Exam 4

Name: ______________________ Section and/or TA: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. The wise student will show work for the multiple choice problems as well.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	15	Total Score
50	15	10	8	8	9	100

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Multiple Choice Questions

1. Let f be a differentiable function. Which of the following expressions equals

$$\int e^x f(x)dx\,?$$

   A. $e^x f(x) + \int e^x f'(x)\,dx.$

   B. $e^x f'(x) - \int e^x f(x)\,dx.$

   C. $e^x f(x) - \int e^x f'(x)\,dx.$

   D. $-e^x f'(x) + \int e^x f(x)\,dx.$

   E. $e^x f'(x) + \int e^x f'(x)\,dx.$

2. The substitution $x = 2\sin(u)$ turns the integral $\int x^3\sqrt{4-x^2}\,dx$ into which of the following?

   A. $\int 16\sin^3(u)\cos(u)\,du$

   B. $\int 32\sin^3(u)\cos^2(u)\,du$

   C. $\int 32\sin^4(u)\cos(u)\,du$

   D. $\int 16\sin^3(u)\cos^2(u)\,du$

   E. $\int 8\sin^3(u)\cos(u)\,du$

3. Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2 2^n}$

   A. $\frac{1}{2}$, $[-\frac{1}{2}, \frac{1}{2}]$
   B. 2, $[-2, 2]$
   C. 2, $(-2, 2]$
   D. $\frac{1}{2}$, $(-\frac{1}{2}, \frac{1}{2}]$
   E. 2, $(-2, 2)$

4. If the Taylor series $(x-2) - \frac{1}{4}(x-2)^2 + \frac{1}{9}(x-2)^3 - \frac{1}{16}(x-2)^4 + \cdots$ converges to a function $f(x)$ for all x in $(1, 3]$, then the value of $f'''(2)$ is

   A. $\frac{1}{9}$
   B. $\frac{1}{3}$
   C. $\frac{2}{3}$
   D. $\frac{1}{54}$
   E. $\frac{2}{27}$

5. Which of the following integrals represents the area of one loop of the curve $r = \sin(5\theta)$.

   A. $\int_0^{2\pi} \frac{1}{2} \sin^2(5\theta)\, d\theta$

   B. $\int_0^{2\pi} \sin^2(5\theta)\, d\theta$

   C. $\int_0^{\pi/5} \sin^2(5\theta)\, d\theta$

   D. $\int_0^{\pi/5} \frac{1}{2} \sin^2(5\theta)\, d\theta$

   E. $\int_0^{\pi/5} \frac{1}{2} \sin(5\theta)\, d\theta$

6. The equation of the tangent line to the parametric curve $x(t) = t^2 + 2t + 2$, $y(t) = t^3 + 1$ at the point $(5, 2)$ is

   A. $y = \frac{4}{3}x - \frac{14}{3}$

   B. $y = \frac{4}{3}x - 3$

   C. $y = \frac{3}{4}x + \frac{7}{4}$

   D. $y = \frac{3}{4}x - 3$

   E. $y = \frac{3}{4}x - \frac{7}{4}$

7. Identify the conic section whose equation is $9x^2 + 4y^2 = 18x + 27$.

   A. An ellipse with foci at $(1, \pm 8)$

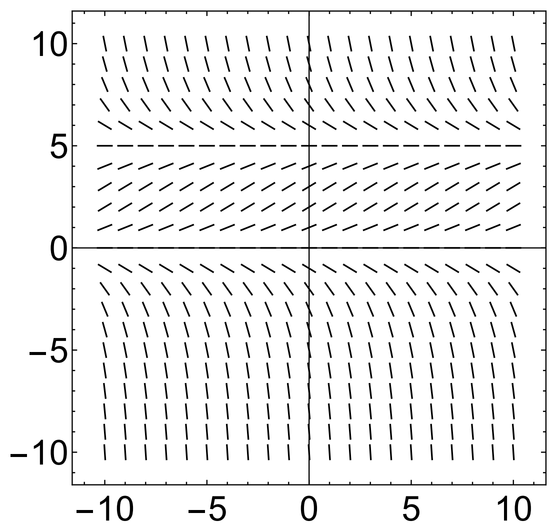
   B. An ellipse with center at $(1, 0)$.

   C. A parabola with focus $(1, 0)$ and directrix $y = -3$.

   D. A hyperbola with foci at $(1, \pm 2)$.

   E. A hyperbola with vertices at $(1, \pm 3)$.

8. Which of the following differential equations generates the given direction field (slope field).



   A. $\dfrac{dy}{dx} = \dfrac{2x}{y}$

   B. $\dfrac{dy}{dx} = x - 2y$

   C. $\dfrac{dy}{dx} = \dfrac{1}{2}y\left(1 - \dfrac{y}{5}\right)$

   D. $\dfrac{dy}{dx} = x - y^2$

   E. $\dfrac{dy}{dx} = \ln(y^2 + 1)$

9. The functions $y(x) = e^{3x}$ and $y = e^{-2x}$ are both solutions to which differential equation?

   A. $y' = y^2 - y - 6$
   B. $y' = y^2 + y - 6$
   C. $y'' + y' - 6 = 0$
   D. $y'' - y' - 6y = 0$
   E. None of these.

10. Find all points (r, θ) with θ in $[0, 2\pi)$ where the tangent line to the polar curve $r = 1 - \sin(\theta)$ is horizontal.

