY$, and so $A_1B_1 = XY$. Apply this rotation to $\Delta A_1B_1C_1$ to get triangle $\Delta A_2B_2C_2$. (Note: If it happens that $B_1 = Y$, then the rotation will just be the identity map fixing all points; no rotation is necessary.) See figure below.



Case I: If $\Delta A_2B_2C_2$ is on the opposite side of $\overleftrightarrow{AB}$ than ΔXYZ (as in the figure above), apply the reflection across $\overleftrightarrow{A_2B_2}$ to $\Delta A_2B_2C_2$ to get $\Delta A_3B_3C_3$. Observe that since since $\text{m}\angle A = \text{m}\angle X$, then $\text{m}\angle A_2 = \text{m}\angle X$. Similarly $\text{m}\angle B_2 = \text{m}\angle Y$. Thus the reflection will map ray $\overrightarrow{A_2C_2}$ to ray $\overrightarrow{XZ}$ and ray $\overrightarrow{B_2C_2}$ to ray $\overrightarrow{YZ}$. Thus the point of intersection (C_2) of the first pair of rays will map to the point of intersection (Z) of the second pair of rays. This means that $\Delta A_3B_3C_3 = \Delta XYZ$.

Case II. Suppose $\Delta A_2B_2C_2$ is on the same side of $\overleftrightarrow{AB}$. Then since $\text{m}\angle A = \text{m}\angle X$ and $\text{m}\angle B = \text{m}\angle Y$, then $\text{m}\angle A_2 = \text{m}\angle X$ and $\text{m}\angle B_2 = \text{m}\angle Y$. Thus $\overrightarrow{A_2C_2}=\overrightarrow{XZ}$ and $\overrightarrow{B_2C_2}=\overrightarrow{YZ}$. So the points of intersection C_2 and Z must be the same. This means that $\Delta A_2B_2C_2 = \Delta XYZ$

In either case we have found a sequence of rigid motions mapping ΔABC to ΔXYZ, so these two triangles are congruent.

10. (SSS Congruence).

    This can now be proved in the traditional way, once you know SAS and ASA congruence.

11. (AA Similarity.) Assume that triangles ΔABC and ΔXYZ are such that $\mathrm{m}\angle A = \mathrm{m}\angle X$, and $\mathrm{m}\angle B = \mathrm{m}\angle Y$. Prove that $\Delta ABC \sim \Delta XYZ$.

    Proof.

    Consider a dilation (choose any center you like) with scaling factor XY/AB. Apply this dilation to ΔABC to get triangle $A_1B_1C_1$. Angle measures are preserved, so $\mathrm{m}\angle A_1 = \mathrm{m}\angle A = \mathrm{m}\angle X$, and $\mathrm{m}\angle B_1 = \mathrm{m}\angle B = \mathrm{m}\angle Y$. Due to the choice of scaling factor, $A_1B_1 = XY$.

    By SAS congruence, $\Delta A_1B_1C_1 \cong \Delta XYZ$, so there is a rigid motion mapping $\Delta A_1B_1C_1$ to ΔXYZ.

    Thus there is a combination of a dilation (a similarity transformation) and a rigid motion mapping ΔABC to ΔXYZ.

    Therefore $\Delta ABC \sim \Delta XYZ$.

12. (SSS Similarity.) Assume you have a correspondence between the vertices of two triangles such that corresponding pairs of sides are proportional. Then the two triangles are similar.

    Sketch of Proof. First apply an appropriate dilation. Then apply SSS congruence.