Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. You should also show your work on the multiple choice questions as it will make it easier for you to check your work. You should give exact answers, rather than a decimal approximation unless the problem asks for a decimal answer. Thus, if the answer is 2π, you should not give a decimal approximation such as 6.283 as your final answer.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Which of the following is equal to $\int x f(x)\, dx$?

   A. $f(x) + \int x^2 f(x)\, dx$

   B. $\frac{x^2}{2} f(x) + \frac{1}{2} \int x^2 f'(x)\, dx$

   C. $\frac{x^2}{2} f(x) - \frac{1}{2} \int x^2 f'(x)\, dx$

   D. $x f(x) - \frac{1}{2} \int x^2 f'(x)\, dx$

   E. $x f(x) + \frac{1}{2} \int x^2 f'(x)\, dx$

2. (5 points) Find $\int x \sin(2x)\, dx$.

   A. $\frac{1}{2} x^2 \sin(2x) + \frac{1}{4} \cos(2x) + C$

   B. $-\frac{1}{2} x \cos(2x) + \frac{1}{4} \sin(2x) + C$

   C. $\frac{1}{2} x \cos(2x) - \frac{1}{4} \sin(2x) + C$

   D. $\frac{1}{2} x \sin(2x) - \frac{1}{4} \cos(2x) + C$

   E. $2x \sin(2x) - \cos(2x) + C$

3. (5 points) Suppose that $x = 2\tan(u)$ and $-\pi/2 < u < \pi/2$. Express $\cos(u)$ in terms of x.

   A. $\dfrac{2}{\sqrt{4+x^2}}$

   B. $\dfrac{1}{\sqrt{4+x^2}}$

   C. $\sqrt{4+x^2}$

   D. $\dfrac{\sqrt{1+x^2}}{2}$

   E. $\dfrac{x}{\sqrt{4+x^2}}$

4. (5 points) Choose the best substitution to evaluate the integral $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$.

   A. $u = 4 - x^2$

   B. $x = 4 - u^2$

   C. $x = 2\tan(u)$

   D. $u = 2\tan(x)$

   E. $u = 2\sin(x)$

5. (5 points) Give the partial fractions decomposition for the function

$$\frac{x+4}{x^2-4}.$$

   A. $\dfrac{3}{x-2} - \dfrac{1}{x+2}$

   B. $\dfrac{3}{2(x-2)} + \dfrac{1}{2(x+2)}$

   C. $\dfrac{3}{2(x-2)} - \dfrac{1}{2(x+2)}$

   D. $\dfrac{1}{2(x-2)} - \dfrac{1}{2(x+2)}$

   E. $\dfrac{1}{2(x-2)} - \dfrac{3}{2(x+2)}$

6. (5 points) Consider the function $f(x) = \dfrac{1}{(x+2)^2(x^2+1)^2}$. Which of the following terms does not appear in the partial fractions decomposition of f?

   A. $\dfrac{A}{(x+2)^3}$

   B. $\dfrac{B}{(x+2)^2}$

   C. $\dfrac{C}{x+2}$

   D. $\dfrac{D_1 x + D_2}{x^2+1}$

   E. $\dfrac{E_1 x + E_2}{(x^2+1)^2}$

7. (5 points) Use the following information and Simpson's rule S_4 to find an approximate value for the integral $\int_1^3 f(x)\,dx$.

x	1	1.5	2	2.5	3
$f(x)$	3	6	12	9	6

   A. 29/2

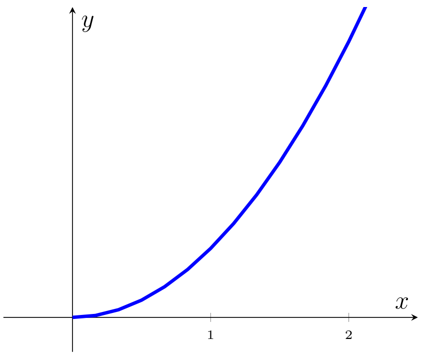
   B. 31/2

   C. 33/2

   D. 15

   E. 16

8. (5 points) We consider R_n, L_n, T_n and the value of the integral $I = \int_0^2 f(x)$ for the function whose graph is pictured. Which of the following statements is correct?



A. $I \leq L_n \leq R_n \leq T_n$

B. $L_n \leq T_n \leq R_n \leq I$

C. $R_n \leq I \leq T_n \leq L_n$

D. $L_n \leq T_n \leq I \leq R_n$

E. $L_n \leq I \leq T_n \leq R_n$

9. (5 points) Determine if the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ exists and, if it exists, give its value.

