Exam 3

Multiple Choice Questions

1. Find the average value of the function $f(x) = 2\sin x - \sin 2x$ on $0 \leq x \leq \pi$.

   A. $\frac{4}{\pi}$

   B. $\frac{5}{\pi}$

   C. 0

   D. 4

   E. 5

2. Find the volume of the solid S whose base is the disk bounded by the circle $x^2 + y^2 = r^2$ and the parallel cross-sections perpendicular to the base are squares.

   A. $\frac{8}{3}r^3$

   B. $\frac{16}{3}r^3$

   C. $\frac{4}{3}r^3$

   D. $\frac{2}{3}r^3$

   E. 0

3. Consider the parametric curve $(x(t), y(t)) = (\sin(\pi t), \cos(\pi t))$. At which value of t does the curve pass through $(0, -1)$?

   A. $t = 0$

   B. $t = 1/2$

   C. $t = 1$

   D. $t = 3/2$

   E. $t = 2$

4. Find the center of mass of a lamina (or thin plate) that occupies the region $R = \{(x, y) | 0 \leq y \leq \sqrt{4 - x^2}\}$.

   A. $(0, 1)$

   B. $\left(0, \frac{1}{2}\right)$

   C. $\left(0, \frac{9}{4\pi}\right)$

   D. $\left(0, \frac{8}{3\pi}\right)$

   E. $\left(0, \frac{5}{6}\right)$

5. Consider the curve of points (x, y) that satisfy the equation $y = x^2$ and lies on the parabola between the points $(2, 4)$ and $(3, 9)$. Write an integral whose value is the length of this curve.

    A. $\int_2^3 \sqrt{1 + 2x^2}\, dx$

    B. $\int_4^9 \sqrt{1 + 4x^2}\, dx$

    C. $\int_4^9 \sqrt{1 + x^4}\, dx$

    D. $\int_2^3 \sqrt{1 + x^4}\, dx$

    E. $\int_2^3 \sqrt{1 + 4x^2}\, dx$

6. An auxiliary fuel tank for a helicopter is shaped like the surface generated by revolving the curve $y = 1 - \frac{x^2}{4}$, $-2 \le x \le 2$, about the x-axis (dimensions are in feet). Find the integral that computes how many cubic feet of fuel the tank will hold.

    A. $\int_{-2}^2 \pi \left(1 - \frac{x^2}{4}\right) dx$

    B. $\int_{-2}^2 2\pi x \left(1 - \frac{x^2}{4}\right) dx$

    C. $\int_{-2}^2 \pi \left(1 - \frac{x^2}{4}\right)^2 dx$

    D. $\int_{-2}^2 2\pi \left(1 - \frac{x^2}{4}\right) \sqrt{1 + \frac{x^2}{4}}\, dx$

    E. $\int_{-2}^2 \pi \sqrt{1 + \frac{x^2}{4}}\, dx$

7. Find $\dfrac{dy}{dx}$ for the curve given by $x = te^t$ and $y = 2t - 3e^t$.

A. $2 - 3e^t$

B. $\dfrac{2-3e^t}{e^t + te^t}$

C. $(1+t)e^t$

D. $\dfrac{e^t + te^t}{2-3e^t}$

E. $\dfrac{2-3e^t}{e^t}$

8. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 0$ and $x = 1$ about the line $y = -3$.

A. $\dfrac{52\pi}{63}$

B. $\dfrac{23\pi}{24}$

C. 7π

D. $\dfrac{23\pi}{14}$

E. 24π

9. Write out the first four terms of the Maclaurin series of $f(x)$ if

$$f(0) = -1,\ f'(0) = 13,\ f''(0) = -1,\ \text{ and } f'''(0) = 3.$$

A. $-1 + 13x - \frac{1}{2}x^2 + \frac{1}{2}x^3$

B. $-1 - 13x - \frac{1}{2}x^2 - \frac{1}{2}x^3$

C. $-1 + 13x + \frac{1}{2}x^2 + \frac{1}{2}x^3$

D. $-1 + 13x + x^2 + 3x^3$

E. $-1 + 13x - x^2 + 3x^3$

10. Using the parameterization $x = a\cos t, y = b\sin t, 0 \le t \le 2\pi$, which of the following definite integrals represents the surface area of the region obtained by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.

