# Table of contents

# Concepts

- Quadratic Functions
  - The Definition of a Quadratic Function
  - Graphs of Quadratic Functions - Parabolas
  - Vertex
  - Absolute Maximum or Absolute Minimum
  - Trasforming the Graph of $f(x) = x^2$ to Obtain the Graph of Any Quadratic Function.
- Polynomial Functions
  - The Definition of a Polynomial
  - Leading Coefficient, Leading Term, Constant Term and the Degree of a Polynomial
  - Graphs of Polynomials
    - Basic Shapes
    - Continuous Graphs
    - Smooth Graphs
    - End Behavior of the Graph
    - Multiplicity of a Root and Behavior of the Graph at $x$-intercepts.
    - How Many Local Extrema Can a Polynomial Graph Have?

# Concepts

- Polynomials-cont.
  - Polynomial Division - The Division Algorithm
  - Roots and Zeros of a Polynomial
  - The Remainder Theorem
  - The Factor Theorem
  - The Rational Root Theorem
  - Finding Rational Roots of a Polynomial
  - Finding Real Roots of a Polynomial

**(Sections 4.1-4.4)**

# Polynomials

We have already studied linear functions and power functions.

- Linear and power functions are special types of polynomial functions.
- Polynomial functions have several very nice properties.
  - Domain of a polynomial function is $(-\infty, \infty)$.
  - The graph of a polynomial function is **continuous** and **smooth**.
- Polynomial functions can be simply built from power functions.
- Linear functions are the simplest of all polynomial functions. Quadratic functions are next in the line of polynomial functions.

# Quadratic Functions

**Definition 1**

A **quadratic function** is a function that is equivalent to a function of the form

$$q(x) = ax^2 + bx + c$$

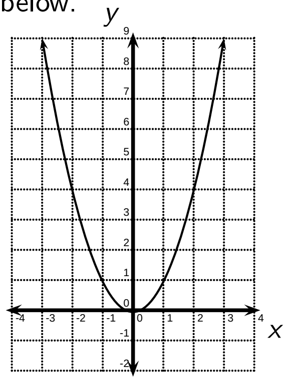
where $a$, $b$, and $c$ are constants and $a \neq 0$.

# Quadratic Functions

**Definition 2**

The graph of a quadratic function is called a **parabola**.

The most basic quadratic function is $f(x) = x^2$. The graph of $f$ is shown below.



# Quadratic Functions

Parabolas are important because they have either an **absolute minimum value** (a smallest output value) or an **absolute maximum value** (a largest output value). The point on the graph that corresponds to the absolute minimum or absolute maximum value is called the **vertex** of the parabola. For the previous graph, the absolute minimum value is 0 and the vertex is $(0, 0)$.

# Quadratic Functions

The graph of every quadratic function can be obtained by transforming the graph of $y = f(x) = x^2$.

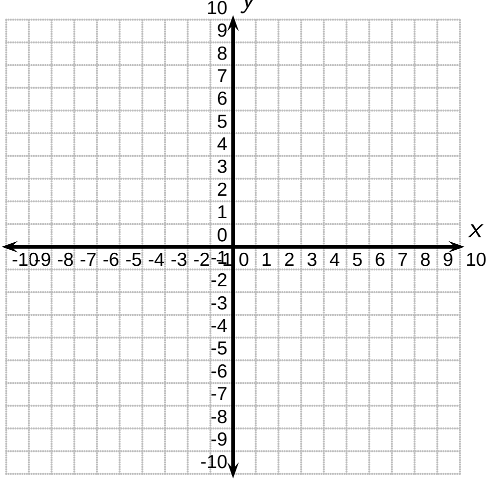
Example 3 (Transformations and Quadratic Functions)

$f(x) = x^2$ and $g(x) = x^2 + 6x + 7$.

- Describe the transformations that could be applied to the graph of $f$ to obtain the graph of $g$.
- Sketch the graph of $g$.
- What is the vertex of the graph of $g$?
- Does $g$ have an absolute minimum value or an absolute maximum value? What is it?

# Quadratic Functions

Let $f(x) = x^2$ and $g(x) = x^2 + 6x + 7$.



# Quadratic Functions

**Definition 4**

A quadratic function $q(x) = ax^2 + bx + c$ can be rewritten in its **standard form**

$$q(x) = a(x - h)^2 + k$$

where $a \neq 0$ and $(h, k)$ is the vertex of the parabola.

