A Framework for CCSSM Geometry with Transformations

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Contents

1 SMSG Postulates **1**

2 New Postulate 15 **4**

3 Some Consequences **6**

1 SMSG Postulates

Here is a list of the SMSG Postulates—basic assumptions in geometry from which all the other results are developed through a sequence of theorems and proofs.

1. Postulate 1. Given any two different points, there is exactly one line which contains both of them.
2. Postulate 2. (The Distance Postulate.) To every pair of different points there corresponds a unique positive number.
3. Postulate 3. (The Ruler Postulate.) The points of a line can be placed in correspondence with the real numbers in such a way that
   (a) To every point of the line there corresponds exactly one real number,
   (b) To every real number there corresponds exactly one point of the line, and
   (c) The distance between two points is the absolute value of the difference of the corresponding numbers.
4. Postulate 4. (The Ruler Placement Postulate.) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.
5. Postulate 5.
   (a) Every plane contains at least three non-collinear points.
   (b) Space contains at least four non-coplanar points.
6. Postulate 6. If two points lie in a plane, then the line containing these points lies in the same plane.
7. Postulate 7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane.
8. Postulate 8. If two different planes intersect, then their intersection is a line.

9. Postulate 9. (The Plane Separation Postulate.) Given a line and a plane containing it. The points of the plane that do not lie on the line form two sets such that (1) each of the sets is convex and (2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the line.

10. Postulate 10. (The Space Separation Postulate.) The points of space that do not lie in a given plane form two sets such that (1) each of the sets is convex and (2) if P is one set and Q is in the other, then the segment $\overline{PQ}$ intersects the plane.

11. Postulate 11. (The Angle Measurement Postulate.) To every angle $\angle BAC$ there corresponds a real number between 0 and 180.

12. Postulate 12. (The Angle Construction Postulate.) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that m$\angle PAB = r$.

13. Postulate 13. (The Angle Addition Postulate.) If D is a point in the interior of $\angle BAC$, then m$\angle BAC = m\angle BAD + m\angle DAC$.

14. Postulate 14. (The Supplement Postulate.) If two angles form a linear pair, then they are supplementary.

15. Postulate 15. (The S.A.S. Postulate.) Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

16. Postulate 16. (The Parallel Postulate.) Through a given external point there is at most one line parallel to a given line.

17. Postulate 17. To every polygonal region there corresponds a unique positive number.

18. Postulate 18. If two triangles are congruent, then the triangular regions have the same area.

19. Postulate 19. Suppose that the region R is the union of two regions R_1 and R_2. Suppose that R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2.

20. Postulate 20. The area of a rectangle is the product of the length of its base and the length of its altitude.

21. Postulate 21. The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

22. Postulate 22. (Cavalieri's Principle.) Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the two intersections have equal areas, then the two solids have the same volume.

2 New Postulate 15

In the spirit of the CCSSM, let's swap out Postulate 15 with a new postulate enumerating assumed properties of transformations.

New Postulate 15.

1. Planar Rigid Motions
   (a) Every rigid motion is one-to-one onto (bijective) mapping of the points of the plane to itself.
   (b) Rigid motions map lines to lines.
   (c) Rigid motions preserve distances.
   (d) Rigid motions preserve angle measures.
   (e) Associated with every line ℓ in the plane is a rigid motion, called a *reflection*, that maps every point A on one side of the line to a point B on the other side of ℓ such that ℓ is the perpendicular bisector of $\overline{AB}$, and fixes the points in the line itself.
   (f) Associated with every point P in the plane and angle measure $0 \leq x < 360$ is a rigid motion, called a *rotation*. The point P is fixed, and otherwise every other point A is mapped to a point B such that $\text{m}\angle AOB = x$ (measured in the counterclockwise direction). Note: If $x = 0$ then the rigid motion is just the identity map.
   (g) Associated with every pair of points P and Q is a rigid motion, called a *translation*. If $P = Q$ then this is just the identity map. If $P \neq Q$ then every point A is mapped to a point B such that either (1) $ABQP$ is a parallelogram, or else (2) all four points lie on a common line such that $b - a = q - p$, where p, q, a, b are the coordinates of P, Q, A, B, respectively, in a coordinate system for ℓ.
2. Planar Similarity Transformations
   (a) Every similarity transformation is a one-to-one onto (bijective) mapping of the points of the plane to itself.
   (b) Similarity transformations map lines to lines.
   (c) Associated with each similarity transformation is a nonzero number k such that all distances are multiplied by $|k|$.

(d) Similarity transformations preserve angle measures.

(e) Associated with every point P and every positive number k is a similarity transformation, called a *dilation*. The point P is fixed, and otherwise every other point A is mapped to a point B such that $PB = kPA$.

(f) A dilation maps any line ℓ to another line either parallel or equal to ℓ.

3 Some Consequences

1. Theorem. A combination (composition) of rigid motions and a dilation is a similarity transformation.
2. Theorem. Every rigid motion is a combination (composition) of translations, rotations, and reflections. (Try tossing two nonsymmetric congruent shapes on the floor.)
3. Theorem. Every rigid motion is a combination (composition) of reflections.
4. Theorem. Every similarity transformation is a combination (composition) of a dilation and a rigid motion.
5. We use these transformations to define congruence and similarity:

   Definition. Two subsets X and Y of points in the plane are said to be *congruent* if there is a rigid motion that maps X to Y. Two subsets X and Y of points in the plane are said to be *similar* if there is a similarity transformation that maps X to Y.
6. Theorem. SASASA Triangle Congruence.
7. Theorem. SAS Triangle Congruence.
8. Theorem. ASA Triangle Congruence.
9. Theorem. SSS Triangle Congruence.
10. Theorem. AA Triangle Similarity.
11. Theorem. SAS Triangle Similarity.
12. Theorem. SSS Triangle Similarity.