Exam 3

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Trig identities

- $\sin^2(x) + \cos^2(x) = 1$,
- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Multiple Choice Questions

1. (5 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$?

   A. $[-1, 1]$
   B. $(-1, 1)$
   C. $[-1, 1)$
   D. $(-1, 1]$
   E. $\{0\}$

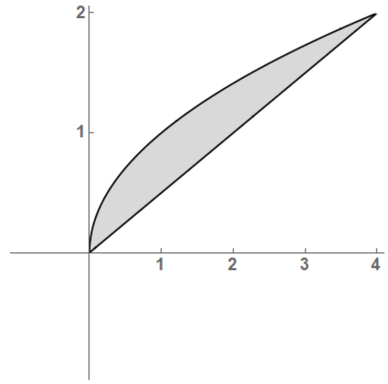
2. (5 points) Find the first 3 terms of the Taylor series for $f(x) = e^{-2t}$ centered at 0.

   A. $0 - \frac{2}{1!}t + \frac{4}{2!}t^2$
   B. $1 - 0t + \frac{4}{2!}t^2$
   C. $1 - \frac{2}{1!}t + \frac{4}{2!}t^2$
   D. $1 - \frac{1}{1!}t + \frac{1}{2!}t^2$
   E. $1 - \frac{2}{2!}t^2 + \frac{4}{4!}t^4$

3. (5 points) What is the average value of the function $h(x) = \cos^4(x)\sin(x)$ on the interval $[0, \pi]$?

   A. $\frac{2}{5\pi}$
   B. $\frac{2\pi}{5}$
   C. 0
   D. $\frac{5\pi}{2}$
   E. $\frac{5}{2\pi}$

4. (5 points) The region bounded by $y = \sqrt{x}$ and $y = \frac{x}{2}$ is shown below.



Consider the solid obtained by rotating this region around the x-axis. Using the washer method, which integral will compute the volume of this solid?

A. $\int_0^4 \pi \left(\sqrt{x} - \frac{x}{2}\right)^2 dx.$

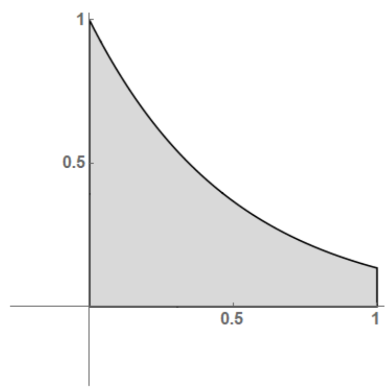
B. $\int_0^2 \pi \left(\left(\sqrt{x}\right)^2 - \left(\frac{x}{2}\right)^2\right) dx.$

C. $\int_0^4 \pi \left(\sqrt{x} - \frac{x}{2}\right) dx.$

D. $\int_0^4 \pi \left((\sqrt{x})^2 - \left(\frac{x}{2}\right)^2\right) dx.$

E. $\int_0^2 \pi \left(\sqrt{x} - \frac{x}{2}\right)^2 dx.$

5. (5 points) The region bouded by the curves $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = 1$ is shown below.



Consider the solid obtained by rotating this region about the y-axis. Using the shell method, which integral will compute the volume of this solid?

A. $\int_0^1 2\pi x e^{-2x} dx.$

B. $\int_0^1 2\pi e^{-2x} dx.$

C. $\int_0^1 2\pi x (e^{-2x})^2 dx.$

D. $\int_0^1 2\pi x^2 e^{-2x} dx.$

E. $\int_0^e 2\pi x^2 e^{-2x} dx.$

6. (5 points) What is the volume of a pyramid that has height 5 and a rectangular base with side lengths 10 and 20?

A. $2(10)^2(5)$.

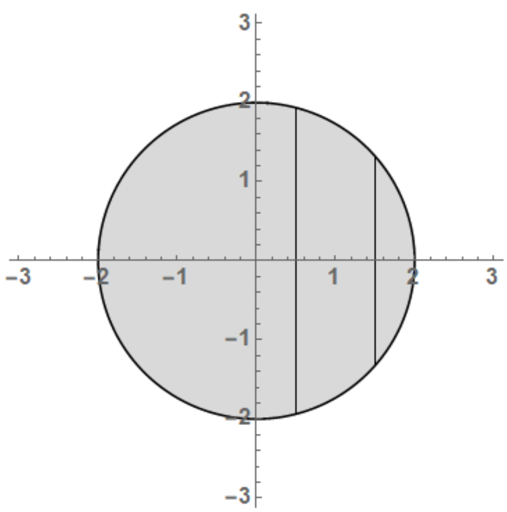
B. $(10)^2(5)$.

