Exam 3

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show <u>all work</u> to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

2 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

3 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

4 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

5 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

6 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

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9 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

10 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Trig identities

- $\sin^2(x) + \cos^2(x) = 1$,
- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $\frac{d}{dx}\cot(x) = -\csc^2(x)$

Multiple Choice Questions

1. (5 points) What is the average value of the function $\csc^2(x)$ on the interval $[\frac{\pi}{4}, \frac{3\pi}{4}]$?

   A. 4π

   B. $\frac{\pi}{4}$

   C. $\frac{\pi}{2}$

   D. $\frac{4}{\pi}$

   E. $\frac{2}{\pi}$

2. (5 points) Which of the following is a parametrization of the circle of radius 1 centered at $(0, 0)$.

   A. $x(t) = \cos(3t)$, $y(t) = 1 + \sin(3t)$

   B. $x(t) = 1 + \sin(t)$, $y(t) = -1 + \cos(t)$

   C. $x(t) = \sin(5t)$**,** $y(t) = \cos(5t)$

   D. $x(t) = 3\cos(t)$, $y(t) = 2\sin(t)$

   E. $x(t) = 2\sqrt{1-t}$, $y(t) = 2\sqrt{1+t}$

3. (5 points) Three masses are located in the plane: 3 grams at $(-5, 1)$, 2 grams at $(1, 1)$, and 6 grams at $(-6, 0)$. Find the center of mass of this system.

   A. $(-4, \frac{5}{12})$

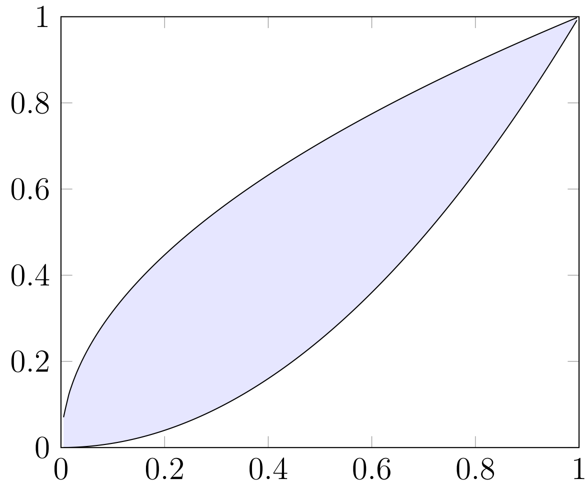
   B. $(\frac{-49}{10}, \frac{1}{2})$

   C. $(-4, \frac{1}{2})$

   D. $(frac{-}4911, \frac{5}{11})$

   E. $(\frac{-1}{4}, 2)$

4. (5 points) The region bounded by $y = \sqrt{x}$ and $y = x^2$ is shown below.



Consider the solid obtained by rotating this region around the x-axis. Using the washer method, which integral will compute the volume of this solid?

A. $\int_0^{\sqrt{2}} 2\pi\left(\sqrt{x} - x^2\right)^2 dx.$

B. $\int_0^1 \pi\left(x - x^4\right) dx.$

C. $\int_0^1 \pi\left(x^2 - x^4\right) dx.$

D. $\int_0^1 \pi\left(\sqrt{x} - x^2\right)^2 dx.$

E. $\int_0^1 \pi\left(\sqrt{x} - x^2\right) dx.$

5. (5 points) Which integral below computes the length of the arc parametrized by (x, x^2) where $0 \leq x \leq 1$?

A. $\int_0^1 1 + 4x^2 dx.$

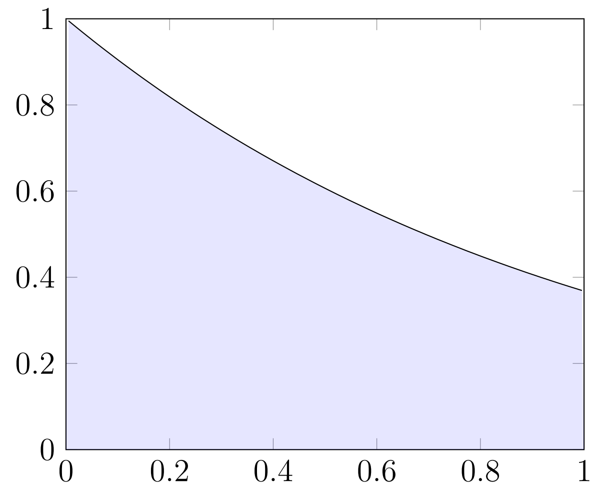
B. $\int_0^1 \sqrt{1 + 4x^2} dx.$

C. $\int_0^1 \sqrt{1 + \frac{x^2}{4}} dx.$

D. $\int_0^1 \sqrt{1 + 4x} dx.$

E. $\int_0^1 2\pi x^2 \sqrt{1 + 4x^2} dx.$

6. (5 points) The region bounded by the curves $y = e^{-x}$, $y = 0$, $x = 0$ and $x = 1$ is shown below.



Consider the solid obtained by rotating this region about the y-axis. Using the shell method, which integral will compute the volume of this solid?

