Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show *all work* on the free response problems. Unsupported answers may not receive credit. You should also show your work on the multiple choice questions as it will make it easier for you to check your work. You should give exact answers, rather than a decimal approximation unless the problem asks for a decimal answer. Thus, if the answer is 2π, you should not give a decimal approximation such as 6.283 as your final answer.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Let $a_n = \dfrac{4n^2+3n}{2n^2+5n}$ and find $\lim_{n\to\infty} a_n$.

   A. 0

   B. 3/5

   C. 2

   D. 4/5

   E. 3/2

2. (5 points) Suppose that $\sum_{n=1}^{\infty} a_n = 3$. Find $\sum_{n=1}^{\infty}(2a_n - 2^{-n})$.

   A. 1

   B. 3

   C. 4

   D. 5

   E. 2

3. (5 points) Determine if the series is convergent and if the series converges, find the sum: $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+2})$.

   A. 3/2

   B. 1

   C. 4/3

   D. 1/2

   E. The series diverges.

4. (5 points) Suppose that we have $|a_n| \leq 1/n^2$. Select the correct statement.

   A. The series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, but not absolutely convergent.

   B. The series $\sum_{n=1}^{\infty} a_n$ is a geometric series.

   C. We have that $\lim_{n\to\infty} a_n = 2$.

   D. The series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

   E. The series $\sum_{n=1}^{\infty} a_n$ is divergent.

5. (5 points) Consider the series $\sum_{n=1}^{\infty} a_n$ with $a_n = 1/n^2$. Compute the quantity $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ and choose the correct statement.

   A. $r = 0$ and the value of the ratio implies the series converges.

   B. $r = 1$ and the value of the ratio implies the series diverges.

   C. $r = 1$ and the value of the ratio gives no information about the convergence of the series.

   D. $r = 0$ and the value of the ratio implies the series diverges.

   E. $r = 0$ and the value of the ratio gives no information about the convergence of of the series.

6. (5 points) Consider the series $s = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. If we approximate s by a partial sum $s_N = \sum_{n=1}^{N} \frac{(-1)^n}{\sqrt{n}}$, what does the alternating series test tell us about the error $|s - s_N|$?

   A. $|s - s_N| \leq 1/(N-1)$

   B. $|s - s_N| \leq 1/\sqrt{N}$

   C. $|s - s_N| \leq 1/\sqrt{N+1}$

   D. $|s - s_N| \leq 1/N$

   E. $|s - s_N| \leq 1/(N+1)$

7. (5 points) Let $f(x) = \frac{1}{1+2x^2}$. Find the first three terms of the power series centered at 0 for $f'(x)$.

   A. $1 + 2x^2 + 4x^4$

   B. $-4x + 16x^3 - 48x^5$

   C. $4x + 16x^3 + 48x^5$

   D. $1 - 2x^2 + 4x^4$

   E. $2x^2 + 4x^4 + 8x^8$

8. (5 points) Select the true statement.

   A. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 1$.

   B. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$.

   C. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   D. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

   E. If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

9. (5 points) Find the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$.

   A. 0

   B. 1

   C. 2

   D. 4

   E. ∞

10. (5 points) If $f(x) = e^{2x}$. Find the first three terms of the MacLaurin Series (or Taylor series centered at 0) for f.

    A. $1 + x + x^2$

    B. $1 + x + x^2/2$

    C. $1 + x^2 + x^4/2$

    D. $1 + 2x + 4x^2$

    E. $1 + 2x + 2x^2$

Free Response Questions

11. Consider the recursive sequence defined by $a_1 = 4$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{8}{a_n})$.

(a) (3 points) Compute a_2 and a_3.

(b) (7 points) Find $A = \lim_{n\to\infty} a_n$. You may assume that the sequence $\{a_n\}$ is convergent.

Solution:

a) $a_2 = 3$, $a_3 = 17/6$.

b) Taking the limit $\lim_{n\to\infty} a_{n+1} = \lim_{\to\infty} \frac{1}{2}(a_n + 8/a_n)$ gives

$$A = \frac{1}{2}(A + \frac{8}{A}).$$

Simplifying we have $A^2 = 8$ and thus $A = \pm\sqrt{8}$. Since the sequence $\{a_n\}$ will have $a_n \geq 0$ for all n, the limit must be $A = \sqrt{8}$.