C. $\frac{1}{3}(10)^2(5)$.

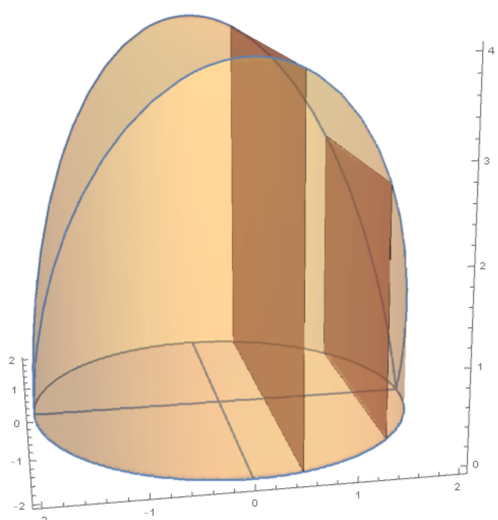
D. $\frac{2}{3}(10)^2(5)$.

E. $\frac{2}{3}(20)(5)$.

7. (5 points) Consider the solid S whose base is bounded by the circle $x^2 + y^2 = 4$, and whose cross sections parallel to the y-axis are squares.



The base of S, with two lines parallel to the y-axis shown.


The solid S, with two of its square cross sections shown

Which integral will calculate the volume of S?

A. $\int_{-2}^{2} (\sqrt{4-x^2})^2 \, dx$

B. $\int_{-2}^{2} (2\sqrt{4-x^2})^2 \, dx$

C. $\int_{-2}^{2} (2\sqrt{4-x^2}) \, dx$

D. $\int_{-2}^{2} \sqrt{4-x^2} \, dx$

E. $\int_{-4}^{4} (2\sqrt{4-x^2})^2 \, dx$

8. (5 points) What is the length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ for $3 \leqslant x \leqslant 8$?

A. $\frac{57}{2}$

B. $\frac{19}{3}$

C. $\frac{19}{2}$

D. $\frac{1}{12}$

E. $\frac{38}{3}$

9. (5 points) The line $y = x\sqrt{3}$ for $1 \leqslant x \leqslant 2$ is rotated about the y-axis. What is the area of the resulting surface?

A. 8π

B. 6π

C. 12π

D. 3π

E. 2π

10. (5 points) The curve $y = 1 + x^2$ for $0 \leqslant x \leqslant 2$ is rotated about the y-axis, producing a surface. Which of the following integrals calculates its surface area?

A. $\int_0^2 2\pi x\sqrt{1 + (1 + x^2)^2}\, dx$

B. $\int_0^2 2\pi(1 + x^2)\sqrt{1 + (2x)^2}\, dx$

C. $\int_1^5 2\pi\sqrt{y - 1}\sqrt{1 + (\sqrt{y - 1})^2}\, dy$

D. $\int_0^2 2\pi x\sqrt{1 + (2x)^2}\, dx$

E. $\int_1^5 2\pi y\sqrt{1 + \left(\frac{1}{2\sqrt{y - 1}}\right)^2}\, dy$

Free Response Questions

11. (a) (5 points) What is the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{7^n} x^n$? What is the interval of convergence?

Solution:

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{7^{n+1}} x^{n+1}}{\frac{n}{7^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{7^{n+1}} x^{n+1} \frac{7^n}{n x^n} \right| = \lim_{n\to\infty} \left| \frac{7^n}{7^{n+1}} \frac{x^{n+1}}{x^n} \frac{n+1}{n} \right|$$
$$= \left| \frac{x}{7} \right| \lim_{n\to\infty} \frac{n+1}{n} = \left| \frac{x}{7} \right|$$

The radius of convergence is 7.

Since $\lim_{n\to\infty} n \neq 0$, $\sum_{n=1}^{\infty} \frac{n}{7^n} 7^n = \sum_{n=1}^{\infty} n$ and $\sum_{n=1}^{\infty} \frac{n}{7^n} (-7)^n = \sum_{n=1}^{\infty} (-1)^n n$ both diverge and the interval of convergence is $(-7, 7)$.