# Quadratic Functions

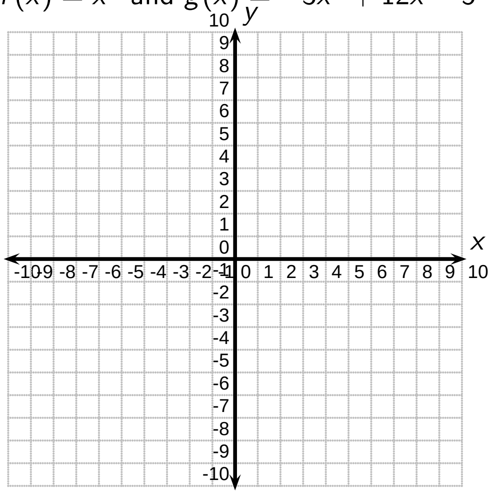
## Example 5 (Transformations and Quadratic Functions)

Let $f(x) = x^2$ and $g(x) = -3x^2 + 12x - 5$.

- Express $g(x)$ in standard form and sketch its graph.
- What is the vertex of the graph of $g$?
- Does $g$ have an absolute minimum value or an absolute maximum value? What is it?

# Quadratic Functions

$f(x) = x^2$ and $g(x) = -3x^2 + 12x - 5$



# Quadratic Functions

## Example 6 (Min or Max?)

Given a quadratic function $f(x) = ax^2 + bx + c$,

- when does a quadratic function have an absolute maximum value?
- when does a quadratic function have an absolute minimum value?
- The vertex of a quadratic function occurs when $x = -\dfrac{b}{2a}$. What is the $y$-coordinate of the vertex?

# Quadratic Functions

## Example 7

Find the maximum value of the function $f(x) = -3x^2 + 10x + 4$.

# Quadratic Functions

## Example 8

Find a quadratic function $f(x) = ax^2 + bx + c$ whose vertex is $(3, -1)$ and goes through the point $(5, 7)$.

# Quadratic Functions

## Example 9 (Optimization)

A farmer has 200 feet of fencing to construct five rectangular pens as shown in the diagram below.

What is the maximum possible area of all five pens?

# Polynomials

## Definition 10

A **polynomial** in $x$ is an algebraic expression that is equivalent to an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n$ is a non-negative integer, $x$ is a variable, and the $a_i$'s are all constants.

In a polynomial, $a_i$ is called the **coefficient** of $x^i$ and $a_0$ is called the **constant term** of the polynomial. If the polynomial contains only a constant term, it is called a **constant polynomial**. If the polynomial equals 0, then it is called the **zero polynomial**.

# Polynomials

If you write the non-zero terms in order so the exponents of $x$ decrease, the first term $a_n x^n$ is the **leading term**. The coefficient of the leading term, $a_n$, is called the **leading coefficient**. The exponent of $x$ in the leading term is called the **degree** of the polynomial. The degree of the zero polynomial is undefined since there is no nonzero term.

# Polynomials

| Polynomial | Not A Polynomial |
|---|---|
| $x^3 + 3x^4 + 2$ | $x^{-3} + 3x^4 + 2$ |
| $\sqrt{2}x + 1$ | $\sqrt{2x} + 1$ |

# Polynomials

## Example 11 (Polynomials)

Determine which of the following expressions are polynomials. If the expression is a polynomial, find its leading term, leading coefficient and degree.

| | Poly? | Leading Term | Leading Coefficient | Degree |
|---|---|---|---|---|
| $\frac{1}{5}x^7 + 2x^3 + \frac{2}{9}x + x^8 + 3x^4$ | | | | |

# Polynomials

Determine which of the following expressions are polynomials. If the expression is a polynomial, find its leading term, leading coefficient and degree.

| | Poly? | Leading Term | Leading Coefficient | Degree |
|---|---|---|---|---|
| $x^{\frac{1}{3}} + 3x^4 + 2$ | | | | |

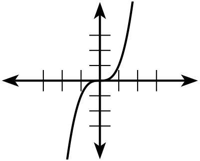
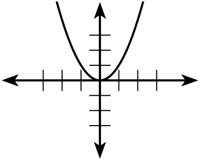
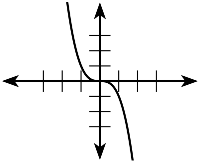
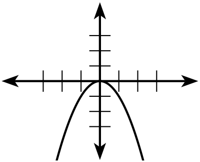
# Polynomials

Determine which of the following expressions are polynomials. If the expression is a polynomial, find its leading term, leading coefficient and degree.

| | Poly? | Leading Term | Leading Coefficient | Degree |
|---|---|---|---|---|
| $(1-x)^4(5+3x)^2$ | | | | |

# Graphs of Polynomials

**Basic Shapes** Let $f(x) = ax^n$.

| | $n$ odd | $n$ even |
|---|---|---|
| $a$ positive |  |  |
| $a$ negative |  |  |

# Graphs of Polynomials

## Example 12

What is the basic shape of $f(x) = -3x^4$?

