MA 114 — Calculus II Fall 2015
Exam 3 November 17, 2015

Name: ____________________

Section: ____________________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:** Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	B	C	~~D~~	E
2	A	B	C	~~D~~	E
3	A	B	C	D	~~E~~
4	~~A~~	B	C	D	E
5	A	B	~~C~~	D	E
6	~~A~~	B	C	D	E
7	A	B	C	D	~~E~~

Exam Scores

Question	Score	Total
MC		28
8		13
9		15
10		15
11		14
12		15
Total		100

Unsupported answers for the free response questions may not receive credit!

Record the correct answer to the following problems on the front page of this exam.

1. Which of the following is the correct form for the partial fraction decomposition of

$$\frac{42x^2+100x-7}{(x-2)^2(x+3)}$$

A. $\dfrac{A}{(x-2)^2}+\dfrac{B}{x+3}$.

B. $\dfrac{A}{(x-2)(x+3)}+\dfrac{B}{(x-2)}$.

C. $\dfrac{Ax+B}{(x-2)(x+3)}+\dfrac{C}{(x-2)}$.

D. $\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{x+3}$.

E. $\dfrac{A}{x-2}+\dfrac{B}{(x+3)}$.

2. Which of the following statements is true. Only one answer is correct.

A. $\displaystyle\int_0^1 \frac{1000}{x^2}dx$ converges.

B. $\displaystyle\int_0^{100} \frac{1000}{x}dx$ converges.

C. $\displaystyle\int_{100}^{\infty} \frac{500}{\sqrt{x}}dx$ converges.

D. $\displaystyle\int_5^{\infty} \frac{1000}{x^2}dx$ converges.

E. $\displaystyle\int_5^{\infty} \frac{e^x}{x^2-1}dx$ converges.

Record the correct answer to the following problems on the front page of this exam.

3. Which of the following integrals represents the arc length of the graph of $f(x) = \ln(2x^3)$ over the interval $[2, 6]$?

$f'(x) = \frac{6x^2}{2x^3} = \frac{3}{x}$

$\int_2^6 \sqrt{1+f'(x)^2}\, dx = \int_2^6 \sqrt{1+\frac{9}{x^2}}\, dx$

A. $\int_2^6 \sqrt{1+\frac{18}{x^2}}\, dx.$

B. $\int_2^6 \sqrt{1+\frac{9}{x}}\, dx.$

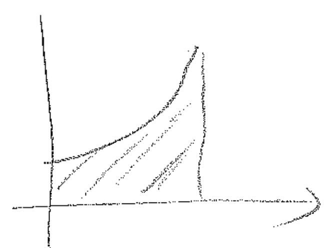
C. $2\pi \int_2^6 \ln(x^3)\sqrt{1+\frac{9}{x^2}}\, dx.$

D. $\int_2^6 \sqrt{1+\ln(2x^3)^2}\, dx.$

E. $\int_2^6 \sqrt{1+\frac{9}{x^2}}\, dx.$

4. Consider the region in the first quadrant enclosed by the graph of $f(x) = e^x$ and the line $x = 1$. Find the y-coordinate of the center of mass of this region when the density is $\rho = 1$.

$M = \int_0^1 e^x\, dx = e-1$



$M_x = \frac{1}{2}\int_0^1 e^{2x}\, dx$

$= \frac{1}{4} e^{2x}\Big|_0^1 = \frac{e^2-1}{4}$

$\frac{M_x}{y} = \frac{e^2-1}{4} \cdot \frac{1}{e-1} = \frac{e+1}{4}$

A. $\frac{e+1}{4}.$

B. $\frac{e-1}{4}.$

C. $e-1.$

D. $\frac{e^2-1}{8}.$

E. $\frac{1}{e-1}.$

Record the correct answer to the following problems on the front page of this exam.

5. Consider the parametrized curve

$$(x, y) = (\sin(t) - t\cos(t),\ \cos(t) + t\sin(t)).$$

Compute the arc length of the curve over the interval $[2, 10]$.

A. 16.

B. 32.

C. 48.

D. 64.

E. 80.

$x'(t) = \cos(t) + t\sin(t) - \cos(t) = t\sin(t)$

$y'(t) = -\sin(t) + \sin(t) + t\cos t = t\cos(t)$

$\int_2^{10} \sqrt{t^2\sin^2(t) + t^2\cos^2(t)}\, dt = \int_2^{10} \sqrt{t^2}\, dt$

$= \int_2^{10} t\, dt = \frac{1}{2}t^2 \Big|_2^{10} = 50 - 2 = \underline{48}$

6. If $f(\theta) = f(-\theta)$, then the curve in the (x, y)-plane with polar equation

$$r = f(\theta) \quad \text{for} \quad \theta \text{ in } [-\pi, \pi]$$

has the following property (only one answer is correct).

