Name: ______________________ Section: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but you may not use a calculator that has symbolic manipulation capabilities of any sort. This forbids the use of TI-89, TI-Nspire CAS, HP 48, TI 92, and many others, as stated on the syllabus. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. You should also show your work on the multiple choice questions as it will make it easier for you to check your work. You should give exact answers, rather than a decimal approximation unless the problem asks for a decimal answer. Thus, if the answer is 2π, you should not give a decimal approximation such as 6.283 as your final answer.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Determine the set of all values of x for which the limit $\lim_{n\to\infty}(x/3)^n$ exists and is finite.

   A. $(-1/3, 1/3)$

   B. $(-3, 3]$

   C. $[-1/3, 1/3]$

   D. $[-3, 3]$

   E. $[-3, 3)$

2. (5 points) Suppose that $\{a_n\}$ is a convergent sequence and

$$\lim_{n\to\infty}\left(\frac{3n}{2n+1}\right)a_n = 12.$$

   Find $\lim_{n\to\infty} a_n$.

   A. 4

   B. 8

   C. 12

   D. 16

   E. 18

3. (5 points) Knowing that

$$\frac{2}{4n^2-1} = \frac{1}{2n-1} - \frac{1}{2n+1},$$

   find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1}.$$

   A. -1

   B. $-1/2$

   C. 0

   D. 1

   E. $1/2$

4. (5 points) Find the sum of the series $\sum_{n=1}^{\infty} 4 \cdot 5^{-n}$.

   A. 1/5
   B. 1/4
   C. 1
   D. 4/5
   E. 5/4

5. (5 points) Use the Limit Comparison Test to test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)^3}.$$

   A. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ gives that the series diverges.
   B. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ gives that the series converges.
   C. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ gives that the series diverges.
   D. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ gives that the series converges.
   E. No conclusion can be drawn from the Limit Comparison Test.

6. (5 points) Choose the correct statement.

   A. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then it is absolutely convergent
   B. If both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} (-a_n)$ are convergent, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent
   C. If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent
   D. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} (-1)^n a_n$ is convergent
   E. None of the statements A–D are correct.

7. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n-1)!}$. Find the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

A. $-\infty$

B. 0

C. 1

D. 2

E. ∞

8. (5 points) Find the radius of convergence for the series $\sum_{n=1}^{\infty} n(x-3)^n$.

A. 0

B. 1/3

C. 1

D. 3

E. ∞

9. (5 points) If $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n}$, find the coefficient of x^6 in the series for $f'(x)$.

A. 0

B. 1

C. 2

D. 5

E. 6

10. (5 points) Find the coefficient of x^{12} in the Maclaurin series for $f(x) = \ln(1 + x^3)$. (The Maclaurin series is another name for the Taylor series centered at 0.)

A. -1

B. $-1/4$

C. 0

D. $1/4$

E. 1

Free Response Questions

11. Find the sums of the following series (exact answers only):

(a) (3 points) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$

Solution: This is a convergent geometric series with initial term $a = 1$ and common ratio $r = 1/3 < 1$. Thus

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{a}{1-r} = \frac{1}{1-(1/3)} = \frac{3}{2}.$$

(b) (3 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}$

Solution: This is a geometric series with initial term $a = 1/3$ and common ratio $r = -1/3$. Since $|r| < 1$, the series converges and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} = \frac{1/3}{1+(1/3)} = \frac{1}{4}.$$

(c) (4 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} e^{-(n+1)}$

Solution: This is a geometric series with initial term $a = e^{-2}$ and common ratio $r = -e^{-1}$. Since $|r| < 1$, the series converges and

$$\sum_{n=1}^{\infty} (-1)^{n-1} e^{-(n+1)} = \frac{e^{-2}}{1+e^{-1}} = \frac{1}{e(e+1)}.$$

12. (a) (4 points) State the ratio test for convergence of a series $\sum_{n=1}^{\infty} a_n$.

Solution:

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ (or $L = \infty$), then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, the ratio test is inconclusive.

(b) (6 points) For each of the series below determine if the ratio test gives convergence, divergence or no information.

