Exam 2

Name: ______________________________ Section: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. The wise student will show work for the multiple choice problems as well.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) By definition, a series $\sum_{n=1}^{\infty} a_n$ converges whenever
   A. $\lim_{n\to\infty} a_n = 0$
   B. $|a_{n+1}/a_n| < 1$ for all n
   C. The sequence of partial sums S_n has a limit as $n \to \infty$
   D. The sequence of partial sums S_n is bounded
   E. The sequence of partial sums S_n is decreasing

2. (5 points) The power series expanded around $a = 0$ for the function $f(x) = \dfrac{1}{1-2x^2}$ is
   A. $\sum_{n=0}^{\infty}(-2)^n x^{2n}$
   B. $\sum_{n=0}^{\infty} 2^n x^{2n}$
   C. $\sum_{n=0}^{\infty}(-2)^n x^n$
   D. $\sum_{n=0}^{\infty}2^n x^n$
   E. $\sum_{n=0}^{\infty}2^{2n} x^n$

3. (5 points) What is the value of $\sum_{n=1}^{\infty}\dfrac{3^n}{2^{n-1}}$?
   A. The series diverges
   B. -6
   C. $2/3$
   D. $9/2.$
   E. $3/2.$

4. (5 points) Choose the correct statement about the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}.$$

   A. The radius of convergence is 1 and the interval of convergence is $(-1, 1]$
   B. The radius of convergence is 1 and the interval of convergence is $(-1, 1)$
   C. The radius of convergence is 1 and the interval of convergence is $[-1, 1)$
   D. The radius of convergence is 1 and the interval of convergence is $[-1, 1]$
   E. The radius of convergence is 0

5. (5 points) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}$$

is convergent or divergent by expressing S_n as a telescoping sum. If convergent, find its sum

A. 1

B. $\frac{1}{4}$

C. 3

D. $\frac{1}{3}$

E. The series diverges.

6. (5 points) Which one of the following power series converges for all real x?

A. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

B. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

C. $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

D. $\sum_{n=1}^{\infty} \frac{x^{2n}}{\sqrt{n}}$

E. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$

7. (5 points) Consider the series

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$$

What can you conclude by applying the root test to this series?

A. The series diverges

B. The series converges conditionally but not absolutely

C. The series converges absolutely

D. No conclusion about the convergence of this series can be drawn from the root test.

8. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{5^n n}{n!}$. What can you conclude about this series using the ratio test?

   A. The series converges absolutely

   B. The series converges conditionally but not absolutely

   C. The series diverges

   D. The ratio test is not conclusive

9. (5 points) Suppose that $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{2}a_n.$$

   Which of the following is the correct listing of a_1, a_2, a_3, a_4?

   A. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

   B. $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}$

   C. $1, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}$

   D. $1, \frac{3}{2}, \frac{7}{4}, \frac{7}{8}$

   E. $1, \frac{3}{2}, \frac{1}{4}, \frac{1}{8}$

10. (5 points) Applying the limit comparison test to the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$$

   by comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ leads to what conclusion about this series?

   A. No conclusion can be drawn from the limit comparison test.

   B. The series converges

   C. The series converges conditionally but not absolutely

   D. The series diverges

Free Response Questions

11. Determine if the sequence is convergent or divergent. If convergent give its limit.

(a) (3 points) $a_n = \dfrac{n^4}{n^3 - 2n}$

Solution: The sequence diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^4}{n^3\left(1-\frac{2}{n^2}\right)} = +\infty$

Scoring: Correct answer, 1 point; reasoning, 2 points.

(b) (3 points) $a_n = e^{n/(n+2)}$

Solution: The sequence converges. Since $\lim_{n\to\infty} \frac{n}{n+2} = 1$, it follows that

$$\lim_{n\to\infty} e^{n/(n+2)} = e.$$

Scoring: $\lim_{n\to\infty} n/(n+2) = 1$, 1 point; use continuity of exponential function, 1 point; correct answer, 1 point. Any other correct and complete solution should also receive full credit.

(c) (4 points) $a_n = \ln(2n^2+1) - \ln(n^2+1)$

Solution: The sequence converges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right) = \ln\left(\lim_{n\to\infty} \frac{2n^2+1}{n^2+1}\right) = \ln 2.$$

Scoring: Combine logarithms, 1 point; exchange logarithm and limit, 1 point; answer, 2 points. Any other correct and complete solution should also receive full credit.

12. Determine if the series is convergent or divergent. Be sure to state which test you are using.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

Solution: Use the ratio test with $a_n = 3^n n^2/(n!)$. We get

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n\, n^2}\right| = 3\left(\frac{n+1}{n}\right)^2 \frac{1}{n+1}.$$

Thus

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

and the series converges.

Scoring: Correctly compute ratio, 2 points; simplify, 1 point; take limit, 1 point; correct conclusion, 1 point.