A. 1

B. 2

C. 1/2

D. -1

E. The integral does not exist.

10. (5 points) Determine if the improper integral $\int_{-1}^{1} \frac{1}{x}\,dx$ exists and, if it exists, give its value.

A. -1

B. 1

C. 0

D. $2e$

E. The integral does not exist.

Free Response Questions

11. (a) (8 points) Find $\int \cos^4(x)\,dx$.

(b) (2 points) Compute the definite integral $\int_0^{\pi/2} \cos^4(x)\,dx$.

Solution:

a) Use the identity $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ to write

$$\int \cos^4(x) = \frac{1}{4}\int (1+\cos(2x))^2\,dx$$

Expand the square to obtain

$$\int \cos^4(x)\,dx = \frac{1}{4}\int (1+2\cos(2x)+\cos^2(2x)\,dx$$

Write $\cos^2(2x) = \frac{1}{2}(1+\cos(4x))$ and collect like terms to obtain

$$\frac{1}{4}\int (1+2\cos(2x)+\cos^2(2x)\,dx = \int \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\,dx$$

Finally, evaluating the integral gives

$$\int \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\,dx = \frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C$$

b) From the fundamental theorem of calculus and the anti-derivative in part a)

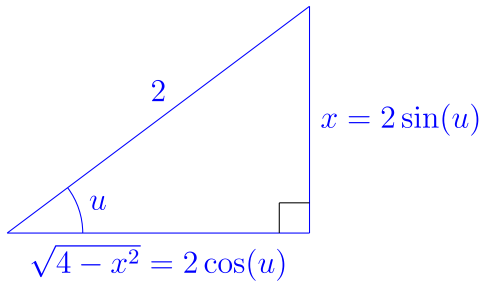
$$\int_0^{\pi/2} \cos^4(x)\,dx = \frac{3}{8}x+\frac{1}{4}\sin(2x)\frac{1}{32}\sin(4x)\Big|_0^{\pi/2} = \frac{3\pi}{16}.$$

Grading: a) Identity for $\cos^2(x)$ (2 points), expand square (1 point), second use of identify (2 points), find anti-derivatives (2 points), simplify (1 point).

b) Use of Fund. Thm. Calculus (1 point), answer (1 point).

12. (10 points) Find the anti-derivative $\int \frac{1}{(4-x^2)^{3/2}}\,dx$. Simplify to give an answer that does not include trigonometric functions.

Solution:



We substitute $x = 2\sin(u)$ and $dx = 2\cos(u)\,du$, $-\pi/2 \le u \le \pi/2$. This implies $\sqrt{4-x^2} = \sqrt{4-4\sin^2(u)} = 2\cos(u)$. We obtain the integral

$$\int \frac{1}{(4-x^2)^{3/2}}\,dx = \int \frac{2\cos(u)}{8\cos^3(u)}\,du.$$

Simplifying gives the integral $\frac{1}{4}\int \frac{1}{\cos^2(u)}\,du = \frac{1}{4}\int \sec^2(u)\,du$.

A basic anti-derivative gives $\frac{1}{4}\int \sec^2(u)\,du = \frac{1}{4}\tan(u) + C$ Finally, we write $x = 2\sin(u)$ and $2\cos(u) = \sqrt{4-x^2}$ to conclude that

$$\frac{1}{4}\tan(u) + C = \frac{1}{4}\frac{\sin(u)}{\cos(u)} + C = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C$$

Grading: Substitution $x = 2\sin(u)$, $dx = 2\cos(u)$ (3 points), simplify $\sqrt{4-x^2} = 2\cos(u)$ (1 point), simplify to integral of $\sec^2(u)$ (3 points), anti-derivative of $\sec^2(u)$ (1 point), write in terms of x (2 points).

13. (a) (7 points) Find the partial fractions decomposition of $\dfrac{x^2+3x+1}{x(x+1)^2}$.

(b) (3 points) Find $\displaystyle\int \frac{x^2+3x+1}{x(x+1)^2}\,dx$.

Solution:

a) We know that for some constants we have

$$\frac{x^2+3x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Find a common denominator and the right-hand side becomes

$$\frac{A(x^2+2x+1) + B(x^2+x) + Cx}{x(x+1)^2}$$

Collecting like terms gives

$$\frac{x^2+3x+1}{x(x+1)^2} = \frac{(A+B)x^2 + (2A+B+C)x + A}{x(x+1)^2}.$$

Equating coefficients in the numerator gives the system of equations

$$\begin{aligned} A &= 1 \\ 2A+B+C &= 3 \\ A+B &= 1 \end{aligned}$$

Solving these equations gives $A = 1$, $B = 0$, and $C = 1$. Thus we obtain the partial fraction decomposition

$$\frac{x^2+3x+1}{x(x+1)^2} = \frac{1}{x} + \frac{1}{(x+1)^2}$$

b) Integrating we find

$$\int \frac{1}{x} + \frac{1}{(x+1)^2}\,dx = \log(|x|) - 1/(x+1) + C$$

Grading: One point per term in form of decomposition (3 points), equations for coefficients (2 points), solve for coefficents and final decomposition (2 points).

b) One point per term in anti-derivative, including $+C$ (3 points).