A. The curve is symmetric about the x-axis.

B. The curve is symmetric about the y-axis.

C. The curve is symmetric about the line $y = x$.

D. The curve is symmetric about the line $y = -x$.

E. The curve is symmetric about the origin $(0, 0)$.

Record the correct answer to the following problems on the front page of this exam.

7. Consider the point with rectangular coordinates $(x, y) = (-3, 4)$. Which of the following are the approximate polar coordinates (r, θ) with θ in $[-\pi, \pi]$?

A. $r = 5$, $\theta \approx -.927$.

B. $r = 5$, $\theta \approx .927$.

C. $r = 1$, $\theta \approx 2.498$.

D. $r = 5$, $\theta \approx -0.643$.

E. $r = 5$, $\theta \approx 2.214$.

$r = \sqrt{x^2+y^2} = 5$

$x < 0$, thus

$\theta = \arctan\left(\frac{4}{-3}\right) + \pi$

≈ 2.214

Also: $(-3, 4)$ is in the 2nd quadrant, therefore θ has to be in $\left(\frac{\pi}{2}, \pi\right) \approx (1.5, 3.1)$

Free Response Questions: Show your work!

8. (a) Find the partial fraction decomposition of $f(x) = \dfrac{4x^3 - 3x^2 + 3x - 2}{x^2(x^2+1)}$.

$$\frac{4x^3-3x^2+3x-2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$4x^3-3x^2+3x-2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$x=0$: $\boxed{-2 = B}$

Now

$$4x^3-3x^2+3x-2+2(x^2+1) = Ax^3+Ax+Cx^3+Dx^2$$

$$4x^3-x^2+3x = (A+C)x^3+Dx^2+Ax$$

So $\boxed{A=3}$, $\boxed{D=-1}$, $A+C=4$, hence $\boxed{C=1}$

Thus $\boxed{\dfrac{4x^3-3x^2+3x-2}{x^2(x^2+1)} = \dfrac{3}{x} - \dfrac{2}{x^2} + \dfrac{x-1}{x^2+1}}$

(b) Evaluate the integral $\displaystyle\int \frac{5}{(x-4)(x+1)}\,dx$.

$$\frac{5}{(x-4)(x+1)} = \frac{A}{x-4} + \frac{B}{x+1} \Rightarrow 5 = A(x+1) + B(x-4)$$

$x=-1$: $5 = -5B \Rightarrow \boxed{B=-1}$

$x=4$: $5 = 5A \Rightarrow \boxed{A=1}$

$$\int \frac{5}{(x-4)(x+1)}\,dx = \int \frac{1}{x-4}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln|x-4| - \ln|x+1| + C$$

Free Response Questions: Show your work!

9. Use the integral test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^3}.$$

converges or diverges. You have to verify all assumptions of the test and show all computations.

$f(x) = \frac{x}{(x^2+1)^3}$ is continuous on $[1,\infty)$, and also positive. For decreasing

$$f'(x) = \frac{(x^2+1)^3 - x\cdot 3(x^2+1)^2 2x}{(x^2+1)^6} = \frac{(x^2+1-6x^2)(x^2+1)^2}{(x^2+1)^6}$$

$$= \frac{-5x^2+1}{(x^2+1)^4} \quad \begin{matrix}]<0 \\ ]>0 \end{matrix} \quad \text{for } x \geq 1.$$

Hence $f'(x) < 0$ and thus f decreasing on $[1,\infty)$.

$$\int_1^{\infty} \frac{x}{(x^2+1)^3}\,dx = \lim_{R\to\infty} \int_1^R \frac{x}{(x^2+1)^3}\,dx \qquad \overline{\overline{\begin{matrix} u = x^2+1 \\ du = 2x\,dx \end{matrix}}}$$

$$= \lim_{R\to\infty} \frac{1}{2} \int_2^{R^2+1} u^{-3}\,du = \lim_{R\to\infty} \left(-\frac{1}{4} u^{-2} \Big|_2^{R^2+1} \right)$$

$$= \lim_{R\to\infty} \left(-\frac{1}{4} \left(\underbrace{\frac{1}{(R^2+1)^2}}_{\to 0} - \frac{1}{4} \right) \right) = \frac{1}{16}$$

Since the improper integral converges, the integral test tells us that also the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^3}$ converges.