(b) (5 points) What is the radius of convergence of the power series $\sum_{n=1}^{\infty} n!(x-3)^n$? What is the interval of convergence?

Solution:

$$\lim_{n\to\infty} \left| \frac{(n+1)!(x-3)^{n+1}}{n!(x-3)^n} \right| = \lim_{n\to\infty} |(n+1)(x-3)| = |(x-3)| \lim_{n\to\infty} (n+1)$$

The radius of convergence is 0 and the interval of converence is $\{3\}$.

12. (a) (3 points) Find the Taylor series centered at 0 for the function $g(x) = \sin(x)$.

Solution:

$$\begin{array}{llll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin(x) & f'''(x) = -\cos x \\ f^{(4n)}(x) = \sin x & f^{(4n+1)}(x) = \cos x & f^{(4n+2)}(x) = -\sin(x) & f^{(4n+3)}(x) = -\cos x \\ f^{(4n)}(0) = 0 & f^{(4n+1)}(0) = 1 & f^{(4n+2)}(0) = 0 & f^{(4n+3)}(0) = -1 \end{array}$$

So the Taylor series is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.

(b) (3 points) Find the Taylor series centered at 0 for the function $f(x) = x\sin(x^2)$.

Solution: We take the Taylor series for $\sin(x)$ and replace x by x^2 and then multiply by x to get $\displaystyle\sum_{n=0}^{\infty} x \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+3}$.

(c) (4 points) Use your answer from part (b) to find $f^{(83)}(0)$. (Reminder: $f^{(83)}(x)$ means the eighty-third derivative of $f(x)$.)

Solution: The value of f^{83} at zero is the 83rd derivative of the degree 83 part of the power series above.
Since $4n + 3 = 83$ if $n = 20$, the degree 83 part of the solution in (b) is

$$\frac{(-1)^{20}}{(2 * 20 + 1)!} x^{4*20+3} = \frac{1}{41!} x^{83}.$$

So $f^{(83)}(0) = \frac{83!}{41!}$.

13. (a) (2 points) Find the Taylor series centered at 0 for the function $g(x) = e^x$.

Solution: Since $f^{(n)} = e^x$ for all n, $f^{(n)}(0) = 1$ for every n. Then the Taylor series centered at 0 is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

(b) (3 points) Find the Taylor series centered at 0 for the function $f(x) = e^{-x^2}$.

Solution: Substitute $-x^2$ for x into the series above to get $\sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n x^{2n}$

(c) (3 points) Find the Taylor series centered at 0 for the antiderivative

$$F(x) = \int e^{-x^2}\, dx.$$

Use $C = 0$ for the constant of integration.

Solution:
$$\int \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{1}{n!}(-1)^n \frac{1}{2n+1} x^{2n+1}$$

(d) (2 points) Write a series that converges to the value of the definite integral

$$\int_0^5 e^{-x^2}\, dx.$$

Solution: $\left(\sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n 5^{2n+1}}{2n+1} \right) - 0$

14. The region between the curves $y = 1 + x^2$ and $y = 2x + 1$ is shown below.

Let S be the solid obtained by rotating this region around the x-axis.

(a) (3 points) Set up but do not evaluate the integral that computes the volume of S using the disk/washer method.

Solution: $\int_0^2 \pi((2x+1)^2 - (1+x^2)^2)dx$

(b) (3 points) Set up but do not evaluate the integral that computes the volume of S using the cylindrical shells method.

Solution: Solving the original equations for x we have $x = \sqrt{y-1}$ and $x = \frac{y-1}{2}$. Then the volume is the integral $\int_1^5 2\pi y \left(\sqrt{y-1} - \frac{y-1}{2}\right) dy$.

(c) (4 points) Find the volume of S by evaluating one of the integrals you found in part (a) or (b).

Solution: (using the integral in part a)

$$\pi \int_0^2 ((2x+1)^2 - (1+x^2)^2)dx = \pi \int_0^2 (4x^2 + 4x + 1 - 1 - 2x^2 - x^4)dx$$
$$= \pi \int_0^2 (2x^2 + 4x - x^4)dx = \left(\frac{2}{3}x^3 + 2x^2 - \frac{x^5}{5}\right)\Big|_0^2$$
$$= \pi\left(\frac{2^4}{3} + 8 - \frac{2^5}{5}\right) = \pi\left(\frac{-16}{15} + 8\right) = \pi\frac{104}{15}$$

