

## 8 Inequalities

### Concepts:

- Equivalent Inequalities
- Linear and Nonlinear Inequalities
- Absolute Value Inequalities

(Sections 4.6 and 1.1)

### 8.1 Equivalent Inequalities

#### Definition 8.1

Two inequalities are **equivalent** if they have the same solution set.

#### Operations that Produce Equivalent Inequalities.

- Add or Subtract the same value on both sides of the inequality.
- Multiply or Divide by the same **positive value** on both sides of the inequality.
- Multiply or Divide by the same **negative value** on both sides of the inequality **AND change the direction of the inequality**.

#### Example 8.2

For each pair of inequalities, determine if the two inequalities are equivalent.

- $x^2 + 2x \leq 5$  and  $x^2 + 2x - 5 \leq 0$ . **Yes.** (Try subtracting 5 from both sides of the first inequality.)
- $6 - x < 5$  and  $x > 1$  **Yes.**
- $\frac{x}{x+2} > 3$  and  $\frac{x}{x+2} - 3 > 0$  **Yes.**
- $\frac{x}{x+2} > 3$  and  $x > 3(x+2)$  **No.**(Notice  $x = -2.5$  is a solution to the first inequality, but not the second.)

You should **NEVER** multiply both sides of an inequality by an expression involving  $x$  that you don't know the sign of. Why?

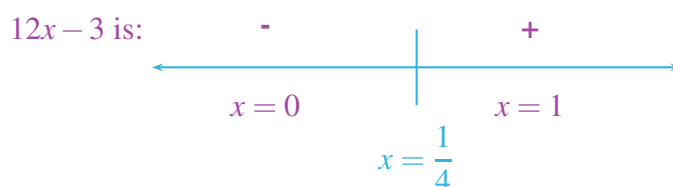
## 8.2 Solving a Linear or Nonlinear Inequality

### Example 8.3 (Linear Inequality)

Solve the inequality  $5x + 3 \leq 6 - 7x$ . Write the solution set in interval notation. By adding  $7x$  and subtracting  $6$  from both sides of the inequality, we obtain  $12x - 3 \leq 0$ . First, we will solve the equality

$$\begin{aligned} 12x - 3 &= 0 \\ 12x &= 3 \\ x &= \frac{1}{4} \end{aligned}$$

Next, we make a sign chart using the solution(s) to the equality as follows:

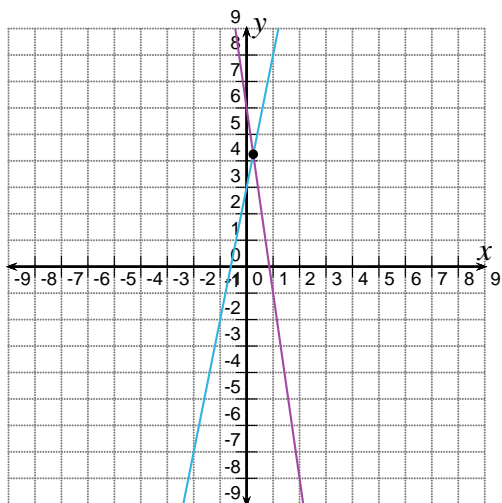


Since we want those values of  $x$  where  $12x - 3 \leq 0$ , then our solution set is  $\left(-\infty, \frac{1}{4}\right]$

#### Note 8.4

There are many ways to solve this inequality algebraically. We will begin by using addition and subtraction to move all the nonzero quantities to one side. This is not necessary for this inequality, but it will help us to understand the process needed for solving more complicated inequalities.

Graph the equations  $y = 5x + 3$  and  $y = 6 - 7x$ . How can you approximate the solutions of an inequality graphically?



$x$	$5x + 3$
0	3
1	8
2	13
-1	-2
-2	-7

$x$	$6 - 7x$
0	6
1	-1
2	-8
-1	13
-2	20

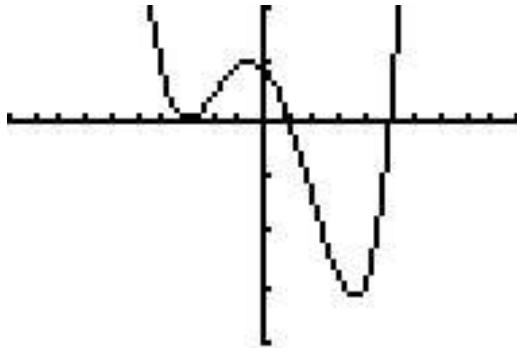
The graphs intersect when  $x = \frac{1}{4}$ . The graph of  $y = 5x + 3$  is below the graph of  $y = 6 - 7x$  when  $x$  is in the interval  $\left(-\infty, \frac{1}{4}\right]$ . Thus, the  $y$ -value of  $y = 5x + 3$  is less than the  $y$  value of  $y = 6 - 7x$ . solving  $5x + 3 \leq 6 - 7x$ .