A. $\int_0^1 2\pi x e^{-x} dx$**.**

B. $\int_0^1 2\pi e^{-x} dx$.

C. $\int_0^1 2\pi x (e^{-x})^2 dx$.

D. $\int_0^1 2\pi x^2 e^{-x} dx$.

E. $\int_0^e 2\pi x^2 e^{-x} dx$.

7. (5 points) The line $y = 2x + 1$ for $1 \leqslant x \leqslant 4$ is rotated about the x-axis. What is the area of the resulting surface?

A. 36π

B. $18\sqrt{3}\pi$

C. $36\sqrt{3}\pi$

D. $\sqrt{3}\pi$

E. $36\sqrt{5}\pi$

8. (5 points) Find the slope of the tangent line to the curve parametrized by $x(t) = t^2 + 1, y(t) = t^3$ at the point $(2, 1)$.

   A. $\frac{3}{2}$

   B. 2

   C. $\frac{2}{5}$

   D. $\frac{4}{5}$

   E. $-\frac{3}{2}$

9. (5 points) Find the length of the curve parametrized by $x(t) = e^t \cos(t)$, $y(t) = e^t \sin(t)$, $0 \leq t \leq \pi$.

   A. $\pi - \sqrt{2}$

   B. $\sqrt{2}(e^\pi - 1)$

   C. $\sqrt{2} - 2$

   D. $e^\pi - e$

   E. $\sqrt{2}\pi$

10. (5 points) The curve $y = 1 + e^x$ for $0 \leqslant x \leqslant 2$ is rotated about the y-axis, producing a surface. Which of the following integrals calculates its surface area?

    A. $\displaystyle\int_2^{1+e^2} 2\pi \ln(y-1)\sqrt{1 + \left(\frac{1}{y-1}\right)^2}\, dy$

    B. $\displaystyle\int_1^{1+e^2} 2\pi y\sqrt{1 + \left(\frac{1}{y-1}\right)^2}\, dy$

    C. $\displaystyle\int_0^2 2\pi(1+e^x)\sqrt{1+e^{2x}}\, dx$

    D. $\displaystyle\int_0^2 2\pi x\sqrt{1+(1+e^x)^2}\, dx$

    E. $\displaystyle\int_0^2 2\pi(1+e^x)\sqrt{1+x^2}\, dx$

Free Response Questions

11. The *cycloid* for the circle of radius 1 is the curve parametrized by the functions

$$x(\theta) = \theta - \sin(\theta),$$
$$y(\theta) = 1 - \cos(\theta).$$

(a) (3 points) Find the coordinates of the point P on the cycloid given by $\theta = \frac{\pi}{4}$.

Solution:
$x(\frac{\pi}{2}) = \frac{\pi}{2} - \sin(\frac{\pi}{2}) = \frac{\pi}{2} - \frac{1}{\sqrt{2}}$ $\qquad$ $y(\frac{\pi}{2}) = 1 - \cos(\frac{\pi}{2}) = 1 - \frac{1}{\sqrt{2}}$

(b) (3 points) Find the slope of the tangent line to the cycloid at P.

Solution:
$\frac{dx}{d\theta} = x'(\theta) = 1 - \cos(\theta),$ $\qquad$ $\frac{dy}{d\theta} = y'(\theta) = \sin(\theta).$

$x'(\frac{\pi}{2}) = 1 - \frac{1}{\sqrt{2}},$ $\qquad$ $y'(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}.$

$$\frac{dy}{dx} = \frac{y'(\frac{\pi}{2})}{x'(\frac{\pi}{2})} = \frac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2} - 1}.$$

(c) (4 points) Find the arc length of the piece of the cycloid parametrized by $0 \leq \theta \leq 2\pi$.

Solution: $L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta =$

$$\int_0^{2\pi} \sqrt{(1 - \cos(\theta))^2 + (\sin(\theta))^2} d\theta = \int_0^{2\pi} \sqrt{1 - 2\cos(\theta) + \cos(\theta)^2 + \sin(\theta)^2} d\theta =$$

$$\int_0^{2\pi} \sqrt{2 - 2\cos(\theta)} d\theta = \int_0^{2\pi} \sqrt{2(1 - \cos(\theta))} d\theta = \int_0^{2\pi} \sqrt{4\sin\left(\frac{\theta}{2}\right)^2} d\theta =$$

$$2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 4\left[-\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = 4\left[1 - (-1)\right] = 8.$$

12. Let S be the region in the plane bounded by the parabola $y = x - x^2$ for $0 \leq x \leq 1$ and the x-axis.

(a) (7 points) Find the total mass M and the moments M_y and M_x for S. Assume that S has uniform density $\rho = 1$.