Other methods are possible. Try to give equivalent credit.

14. (a) (7 points) Find the anti-derivative $\int x^2 e^{-x}\, dx$.

(b) (3 points) Determine if the improper integral $\int_0^\infty x^2 e^{-x}\, dx$ exists and, if it exists, find its value.

Solution:

a) Let $u = x^2$, $dv = e^{-x}\, dx$ so that $du = 2x\, dx$ and $v = -e^{-x}$. Integrate by parts to obtain

$$\begin{aligned} \int x^2 e^{-x}\, dx &= -x^2 e^{-x} - \int 2x(-e^{-x})\, dx \\ &= -x^2 e^{-x} + 2\int x e^{-x}\, dx \end{aligned}$$

We integrate by parts again with $u = x$ and $dv = e^{-x}\, dx$ to obtain

$$\begin{aligned} &= -x^2 e^{-x} - 2x e^{-x} + 2\int e^{-x}\, dx \\ &= (-x^2 - 2x - 2)e^{-x} + C \end{aligned}$$

Thus we obtain

$$\int x^2 e^{-x}\, dx = (-x^2 - 2x - 2)e^{-x} + C$$

b) We use the definition of improper integral to write

$$\int_0^\infty x^2 e^{-x}\, dx = \lim_{N\to\infty} \int_0^N x^2 e^{-x}\, dx.$$

We use the anti-derivative from part a) to write

$$\int_0^N x^2 e^{-x}\, dx = (-N^2 - 2N - 2)e^{-N} + 2.$$

From L'Hôpital's rule we have $\lim_{N\to\infty} (-N^2 - 2N - 2)e^{-N} = 0$.

Grading: a) Choice of $u = x^2$, $dv = e^{-x} dx$ (2 points), first integration by parts (2 points), choice of u and dv for second integration parts (1 point), second integration by parts (1 point), simplify answer (1 point).

b) Write improper integral as limit (1 point), use anti-derivative from part a) (1 point), find limit and obtain value (1 point).

15. Consider the integral $\int_1^3 \frac{1}{x}\,dx$.

(a) (4 points) Use the Trapezoid rule with $n = 4$ to find an approximate value for this integral.

(b) (6 points) If we use the Trapezoid rule with n, then the error satisfies

$$|T_n - \int_a^b f(x)\,dx| \le \frac{K(b-a)^3}{12n^2}$$

where K is a number that satisfies $|f''(x)| \le K$ for $a \le x \le b$.

Use this estimate to find a value for n for which we have $|T_n - \int_1^3 \frac{1}{x}\,dx| \le \frac{1}{500}$.

Solution:

a) Use division points $1, 3/2, 2, 5/2, 3$ with $h = (3-1)/4 = 1/2$. With $f(x) = 1/x$, we have

$$\begin{aligned} T_4 &= \frac{h}{2}(f(1) + 2f(3/2) + 2f(2) + 2f(5/2) + f(3)) \\ &= \frac{1}{4}(1 + \frac{4}{3} + \frac{1}{2} + \frac{4}{5} + \frac{1}{3}) = 67/60 \approx 1.117 \end{aligned}$$

b) Compute $f''(x) = 2/x^3$. Maximum value of $|f''(x)|$ for $1 \le x \le 3$ is $K = 2$. Substitute $a = 1$, $b = 3$ and $K = 2$ to obtain inequality for n,

$$\frac{2 \cdot 2^3}{12n^2} = \frac{4}{3n^2} \le 1500.$$

Solving for n gives $n \ge \sqrt{2000/3} = \sqrt{666.\overline{6}} \approx 25.8$. Since n must be an integer, the smallest correct choice for n is 26.

Grading: a) Value of h (1 point), Expression for T_4 (2 points), value of T_4 (1 point). Accept decimal approximations, but indicate exact answer is preferred.

b) Compute f'' (1 point), Value of K (1 point), inequality for n (2 points), solution to inequality (1 point), choosing next integer (1 point).

Accept larger values for K and n with appropriate justification.