Thinking graphically can help us understand the algebraic procedure for solving Nonlinear Inequalities. Given an expression, such as  $(x+3)^2(x-1)(x-5)$ , the expression is positive when the graph of  $y = (x+3)^2(x-1)(x-5)$  is above the x-axis. The expression is negative when the graph of  $y = (x+3)^2(x-1)(x-5)$  is below the x-axis.

**Example 8.5 (Polynomial Inequality)**

The graph of  $y = (x+3)^2(x-1)(x-5)$  is shown below. The viewing window is  $[-10, 10] \times [-200, 100]$ . Use the graph to help you approximate the solutions of the inequality.

$$(x+3)^2(x-1)(x-5) > 0$$



$$(-\infty, -3) \cup (-3, 1) \cup (5, \infty)$$

The algebraic procedure for solving an inequality is based on the intuition we gain from the graphical solution.

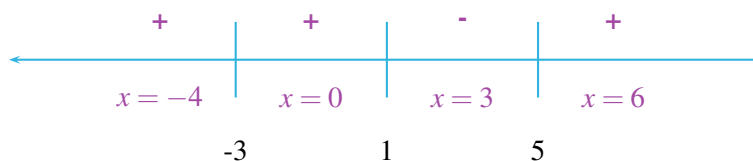
**Algebraic Procedure for Solving Linear and Nonlinear Inequalities**

1. Use addition and subtraction to move all nonzero quantities to one side. If fractional expressions are involved, simplify so the nonzero side is a single fractional expression.
2. Find the zeros of the expression AND the zeros of all denominators. (If you can factor the expression, this can help. Finding a zero means find when the expression equals zero.)
3. Make a sign chart to determine if the values between the zeros from step 2 lead to positive or negative values of the polynomial.
4. Answer the question.

**Example 8.6**

Use the algebraic approach to solve  $(x+3)^2(x-1)(x-5) > 0$ . Be sure to write you answer in interval notation.

$(x+3)^2(x-1)(x-5)$  is:



Since we want the values of  $x$  such that  $(x+3)^2(x-1)(x-5) > 0$ , then are solution set contains the positive intervals. So, the solution set is  $(-\infty, -3) \cup (-3, 1) \cup (5, \infty)$ .

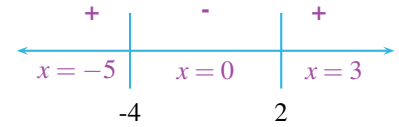
**Example 8.7 (Quadratic Inequality)**

Solve the inequality below. Be sure to write your answer in interval notation.

$$x^2 + 2x > 8$$

Rewrite as  $x^2 + 2x - 8 > 0$ , then solve the equality

$$\begin{aligned} x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \\ x+4=0 &\text{ or } x-2=0 \\ x=-4 &\quad x=2 \end{aligned}$$

 $(x+4)(x-2)$  is:The solution set is  $(-\infty, -4) \cup (2, \infty)$ **Example 8.8 (Rational Inequality)**

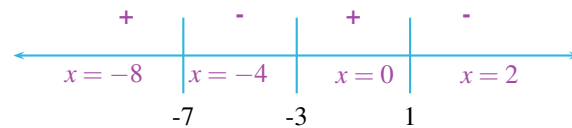
Solve the inequality below. Be sure to write your answer in interval notation.

$$\frac{2}{x+3} \leq \frac{4}{x-1}$$

$$\begin{aligned} \frac{2}{x+3} &\leq \frac{4}{x-1} \\ \frac{2}{x+3} - \frac{4}{x-1} &\leq 0 \\ \frac{2(x-1) - 4(x+3)}{(x+3)(x-1)} &\leq 0 \\ \frac{2x-2-4x-12}{(x+3)(x-1)} &\leq 0 \\ \frac{-2x-14}{(x+3)(x-1)} &\leq 0 \end{aligned}$$

$$\begin{aligned} -2x - 14 &= 0 \\ -2x &= 14 \\ x &= -7 \end{aligned}$$

$$\begin{aligned} (x+3)(x-1) &= 0 \\ x+3=0 &\quad x-1=0 \\ x=-3 &\quad x=1 \end{aligned}$$