Solution: $M = \displaystyle\int_0^1 x - x^2 dx = \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1 = \frac{1}{6}.$

$$M_y = \int_0^1 x(x - x^2)dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^1 = \frac{1}{12}.$$

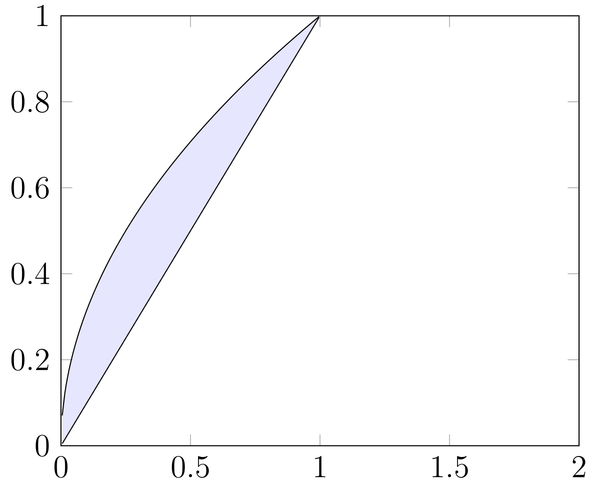
$$M_x = \int_0^1 \frac{1}{2}(x - x^2)^2 dx = \frac{1}{2}\int_0^1 x^2 - 2x^3 + x^4 dx = \frac{1}{2}\left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{1}{60}$$

(b) (3 points) Find the center of mass of S.

Solution:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{6}{12}, \frac{6}{60}\right) = \left(\frac{1}{2}, \frac{1}{10}\right).$$

13. The region between the curves $y = \sqrt{x}$ and $y = x$ is shown below.



Let V be the solid obtained by rotating this region around the line $x = 2$.

(a) (3 points) Set up but do not evaluate the integral that computes the volume of V using the disk/washer method.

Solution: $\int_0^1 \pi((2-y^2)^2 - (2-y)^2)dy$

(b) (3 points) Set up but do not evaluate the integral that computes the volume of V using the cylindrical shells method.

Solution: $\int_0^1 2\pi(2-x)(\sqrt{x}-x)dx.$

(c) (4 points) Find the volume of V by evaluating one of the integrals you found in parts (a) or (b).

Solution: (using the integral in part a)

$$\int_0^1 \pi((2-y^2)^2-(2-y)^2)dy = \pi\int_0^1 4y-5y^2+y^4dy = \pi\left[2y^2 - \frac{5}{3}y^3 + \frac{1}{5}y^5\right]_0^1 = \frac{8\pi}{15}$$

Solution:

$$\int_0^1 2\pi(2-x)(\sqrt{x}-x)dx = 2\pi\int_0^1 2x^{\frac{1}{2}} - 2x - x^{\frac{3}{2}} + x^2dx =$$

$$2\pi\left[\frac{4}{3}x^{\frac{3}{2}} - x^2 - \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{3}x^3\right]_0^1 = 2\pi[\frac{4}{3} - 1 - \frac{2}{5} + \frac{1}{3}] = \frac{8\pi}{15}$$

14. Let S be the surface obtained by revolving the arc L formed by the graph of $\sqrt{x+1}$ with $0 \leq x \leq 5$ around the x axis.

(a) (2 points) Set up but do not evaluate an integral which computes the arc length of L.

Solution:

$$L = \int_0^5 \sqrt{1 + \left(\frac{1}{2\sqrt{x+1}}\right)^2}\, dx$$

(b) (3 points) Set up but do not evaluate an integral which computes the surface area of S.

Solution:

$$A = \int_0^5 2\pi\sqrt{x+1}\sqrt{1 + \left(\frac{1}{2\sqrt{x+1}}\right)^2}\, dx$$

(c) (5 points) Find the surface area of S.

Solution:

$$A = \int_0^5 2\pi\sqrt{x+1}\sqrt{1 + \left(\frac{1}{2\sqrt{x+1}}\right)^2}\, dx = 2\pi \int_0^5 \sqrt{x + \frac{5}{4}}$$

Do a substitution with $x + \frac{5}{4} = u$

$$= 2\pi \int_{\frac{5}{4}}^{\frac{25}{4}} u^{\frac{1}{2}} du = 2\pi \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{\frac{5}{4}}^{\frac{25}{4}} = \frac{4\pi}{3}\left[\left(\frac{25}{4}\right)^{\frac{3}{2}} - \left(\frac{5}{4}\right)^{\frac{3}{2}}\right] = \frac{2\pi}{3}(125 - \sqrt{125})$$

15. Let P be the pyramid of height 10 and rectangular base of length 15 and width 20.

(a) (5 points) Find a function giving the area of the cross-section of P at height z.

Solution: Using similar triangles, $L(z) = \frac{15}{10}(10-z)$ and $W(z) = \frac{20}{10}(10-z)$, so $A(z) = 3(10-z)^2$.

(b) (3 points) Set up an integral which computes the volume of P.

Solution: $V = \displaystyle\int_0^{10} 3(10-z)^2 dz$

(c) (2 points) Find the volume of P.

Solution: $V = \displaystyle\int_0^{10} 3(10-z)^2 dz = 3\int_0^{10} 100 - 20z + z^2 dz =$

$3\left[100z - 10z^2 + \frac{1}{3}z^3\right]_0^{10} = 1000.$