# Graphs of Polynomials

**Special Properties of Polynomials Graphs**

**(P-1)** Polynomial Graphs are **continuous**.

The graph of a function is **continuous** if there are no breaks in the graph. Intuitively, this means that you can sketch the graph without picking up your pencil. (The graph of the greatest integer function is not continuous because there are breaks in the graph.)

# Graphs of Polynomials

**(P-2)** Polynomial graphs are **smooth**.

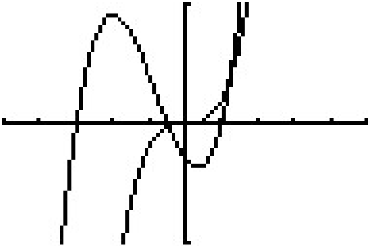
The graph of a polynomial does not have any sharp corners. (The graph of the absolute value function is not smooth because there is a sharp corner at the tip of the vee.)

# Graphs of Polynomials

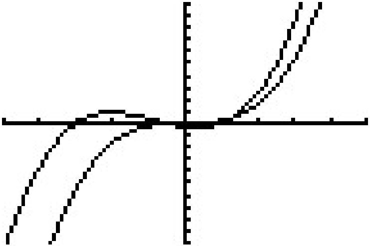
**(P-3)** The **shape of a polynomial graph is dominated by the leading term when $|x|$ is large**.

For example, let's consider the graphs of $y = 2x^3$ and $y = 2x^3 + 5x^2 - 4x - 3$.

Viewing Window: $[-5, 5] \times [-10, 10]$

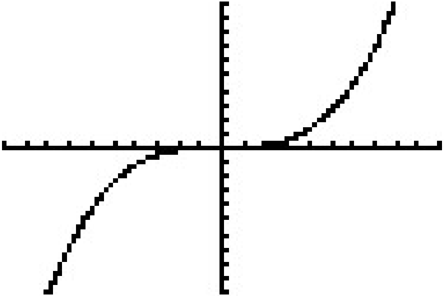


Viewing Window: $[-5, 5] \times [-100, 100]$



# Graphs of Polynomials

Viewing Window $[-100, 100] \times [-1000000, 1000000]$



We say that the graphs of $y = 2x^3$ and $y = 2x^3 + 5x^2 - 4x - 3$ have the same **end behavior**.

# Graphs of Polynomials

Eventually, the leading term of any polynomial will dominate the shape of its graph, so the end behavior of a polynomial graph can be determined by examining the graph of the leading term.

**END BEHAVIOR:**

We have some notation to help us describe the end behavior of graphs.

- If the $y$ values become very large as $x$ becomes very large, we denote this by

$$y \to \infty \text{ as } x \to \infty.$$

- If the $y$ values become very large and negative as $x$ becomes very large, we denote this by

$$y \to -\infty \text{ as } x \to \infty.$$

# Graphs of Polynomials

- If the $y$ values become very large as $x$ becomes very large and negative, we denote this by

$$y \to \infty \text{ as } x \to -\infty.$$

- If the $y$ values become very large and negative as $x$ becomes very large and negative, we denote this by

$$y \to -\infty \text{ as } x \to -\infty.$$

# Graphs of Polynomials

## Example 13 (End Behavior)

Describe the end behavior of the graph of $y = 7 - x + 3x^8$.

# Graphs of Polynomials

## Example 14 (End Behavior)

Describe the end behavior of the graph of $y = ax^n$ by examining the basic shapes.

- If $a > 0$ and $n$ even,

$$y \to \underline{\hspace{4cm}} \text{ as } x \to \infty \text{ and}$$

$$y \to \underline{\hspace{4cm}} \text{ as } x \to -\infty.$$

# Graphs of Polynomials

- If $a < 0$ and $n$ even,

$$y \to \underline{\hspace{4cm}} \text{ as } x \to \infty \text{ and}$$

$$y \to \underline{\hspace{4cm}} \text{ as } x \to -\infty.$$

# Graphs of Polynomials

- If $a > 0$ and $n$ odd,

$$y \to \underline{\hspace{4cm}} \text{ as } x \to \infty \text{ and}$$

$$y \to \underline{\hspace{4cm}} \text{ as } x \to -\infty.$$

# Graphs of Polynomials

- If $a < 0$ and $n$ odd,

$$y \to \underline{\hspace{4cm}} \text{ as } x \to \infty \text{ and}$$

$$y \to \underline{\hspace{4cm}} \text{ as } x \to -\infty.$$

# Graphs of Polynomials

### Example 15

Describe the end behavior of the graph of $y = (5 - x)^3(2x + 15)^{71}$.