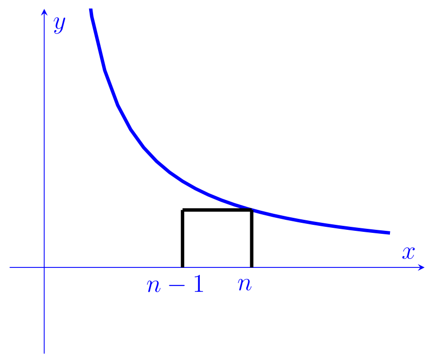
Grading: a) 2 points for one value, 1 point for second value. Correct method with arithmetic error should result in 2 points.

b) Equation for A 2 points, simplify 1 point, find two roots 2 points, choose $A > 0$ 1 point, explain why $A > 0$ 1 point.

12. (a) (3 points) Find an improper integral so that we have $\sum_{n=N}^{\infty} \frac{1}{n^3} \leq \int_A^{\infty} f(x)\,dx$.

(b) (7 points) Use your answer to part a) to find a value N so that $\sum_{n=N}^{\infty} \frac{1}{n^3} \leq 1/72$.

Solution: Need to work on graph.



We have $1/n^3 \leq 1/x^3$ if $x \leq n$ and thus we have $1/n^3 \leq \int_{n-1}^{n} 1/x^3\,dx$. In the sketch above, the area of the box is $1/n^3$ and is smaller than the area represented by the integral. This implies that

$$\sum_{n=N}^{\infty} \frac{1}{n^3} \leq \int_{N-1}^{\infty} \frac{1}{x^3}\,dx.$$

b) The improper integral $\int_{N-1}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2(N-1)^2}$.

Solving $1/(2(N-1)^2) \leq 1/72$ gives $36 \leq (N-1)^2$. Or $N \geq 7$.

Grading: a) Interval $[N-1, \infty)$ 2 points, function $f(x) = 1/x^3$ 1 point.

b) Evaluate improper integral 3 points. equation $1/(2(N-1)^2) < 1/72$ 2 points, solve to obtain $N > 7$ 2 points. Award credit for students who apply a correct method to an incorrect answer in part a).

13. For each of the following series, use the limit comparison test to test convergence. Your answer should give the series you are using for comparison.

(a) (5 points) $\sum_{n=1}^{\infty} a_n$ with $a_n = \frac{1}{3^n} \cdot \frac{n^2+4n}{4n^2+3}$.

(b) (5 points) $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{3n^4+1}{n^5+1}$.

Solution: a) Compare with $c_n = 1/3^n$. We have $\lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} \frac{3^n}{1} \cdot \frac{n^2+4n}{3^n(4n^2+3)} = 1/4$. Since the limit is a nonzero real number, the convergence of the geometric series $\sum 1/3^n$ implies the convergence of the given series.

b) Compare with $d_n = 1/n$. We have

$$\lim_{n\to\infty} \frac{b_n}{d_n} = \lim_{n\to\infty} \frac{n}{1} \cdot \frac{3n^4+1}{n^5+1} = 3$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the limit is a nonzero real number, the given series diverges.

a) Choice of series to compare 2 points, compute limit 1 point, convergence of chosen comparison series 1 point, answer 1 point.

b) Choice of series to compare 2 points, compute limit 1 point, divergence of chosen comparison series 1 point, answer 1 point.

14. (10 points) Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$ and find the radius and interval of convergence. Be sure to test the endpoints.

Solution: Using the ratio test we find the radius of convergence is 2.

Substituting $x = 2$, gives $\sum_{n=1}^{\infty} \frac{x^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series.

Substituting $x = -2$, gives $\sum_{n=1}^{\infty} \frac{x^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is a convergent alternating series.

The interval of convergence is $[-2, 2) = \{x : -2 \leq x < 2\}$.

Grading: Compute ratio a_{n+1}/a_n 1 point, find limit 2 points, radius of convergence 2 points, divergence at $x = 2$ 2 points, convergence at $x = -2$ 2 points, interval 1 point.

15. (a) (2 points) Give the Taylor series centered at 0 (or MacLaurin series) for the function $\sin(x)$.

(b) (5 points) Give the Taylor series centered at 0 for the function $\int \sin(x^2)\,dx$.

(c) (3 points) Use your answer to part b) to find a series for the definite integral $\int_0^1 \sin(x^2)\,dx$.

Solution:

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

b) $\displaystyle\int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)(2n+1)!}$.

c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!}$

a) Series for $\sin(x)$ 2 points,

b) Series for $\sin(x^2)$ 2 points, series for anti-derivative including C 3 points.

c) Evaluate integral to obtain series 3 points.