A. $\int_0^{\pi} 2\pi a\cos t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$

B. $\int_0^{\pi} 2\pi b\sin t\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt$

C. $\int_0^{\pi} 2\pi a\cos t\sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt$

D. $\int_0^{\pi} 2\pi ab\sin t\cos t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$

E. $\int_0^{\pi} 2\pi b\sin t\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$

Free Response Questions

11. Consider the function $f(x) = x^3 + x + 1$.

(a) (5 points) Find all derivatives, $f^{(n)}(x)$ of $f(x)$.

Solution:

$$\begin{aligned} f(x) &= x^3 + x + 1 \\ f'(x) &= 3x^2 + 1 \\ f''(x) &= 6x \\ f'''(x) &= 6 \\ f^{(n)}(x) &= 0, \quad n \geq 4 \end{aligned}$$

(b) (5 points) Find the Taylor series for $f(x)$ centered at $a = 1$.

Solution:

$$f(1) = 3, \quad f'(1) = 4, \quad f''(1) = 6, \quad f'''(1) = 6$$

$$\begin{aligned} T(x) &= f(1) + f'(1)(x-1) + \frac{f''(x)}{2!}(x-1)^2 + \frac{f'''x}{3!}(x-1)^3 \\ &= 3 + 4(x-1) + 3(x-1)^2 + (x-1)^3 \end{aligned}$$

(c) (3 points) Find the radius of convergence for this Taylor series.

Solution: The radius of convergence is ∞ since the Taylor series is a polynomial.

12. Let R be the region bounded by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$.

(a) (5 points) Using the method of cylindrical shells write down the integral needed to find the volume generated by rotating R about the y-axis.

Solution:

$$V = \int_0^1 2\pi x f(x)\, dx = 2\pi \int_0^1 x e^{-x^2} dx.$$

(b) (5 points) **Showing all of your work**, evaluate this integral.

Solution:

$$2\pi \int_0^1 x e^{-x^2} dx = -\pi e^{-x^2}\Big|_0^1 = \pi \left(1 - \frac{1}{e}\right).$$

13. Consider the parametric curve C given by $\{(t, \frac{2}{3}t^{3/2}) : 0 \leq t \leq T\}$.

(a) (4 points) Write an integral for the length of the curve C.

Solution: $x'(t) = 1$ and $y'(t) = t^{1/2} = \sqrt{t}$.

$$L = \int_0^T \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_0^T \sqrt{1+t}\, dt$$

(b) (5 points) Evaluate the integral you found to express the length as a function of T.

Solution:

$$L = \int_0^T \sqrt{1+t}\, dt = \frac{2}{3}(1+t)^{3/2}\Big|_0^T = \frac{2}{3}\left((1+T)^{3/2} - 1\right)$$

(c) (4 points) Find the value of T for which the curve has length $14/3$.

Solution:

$$\begin{aligned} \frac{2}{3}\left((1+T)^{3/2} - 1\right) &= \frac{14}{3} \\ (1-T)^{3/2} - 1 &= 7 \\ (1-T)^{3/2} &= 8 \\ T &= 3 \end{aligned}$$

14. Consider the parametric curve C given by $(x(t), y(t)) = (t^2 + t, t^2 - t)$ for t in the real numbers.

(a) (5 points) Find the tangent line to the curve at the point $(x, y) = (2, 0)$. Give your answer in the form $y = mx + b$.

Solution:

$$\begin{aligned}
\frac{dx}{dt} &= 2t + 1 \\
\frac{dy}{dt} &= 2t - 1 \\
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{2t-1}{2t+1} \\
m &= \left.\frac{dy}{dx}\right|_{t=1} = \frac{1}{3} \\
y - 0 &= \frac{2}{3}(x - 2) \\
y &= \frac{2}{3}x - \frac{4}{3}
\end{aligned}$$

(b) (4 points) Find the point(s) (x, y) on the curve where the tangent line is horizontal. Show your work.

Solution: Find where $\frac{dy}{dt} = 0$.

$$\begin{aligned}
\frac{dy}{dt} &= 0 \\
2t - 1 &= 0 \\
t &= \frac{1}{2}
\end{aligned}$$

The point on the curve is $\left(x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right) = \left(\frac{3}{4}, -\frac{1}{4}\right)$.

15. (5 points) Set up but do not evaluate the integral that gives the surface area of revolution generated by rotating the graph of $y = x - x^2$, $0 \leq x \leq 1$, around the x-axis.

Solution:

$$\begin{aligned} SA &= \int_0^1 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx \\ &= \int_0^1 2\pi(x-x^2)\sqrt{1+(1-2x)^2}\,dx \\ &= \int_0^1 2\pi(x-x^2)\sqrt{2-4x+4x^2}\,dx \end{aligned}$$