   A. $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$
   B. $\left(\frac{1}{2}, \frac{7\pi}{6}\right), \left(\frac{1}{2}, \frac{11\pi}{6}\right)$
   C. $\left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right), \left(2, \frac{3\pi}{2}\right)$
   D. $\left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right)$
   E. $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), \left(2, \frac{3\pi}{2}\right)$

Free Response Questions

11. Let R be the region bounded by the graph of $y = \sin(x)$ and the x-axis for $0 \leq x \leq \pi$.

(a) (3 points) Find the area of R.

Solution:

$$\text{area} = \int_0^\pi \sin(x)\,dx = -\cos(x)|_0^\pi = 2$$

(b) (6 points) Find the volume of the solid obtained by rotating the region R about the x-axis. SHOW ALL OF YOUR WORK!

Solution:

$$V = \int_0^\pi \pi \sin^2(x)\,dx = \pi \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right|_0^\pi = \frac{\pi^2}{2}$$

(c) (6 points) Find the volume of the solid obtained by rotating the region R about the y-axis. SHOW ALL OF YOUR WORK!

Solution:

$$V = \int_0^\pi 2\pi x \sin(x)\,dx = 2\pi\left(\sin(x) - x\cos(x)\right|_0^\pi = 2\pi^2$$

12. Let $f(x) = \dfrac{1}{(x-1)(x-2)}$.

(a) (6 points) Decompose $f(x)$ by partial fractions

Solution:

$$\frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$
$$1 = A(x-2) + B(x-1)$$

Substituting $x = 1$, we get

$$1 = -A + 0 \quad \text{or } A = -1$$

Substituting $x = 2$, we get

$$1 = 0 + B \quad \text{or } B = 1$$
$$\frac{1}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{1}{x-2}$$

(b) (4 points) Find $\int f(x)\,dx$.

Solution: From above we have

$$\int \frac{1}{(x-1)(x-2)}dx = \int \left(\frac{-1}{x-1} + \frac{1}{x-2}\right)\,dx = -\ln|x-1| + \ln|x-2| + C$$

13. (8 points) Use Euler's method with step size 0.25 to find $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = y - x$, $y(0) = 1$. Show all of your work.

Solution: Here $\Delta x = 0.25$ and $\frac{dy}{dx} = y - x$ with $y(0) = 1$.

x_{old}	y_{old}	dy/dx	Δx	Δy	x_{new}	y_{new}
0.0	1.00	1.0	0.25	0.25	0.25	$y(0.25) = 1.25$
0.25	1.25	1.0	0.25	0.25	0.50	$y(0.50) = 1.50$
0.50	1.50	1.0	0.25	0.25	0.75	$y(0.75) = 1.75$
0.75	1.75	1.0	0.25	0.25	1.0	$y(1.0) = 2.0$

Thus, $y(1.0) = 2.0$.

14. (8 points) Solve the differential equation $xy' = y(2x^2 - 1)$ subject to the condition that $y(1) = 1$.

Solution: This is a separable equation, so

$$\begin{aligned} x\frac{dy}{dx} &= y(2x^2 - 1) \\ \frac{dy}{y} &= \left(2x - \frac{1}{x}\right) dx \\ \ln|y| &= x^2 - \ln|x| + C \end{aligned}$$

Using the initial condition, $y(1) = 1$, we have

$$\begin{aligned} \ln(1) = 0 = 1 - \ln(1) + C &\longrightarrow C = -1 \\ \ln|y| &= x^2 - \ln|x| - 1 \\ y &= e^{x^2 - \ln|x| - 1} \\ &= \frac{e^{x^2-1}}{x} \end{aligned}$$

15. Let the function $f(x)$ be given by $f(x) = (x-3)^2$.

(a) (6 points) Find the average value of $f(x)$ on the interval $[2, 5]$.

Solution:

$$f_{ave} = \frac{1}{5-2}\int_2^5 (x-3)^2 dx = \frac{1}{3}\left(\frac{1}{3}(x-3)^3\right|_2^5 = 1.$$

(b) (3 points) Find all c in $(2, 5)$ so that the average value is equal to $f(c)$.

Solution: Solve $f(x) = 1$ or

$$\begin{aligned} (x-3)^2 &= 1 \\ x^2 - 6x + 9 &= 1 \\ x^2 - 6x + 8 &= 0 \\ (x-2)(x-4) &= 0 \\ x &= 2 \quad \text{or} \quad x = 4 \end{aligned}$$

The only solution in $(2, 5)$ is $x = 4$.