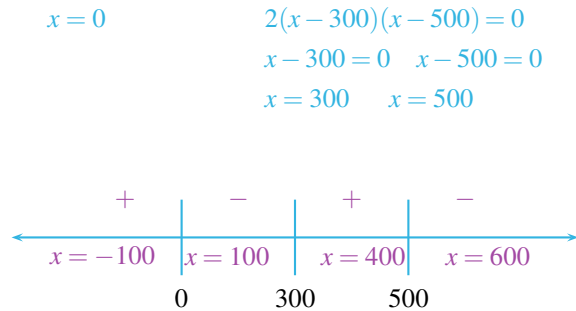
The solution set is  $[-7, -3) \cup (1, \infty)$ . Note, the points  $x = -3$  and  $x = 1$  are excluded because division by zero is undefined.

**Example 8.9 (Inequality Application)**

A computer store has determined that the cost  $C$  of ordering and storing  $x$  laser printers is given by  $C = 2x + \frac{300,000}{x}$ . If the delivery truck can bring at most 450 printers per order, how many printers should be ordered at a time to keep the cost below \$1600?

First we need  $0 \leq x \leq 450$ . The question is asking what  $x$  values ensure  $C \leq 1600$ ?

$$\begin{aligned} 2x + \frac{300,000}{x} &\leq 1600 \\ 2x + \frac{300,000}{x} - 1600 &\leq 0 \\ \frac{2x^2 - 1600x + 300,000}{x} &\leq 0 \\ \frac{2(x^2 - 800x + 150,000)}{x} &\leq 0 \\ \frac{2(x - 300)(x - 500)}{x} &\leq 0 \end{aligned}$$



The solution set to the inequality is  $(-\infty, 0) \cup [300, 500]$ . However, for this word problem, the number of printers  $x$  must be positive, (i.e.  $x \geq 0$ ) and less than 450 to fit on the truck. Thus, to answer the question, between 300 and 450 printers should be ordered.

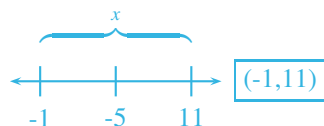
### 8.3 Absolute Value Inequalities

Number lines can be really insightful when working with absolute value equations and inequalities. Recall that we think of the absolute value as a distance.

**Example 8.10 (A Distance Example)**

Solve  $|5 - x| < 6$  geometrically. Be sure to write your answer in interval notation.

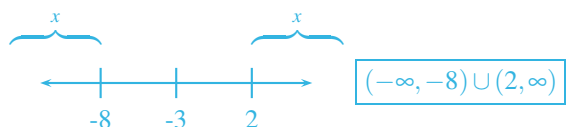
The distance from 5 to  $x$  is less than 6.



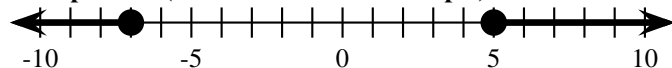
**Example 8.11 (Another Distance Example)**

Solve  $|x + 3| > 5$  geometrically. Be sure to write your answer in interval notation.

The distance from  $x$  to -3 is greater than 5.



**Example 8.12 (Another Distance Example)**



- (a) Write a distance sentence that corresponds to this number line. The distance from  $x$  to -1 is greater than or equal to 6.
- (b) Write an absolute value equation or inequality that corresponds to this number line.  $|x + 1| \geq 6$

**Example 8.13 (The Algebraic Approach to Absolute Values)**

Solve each inequality algebraically.

(a)  $|2x + 3| \geq 7$

$$\begin{aligned} 2x + 3 &\geq 7 \\ 2x - 4 &\geq 0 \end{aligned}$$

or

$$\begin{aligned} 2x + 3 &\leq -7 \\ 2x + 10 &\leq 0 \end{aligned}$$



The solution set is  $(-\infty, -5] \cup [2, \infty)$ .

(b)  $|x + 2| + 1 < 3$

$$\begin{aligned} |x + 2| + 1 &< 3 \\ |x + 2| &< 2 \\ x + 2 &< 2 \quad \text{or} \quad x + 2 > -2 \\ x &< 0 \quad \quad x > -4 \end{aligned}$$

The solution set is  $(-4, 0)$ .