(i). $\sum_{n=1}^{\infty} \frac{1}{2^n n!}$

Solution: We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^n\, n!}{2^{n+1}(n+1)!} = \frac{1}{2(n+1)}.$$

Since $\lim_{n\to\infty} \frac{1}{2(n+1)} = 0 < 1$, this series is absolutely convergent.

(ii). $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

Solution: We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)\sqrt{n+1}}{\sqrt{n}\,(n+2)}.$$

Since

$$\lim_{n\to\infty} \frac{(n+1)\sqrt{n+1}}{\sqrt{n}\,(n+2)} = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) \cdot \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = 1,$$

the ratio test is inconclusive.

13. (a) (4 points) Find the Maclaurin series for $f(x) = \dfrac{1}{1+x^2}$. [Hint: Recall the sum of a geometric series. You don't need to write your answer in sigma notation.]

Solution: Since the Maclaurin series for $g(x) = \dfrac{1}{1-x}$ is

$$g(x) = 1 + x + x^2 + x^3 + \cdots$$

(geometric series), the Maclaurin series for $f(x) = \dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$ is

$$f(x) = 1 - x^2 + x^4 - x^6 + \cdots .$$

(b) (3 points) Find the Maclaurin series for $\arctan(x) = \displaystyle\int_0^x \frac{1}{1+t^2}\,dt$.

Solution: Integrating

$$f(t) = \frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots$$

term by term from 0 to x gives

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .$$

(c) (3 points) Use the series found in part (b) to represent $\pi/6 = \arctan(\sqrt{3}/3)$ as a sum of an alternating series.

Solution: We have

$$\frac{\pi}{6} = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3^2 \cdot 3} + \frac{\sqrt{3}}{3^3 \cdot 5} - \frac{\sqrt{3}}{3^4 \cdot 7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{3}}{3^{n+1}(2n+1)}.$$

14. Consider the alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$.

(a) (5 points) How many terms n of the series would you need to add to find its sum s to within 0.004?

Solution: We have
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \sum_{n=1}^{\infty} (-1)^{n-1} b_n,$$
where $b_n = 1/n^4 > 0$. Since the sequence $\{b_n\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}$ converges by the alternating series test and
$$|s - s_n| \leq b_{n+1} = \frac{1}{(n+1)^4}.$$
We need $\dfrac{1}{(n+1)^4} \leq 0.004$, i.e. $(n+1)^4 \geq \dfrac{1}{0.004} = 250$. Since $4^4 = 256$, it is enough to take the first 3 terms ($n = 3$).

(b) (5 points) For the value of n found in (a), compute the value of the partial sum s_n (**exact answer** written as a fraction).

Solution:
$$s_3 = 1 - \frac{1}{16} + \frac{1}{81} = \frac{1231}{1296}.$$

15. (10 points) Determine if each of the series converges. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{2n}\right)^n$

Solution: *Root Test.* If $a_n = \left(\dfrac{n+1}{2n}\right)^n$, we have $\sqrt[n]{|a_n|} = \dfrac{n+1}{2n}$ and

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2} < 1.$$

Thus $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{2n}\right)^n$ is absolutely convergent by the root test.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{e^{1/n}}{\sqrt{n}}$

Solution: *Limit Comparison Test.* Let $a_n = \dfrac{e^{1/n}}{\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$. Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}e^{1/n} = e^0 = 1 \neq 0.$$

Since $\displaystyle\sum_{n=1}^{\infty}b_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series with $p = 1/2 < 1$, we conclude that $\displaystyle\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\frac{e^{1/n}}{\sqrt{n}}$ is also divergent by the limit comparison test.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\ln(n+1)}$

Solution: *Alternating Series Test.* Since $\ln(x+1) > 0$ is increasing for $x \geq 1$ and $\displaystyle\lim_{x\to\infty}\ln(x+1) = \infty$, then $\dfrac{1}{\ln(x+1)}$ is decreasing and $\displaystyle\lim_{x\to\infty}\frac{1}{\ln(x+1)} = 0$. Thus if $b_n = \dfrac{1}{\ln(n+1)}$, then $\{b_n\}$ is a decreasing sequence and $\displaystyle\lim_{n\to\infty}b_n = 0$. Thus $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\ln(n+1)}$ is convergent by the alternating series test.