Free Response Questions: Show your work!

10. Consider the parametrized curve

$$x(t) = t^3 - 6t^2 + 6, \; y(t) = 2t^2 + 8t + 10, \text{ where } -\infty < t < \infty.$$

(a) Find the value(s) of t that corresponds to the point $(-1, 4)$.

$2t^2+8t+10 = 4 \Rightarrow 2t^2+8t+6=0$

$\Rightarrow t^2+4t+3 = (t+3)(t+1) = 0, \; t=-3, t=-1.$

$x(-3) = -27-54+6 \neq -1$

$x(-1) = -1-6+6 = -1.$

$\boxed{\text{Hence } t=-1 \text{ is the only value.}}$

(b) Find all points on the curve where the curve has a horizontal tangent line.

slope of TL is $\frac{y'(t)}{x'(t)} = \frac{4t+8}{3t^2-12t}$.

$4t+8=0$ for $t=-2$ and $3(-2)^2-12(-2) \neq 0.$

Hence the curve has a horizontal TL

at $(x(-2), y(-2)) = \boxed{(-26, 2)}$

(c) Find all points on the curve where the curve has a vertical tangent line.

$3t^2-12t = 3t(t-4) = 0$ for $t=0$ and $t=4$,

$4t+8 \neq 0$ for $t=0$ and $t=4$. Thus

the curve has a vertical TL at the

points $(x(0), y(0)) = \boxed{(6, 10)}$ and $(x(4), y(4)) = \boxed{(-26, 74)}$

(d) Compute the speed of the parametrization at time $t = 1$.

$\sqrt{x'(1)^2+y'(1)^2} = \sqrt{81+144} = \sqrt{225}$

$= \underline{\underline{15}}$

11. Given the polar curve

$$r = \frac{6}{\cos(\theta) + 3} \quad \text{for } \theta \text{ in } [0, 2\pi].$$

(a) Find the polar coordinates and the rectangular coordinates of all intersection points of the curve with the y-axis.

The points on the y-axis have angle $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$. For these values $\cos(\theta) = 0$, and thus $r = 2$.
Hence the points are

polar	rectangular
$(2, \frac{\pi}{2})$	$(0, 2)$
$(2, \frac{3\pi}{2})$	$(0, -2)$

(b) Find the polar coordinates and the rectangular coordinates of all intersection points of the curve with the circle about $(0, 0)$ of radius 3.

The points on this circle have radial coordinate 3. Thus

$$3 = \frac{6}{\cos\theta + 3}$$

$$\Rightarrow 3\cos\theta = -3$$

$$\Rightarrow \cos\theta = -1$$

$\theta = \pi$ is the only solution. Hence the point is

polar	rectangular
$(3, \pi)$	$(-3, 0)$

Free Response Questions: Show your work!

12. Consider the lamina with density $\rho = 1$ and region enclosed by the graphs of

$$f(x) = 6\sqrt{x} \text{ and } g(x) = 2x.$$

(a) Compute the points of intersection of the two graphs.

$f(x) = g(x) \Rightarrow 36x = 4x^2 \Rightarrow 9x = x^2$.

$\Rightarrow x = 0$ or $x = 9$.

Intersection points $(0, 0)$, $(9, 18)$

(b) Compute the moment M_x.

$$M_x = \frac{1}{2}\int_0^9 f^2(x) - g^2(x)\,dx = \frac{1}{2}\int_0^9 (36x - 4x^2)\,dx$$

$$= \frac{1}{2}\left(18x^2 - \frac{4}{3}x^3\right)\Big|_0^9 = 3^5 = 243$$

(c) Compute the moment M_y.

$$M_y = \int_0^9 x(f(x) - g(x))\,dx = \int_0^9 6x^{\frac{3}{2}} - 2x^2\,dx$$

$$= \left(\frac{12}{5}x^{\frac{5}{2}} - \frac{2}{3}x^3\right)\Big|_0^9 = \frac{2}{5}\cdot 3^5$$

(d) Compute the center of mass. **Give the exact answer and simplify it.**

$$M = \int_0^9 (6x^{\frac{1}{2}} - 2x)\,dx = \left(4x^{\frac{3}{2}} - x^2\right)\Big|_0^9 = 3^3 = 27$$

$$\text{COM} = \left(\frac{\frac{2}{5}\cdot 3^5}{3^3}, \frac{3^5}{3^3}\right) = \underline{\left(\frac{18}{5}, 9\right)}$$