Solution: (using the integral in part b) To compute $\int_1^5 2\pi y\left(\sqrt{y-1}-\frac{y-1}{2}\right)dy$ first use the substitution $u=y-1$ then $du=dy$ and

$$\begin{aligned}\int y\left(\sqrt{y-1}-\frac{y-1}{2}\right)dy &= \int (u+1)(\sqrt{u}-\frac{u}{2})du\\ &= \int u^{\frac{3}{2}}-\frac{u^2}{2}+\sqrt{u}-\frac{u}{2}du+C\\ &= \frac{2}{5}u^{\frac{5}{2}}-\frac{u^3}{6}+\frac{2}{3}u^{\frac{3}{2}}-\frac{u^2}{4}\\ &= \frac{2}{5}(y-1)^{\frac{5}{2}}-\frac{(y-1)^3}{6}+\frac{2}{3}(y-1)^{\frac{3}{2}}-\frac{(y-1)^2}{4}\end{aligned}$$

Then

$$\begin{aligned}&\int_1^5 2\pi y(\sqrt{1-y}-\frac{y-1}{2})dy\\ &\quad= 2\pi\left(\frac{2}{5}(5-1)^{\frac{5}{2}}-\frac{(5-1)^3}{6}+\frac{2}{3}(5-1)^{\frac{3}{2}}-\frac{(5-1)^2}{4}\right)\\ &\qquad-2\pi\left(\frac{2}{5}(1-1)^{\frac{5}{2}}-\frac{(1-1)^3}{6}+\frac{2}{3}(1-1)^{\frac{3}{2}}-\frac{(1-1)^2}{4}\right)\\ &\quad= 2\pi\left(\frac{2}{5}(4)^{\frac{5}{2}}-\frac{(4)^3}{6}+\frac{2}{3}(4)^{\frac{3}{2}}-\frac{(4)^2}{4}\right)=2\pi\left(\frac{2^6}{5}-\frac{2^5}{3}+\frac{2^4}{3}-4\right)\\ &\quad= 2\pi\left(\frac{112}{15}-4\right)=2\pi\frac{52}{15}\end{aligned}$$

15. (a) (5 points) Find the length of the curve $y = \ln(\cos(x))$ for $0 \leq x \leq \frac{\pi}{3}$. (You may find it helpful to know that $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$)

Solution: Since $y = \ln(\cos(x))$, $y' = \frac{1}{\cos(x)}(-\sin(x))$. Then

$$\int_0^{\frac{\pi}{3}} \sqrt{1 + \frac{\sin^2 x}{\cos^2 x}} = \int_0^{\frac{\pi}{3}} \sqrt{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}} = \int_0^{\frac{\pi}{3}} \sec x$$

$$= \ln|\tan x + \sec x||_0^{\frac{\pi}{3}} = \ln(\sqrt{3} + 2) - \ln|0 - 1|$$

(b) (5 points) Find the area of the surface obtained by rotating the curve $y = \sqrt{5 - x}$, $3 \leq x \leq 5$ around the x-axis.

Solution: $y' = \frac{1}{2}(5 - x)^{\frac{-1}{2}}(-1)$

$$\int_3^5 2\pi\sqrt{5 - x}\sqrt{1 + \left(\frac{1}{2}(5 - x)^{\frac{-1}{2}}(-1)\right)^2}\,dx = \int_3^5 2\pi\sqrt{5 - x}\sqrt{1 + \frac{1}{4(5 - x)}}dx$$

$$= \int_3^5 2\pi\sqrt{5 - x + \frac{1}{4}}dx = \int_3^5 2\pi\sqrt{\frac{21}{4} - x}dx$$

$$= 2\pi\left(\frac{21}{4} - x\right)^{\frac{3}{2}}\frac{2}{3}(-1)\Bigg|_3^5 = 2\pi\left(\left(\frac{21}{4} - 5\right)^{\frac{3}{2}}\frac{-2}{3} - \left(\frac{21}{4} - 3\right)^{\frac{3}{2}}\frac{-2}{3}\right)$$

$$= 2\pi\left(\left(\frac{1}{4}\right)^{\frac{3}{2}}\frac{-2}{3} - \left(\frac{9}{4}\right)^{\frac{3}{2}}\frac{-2}{3}\right)\frac{-4\pi}{3}\left(\frac{1}{8} - \frac{27}{8}\right) = \frac{2^2 13\pi}{3 2^3} = \frac{13\pi}{6}$$