(b) (5 points) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

Solution: Let $a_n = (n+1)/(n^2+1)$ and $b_n = 1/n$. Note that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n(n+1)}{n^2+1} = 1$$

so the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together by the Limit Comparison Test. Since $\sum_n b_n$ is the harmonic series and is known to diverge, it follows that $\sum_{n=1}^{\infty} a_n$ also diverges.

Scoring: Valid choice of comparison sequence, 1 point; computation of nonzero limit for ratio, 1 point; correct statement of ratio test and reason for divergence, 2 points; conclusion, 1 point.

13. (a) (5 points) State the root test. Be sure your conclusion describes all three cases.

Solution: Given a series $\sum_{n=1}^{\infty} a_n$ let

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

1. If $L < 1$ the series is absolutely convergent (and therefore convergent).
2. If $L > 1$ the series is divergent.
3. If $L = 1$ the test is inconclusive.

Scoring: Correct statement of subject ("Given a series..." or some other phrase that indicates the purpose of the test), 1 point; definition of L, 1 point; 1 point for each of alternatives (1)–(3)

(b) (5 points) Use the root test to determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)3^n}{2^{2n+1}}$ converges. Explain your reasoning. It may be useful to recall that

$$\lim_{n\to\infty} (n+1)^{1/n} = 1.$$

Solution: First,

$$|a_n|^{1/n} = \left| \frac{(n+1)3^n}{2^{2n+1}} \right|^{1/n} = \frac{3}{2^{2+1/n}} (n+1)^{1/n}.$$

Hence

$$\lim_{n\to\infty} |a_n|^{1/n} = \frac{3}{4}.$$

Since $L < 1$, the series converges absolutely.

Scoring: correct computation of $|a_n|^{1/n}$, 1 point; correct computation of limit, 2 points; correct conclusion, 2 points.

14. (10 points) Consider the series

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$$

Use the integral test to show that this series converges. Be sure to verify the hypotheses of the integral test and to show all work. You may use the following integral formula:

$$\int \frac{\ln(x)}{x^2}\,dx = -\frac{1+\ln(x)}{x} + C$$

Solution: Let $f(x) = \ln(x)/x^2$. The function f is continuous on $[2, \infty)$, and positive on $[2, \infty)$ since $\ln(x) > 0$ for $x > 1$. Finally, the function f is decreasing since $f'(x) = (1 - 2\ln x)/x^3$ and $\ln x > 1/2$ for $x \geq 2$. Using the given formula, we get

$$\int_2^N f(x)\,dx = \left(\frac{1+\ln(2)}{2}\right) - \left(\frac{1+\ln(N)}{N}\right).$$

Thus

$$\int_2^\infty f(x)\,dx = \lim_{N\to\infty}\left[\left(\frac{1+\ln(2)}{2}\right) - \left(\frac{1+\ln(N)}{N}\right)\right] = \left(\frac{1+\ln(2)}{2}\right)$$

Since the integral converges, the series converges also by the integral test.

Scoring: Identification of $f(x) = \ln(x)/x^2$ 1 point; check that f is positive, increasing and continuous, 1 point each for 3 point total; correct computation of $\int_1^N f(x)\,dx$, 3 points; correct computation of limit to find improper integral, 2 points; correct conclusion, 1 point.

15. (a) (3 points) Determine the power series expansion about $a = 0$ for the function $f(x) = \dfrac{1}{1+x^2}$.

Solution: Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, replacing x by $-x^2$ we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Scoring: substitution $x \to -x^2$, 1 point; answer 2 points. $\sum_{n=1}^{\infty} (-x^2)^n$ is an acceptable final answer.

(b) (3 points) Determine the power series expansion about $a = 0$ for $\displaystyle\int_0^x \frac{1}{1+t^2}\, dt$ by integrating the power series expansion for $\dfrac{1}{1+t^2}$.

Solution: Integrating term-by-term we get

$$\begin{aligned}\int_0^x \frac{1}{1+t^2}\, dt &= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n}\, dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\end{aligned}$$

Scoring: Correct power series expansion for $1/(1+t^2)$, 1 point; correct integration, 1 point; final answer, 1 point. The form $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ is an acceptable final answer; however, giving only the first few terms of the expansion for $(1+t^2)^{-1}$ and the first few terms of the integrated series should be worth only 2 out of 3 points.

(c) (4 points) Determine the radius of convergence and the interval of convergence of the series you found in part (b).

Solution: Since the series for $1/(1-x)$ converges for $|x| < 1$, the series from part (a) will converge for $|x^2| < 1$. Hence, the radius of convergence is 1. At the endpoint $x = 1$, the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

converges by the alternating series test. At the endpoint $x = -1$, the series

$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$$

again converges.

Hence, the interval of convergence is $[-1, 1]$.

Scoring: Correct radius of convergence for series expansion of $(1+x^2)^{-1}$, 1 point; correct deduction of radius of convergence for the integrated series, 1 point; correct determination of convergence at $x = 1$, 1 point; correct determination of divergence at $x = -1$, 1 point.