# Graphs of Polynomials

**(P-4)** The graph of a polynomial of degree $n$ can have **at most** $n$ $x$-**intercepts**.

# Graphs of Polynomials

### Definition 16 (Roots and Zeros)

Let $P(x)$ be a polynomial. The number $c$ is called a **root** or a **zero** of $P$ if and only if $P(c) = 0$. (*Note that a zero is the same as an* $x$*-intercept.*)

# Graphs of Polynomials

## Example 17 (To Cross or To Touch and Turn Around?)

Use your graphing calculator to look at the following graphs and describe the behavior of the graph at $x = 3$ and $x = -5$.

1. $y = (x - 3)(x + 5)$
2. $y = (x - 3)^2(x + 5)$
3. $y = (x - 3)^3(x + 5)$
4. $y = (x - 3)^4(x + 5)$
5. $y = (x - 3)(x + 5)^2$
6. $y = (x - 3)^2(x + 5)^3$

# Graphs of Polynomials

Definition 18

If $(x-c)^k$ is a factor of a polynomial $P(x)$ an no higher power of $(x-c)$ is a factor of the polynomial, then $c$ is called a **root of multiplicity** $k$ of the polynomial $P$.

# Graphs of Polynomials

## Example 19 (Multiplicity of a root)

Let $P(x) = (x-1)^2(x-2)(x+4)^7(x+7)^9$. List the roots of $P$ and their multiplicities. Describe the behavior of the graph of $P$ at each $x$-intercept.

# Graphs of Polynomials

**Theorem 20**

*Let $P(x)$ be a polynomial and let $c$ be a root of multiplicity $k$ of $P$.*

- *If $k$ is odd, then the graph of $P$ crosses the $x$-axis at $(c, 0)$.*
- *If $k$ is even, then the graph of $P$ touches the $x$-axis at $(c, 0)$ but does not cross the $x$-axis at $(c, 0)$.*

# Graphs of Polynomials

**(P-5)** The graph of a polynomial of degree $n$ has **at most** $n-1$ **local extrema**.

> **Definition 21**
> A **local extremum** is a local minimum or a local maximum. The plural of "local extremum" is "**local extrema**"

On a graph, a local minimum appears as the point at the bottom of of a valley and a local maximum appears as the point at the top of a mountain. This does not mean that it is the lowest valley or the highest mountain. You are looking for all valleys and all mountains.

# Graphs of Polynomials

## Example 22 (Local Extrema)

- Could the graph of a $7^{th}$ degree polynomial have 8 local extrema?
- Could the graph of a $7^{th}$ degree polynomial have 5 local extrema?
- Could the graph of a $7^{th}$ degree polynomial have 4 local extrema?

# Graphs of Polynomials

## Example 23 (Local Extrema)

How many local extrema could the graph of a polynomial of degree 10 have?

# Polynomial Division

## Example 24 (Review of Long Division)

Use long division to find the quotient and the remainder.

$$\frac{794}{3}$$

# Polynomial Division

## Theorem 25 (The Division Algorithm)

*Let $P(x)$ and $D(x)$ be polynomials. Then there are unique polynomials $Q(x)$ and $R(x)$ such that*

$$P(x) = D(x)Q(x) + R(x)$$

*and either $R(x)$ is the zero polynomial or the degree of $R(x)$ is less than the degree of $D(x)$.*

## Definition 26

In the Division Algorithm:

- $P(x)$ is the **dividend**.
- $D(x)$ is the **divisor**.
- $Q(x)$ is the **quotient**.
- $R(x)$ is the **remainder**.

# Polynomial Division

## Example 27 (Polynomial Division)

Find the quotient and the remainder.

$$\frac{3x^3 - 2x^2 + 4x - 3}{x + 4}$$

# Polynomial Division

## Example 28 (Polynomial Division)

Find the quotient and the remainder.

$$\frac{6x^5 + 18x^2 + 6}{3x^2 + 9}$$

# Polynomial Division

Example 29 (A Preview of the Factor and Remainder Theorems)

Let $P(x) = x^2 + 5x + 6$. Find the quotient and remainder of

$$\frac{P(x)}{x+3}.$$

# Polynomial Division

- $P(x) = (x+3) \cdot$ ____________ $+$ ______
- What does the remainder tell you about the factors of $P$?
- What does the remainder tell you about $P(-3)$?
- What does the remainder tell you about the graph of $P$?

# Polynomial Division

Example 30 (A Preview of the Factor and Remainder Theorems)

Let $P(x) = 2x^2 + 8$. Find the quotient and remainder of

$$\frac{P(x)}{x-2}.$$

# Polynomial Division

- $P(x) = (x - 2) \cdot$ _____________ $+$ _______
- What does the remainder tell you about the factors of $P$?
- What does the remainder tell you about $P(2)$?
- What does the remainder tell you about the graph of $P$?

# Polynomial Division

**Theorem 31**

*Let $P(x)$ and $D(x)$ be polynomials. Then $D(x)$ is a factor of $P(x)$ if and only if the remainder of the division problem $\dfrac{P(x)}{D(x)}$ is the zero polynomial.*

# Polynomial Division

**Theorem 32 (The Remainder Theorem)**

*Let $P(x)$ be a polynomial. Then*

$$P(c) = \textit{the remainder of the division problem } \frac{P(x)}{x-c}.$$

# Polynomial Division

## Example 33

What is the remainder of $\dfrac{x^5+7}{x+2}$?

# Polynomial Division

The next theorem includes the phrase, "The following are equivalent. . ." This means that all the statements are true or all of them are false. It is never the case that some are true and some are false.

**Theorem 34**

*The following are equivalent for the polynomial $P(x)$:*

- $(x - c)$ *is a factor of* $P(x)$.
- _____________ *is a root (or zero) of* $P(x)$.
- $P(c) =$ _____________.
- _____________ *is an* $x$*-intercept of the graph of* $P$.

# Polynomial Division

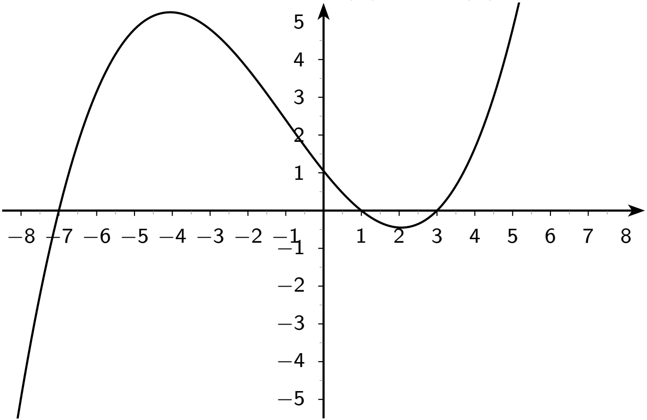
**Theorem 35 (Number of Roots)**

*A polynomial of degree $n$ has at most $n$ distinct roots.*

# Polynomial Division

## Example 36

Below is the graph of $y = f(x)$ and $f(x)$ is a polynomial function.




What is the remainder when $f(x)$ is divided by $(x + 7)$?

# Finding Roots of Polynomials

The roots or zeros of a polynomial are often important in applications. When a polynomial has integer coefficients, the Rational Roots Theorem allows us to narrow the search for roots which are rational numbers.

**Theorem 37 (The Rational Roots Theorem)**

*Let*

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad (a_n \neq 0)$$

*be a polynomial* **with integer coefficients**. *If $\frac{r}{s}$ is a rational number in lowest terms and $\frac{r}{s}$ is a root of $P$ then*

- *$r$ is a factor of $a_0$ AND*
- *$s$ is a factor of $a_n$.*

# Finding Roots of Polynomials

## Example 38 (The Rational Roots Theorem)

Let $P(x) = 3x^3 - 8x^2 - x + 10$. List all of the possible rational roots of $P(x)$ as given by the Rational Roots Theorem. (Do not check to see which are actually zeros.)

# Finding Roots of Polynomials

Example 39 (Finding all Rational Roots of a Polynomial)

Let $P(x) = 2x^4 + 8x^3 + 2x^2 - 16x - 12$. Find all the rational roots of $P(x)$. What does this tell you about the factors of $P(x)$? *(Hint: This may seem like a long list of values at which to evaluate $P$, but you can use the table function on your calculator to expedite the process.)*

# Finding Roots of Polynomials

Example 40

Completely factor 6160.

# Finding Roots of Polynomials

Suppose you know that $(x - c)$ is a factor of $P(x)$. What does this mean? How can you use this information to completely factor $P(x)$?

# Finding Roots of Polynomials

Example 41 (Finding all Real Roots of a Polynomial)

Let $P(x) = 2x^4 + 8x^3 + 2x^2 - 16x - 12$. Find all the real roots of $P(x)$.