Exam 4

Name: _____   Section: _____

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show <u>all work</u> to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1   (A)   (B)   (C)   (D)   (E)   **6**   (A)   (B)   (C)   (D)   (E)

2   (A)   (B)   (C)   (D)   (E)   **7**   (A)   (B)   (C)   (D)   (E)

3   (A)   (B)   (C)   (D)   (E)   **8**   (A)   (B)   (C)   (D)   (E)

4   (A)   (B)   (C)   (D)   (E)   **9**   (A)   (B)   (C)   (D)   (E)

5   (A)   (B)   (C)   (D)   (E)   **10**   (A)   (B)   (C)   (D)   (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Trig identities

- $\sin^2(x) + \cos^2(x) = 1$,

- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Multiple Choice Questions

1. (5 points) Use the midpoint rule with 4 intervals to approximate $\int_0^2 (4 - x^2)dx$.

     A. $\frac{1}{2}\left(\frac{16}{4} + \frac{15}{4} + \frac{12}{4} + \frac{7}{4}\right)$.

     B. $\frac{1}{2}\left(\frac{63}{16} + \frac{55}{16} + \frac{39}{16} + \frac{15}{16}\right)$

     C. $\frac{1}{4}\left(\frac{16}{4} + \frac{30}{4} + \frac{24}{4} + \frac{14}{4}\right)$.

     D. $\frac{1}{6}\left(\frac{16}{4} + \frac{60}{4} + \frac{24}{4} + \frac{28}{4}\right)$.

     E. 0.

2. (5 points) The improper integral $\int_2^\infty \frac{1}{x^p \ln x} dx$ ...

     A. converges for $p = 2$ and diverges for $p = 0, 1$.

     B. converges for $p = 2, 1$ and diverges for $p = 0$.

     C. converges for $p = 2, 1, 0$.

     D. diverges for $p = 0, 1, 2$.

     E. converges for $p = 1$ and diverges for $p = 0, 2$.

3. (5 points) Which of the following sequences converge?

     A. $b_n = 5^n$.

     B. $c_n = \dfrac{(-1)^n n}{n+1}$.

     C. $a_n = \dfrac{5n-1}{12n+9}$.

     D. None of the above.

     E. All of the above.

4. (5 points) Find the Taylor polynomial $T_3(x)$ for e^{3x} centered at 0. What is $T_3(1)$?

     A. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6}$.

     B. $1 + 3 + 9 + 27$.

     C. $1 + 3 + \dfrac{9}{2} + \dfrac{27}{3}$.

     D. $1 + 3 + \dfrac{9}{2} + \dfrac{27}{6}$.

     E. 1.

5. (5 points) Let $f(x) = \sqrt{x}$. Find a value of $c \in [4, 9]$ so that $f(c)$ is the average value of $f(x)$ on the interval $[4, 9]$.

     A. $\left(\dfrac{38}{15}\right)^2$.

     B. $\dfrac{38}{15}$.

     C. $\left(\dfrac{27}{15}\right)^2$.

     D. $\dfrac{-8}{15}$.

     E. $\left(\dfrac{38}{5}\right)^2$.

6. (5 points) Find the volume of a solid whose base is the unit circle $x^2 + y^2 = 1$ and the cross sections perpendicular to the x-axis are triangles whose height and base are equal.

  A. $\dfrac{4}{3}$.

  B. $\dfrac{8}{3}$.

  C. 0

  D. $\dfrac{2}{3}$.

  E. $\dfrac{16}{3}$.

7. (5 points) A surface is created by rotating the graph of $f(x) = 5 + \sin(6x)$ from $x = 0$ to $x = 100$ around the x-axis. What is the integral that computes the area of this surface?

  A. $\displaystyle\int_0^{100} 2\pi(5 + \sin(6x))dx$.

  B. $\displaystyle\int_0^{100} 2\pi\sqrt{1 + (6\cos(6x))^2}dx$.

  C. $\displaystyle\int_0^{100} 2\pi(5 + \sin(6x))\sqrt{1 + 6\cos(6x)}dx$.

  D. $\displaystyle\int_0^{100} 2\pi(5 + 6\cos(6x))\sqrt{1 + \sin(6x)}dx$.

  E. $\displaystyle\int_0^{100} 2\pi(5 + \sin(6x))\sqrt{1 + (6\cos(6x))^2}dx$.

8. (5 points) Find the center of mass of the region between the curves $y = x$ and $y = x^2$ (Assume the region has constant density).

  A. $\left(\dfrac{1}{2}, \dfrac{2}{5}\right)$.

  B. $\left(\dfrac{1}{12}, \dfrac{1}{15}\right)$.

  C. $\left(\dfrac{2}{5}, \dfrac{1}{2}\right)$.

  D. $\left(\dfrac{1}{15}, \dfrac{1}{12}\right)$.

  E. $\left(1, \dfrac{1}{5}\right)$.

9. (5 points) Which of the following is the equation for a hyperbola with vertices $(0, \pm 2)$ and foci $(0, \pm 5)$?

A. $\dfrac{x^2}{4} - \dfrac{y^2}{21} = 1.$

B. $\dfrac{y^2}{4} + \dfrac{x^2}{21} = 1.$

C. $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1.$

D. $\dfrac{x^2}{4} + \dfrac{y^2}{21} = 1.$

E. $\dfrac{x^2}{21} - \dfrac{y^2}{4} = 1.$

10. (5 points) Which of the following is the direction field for the equation $y' = y - 2x$?

A.

B.

C.

D.

E.
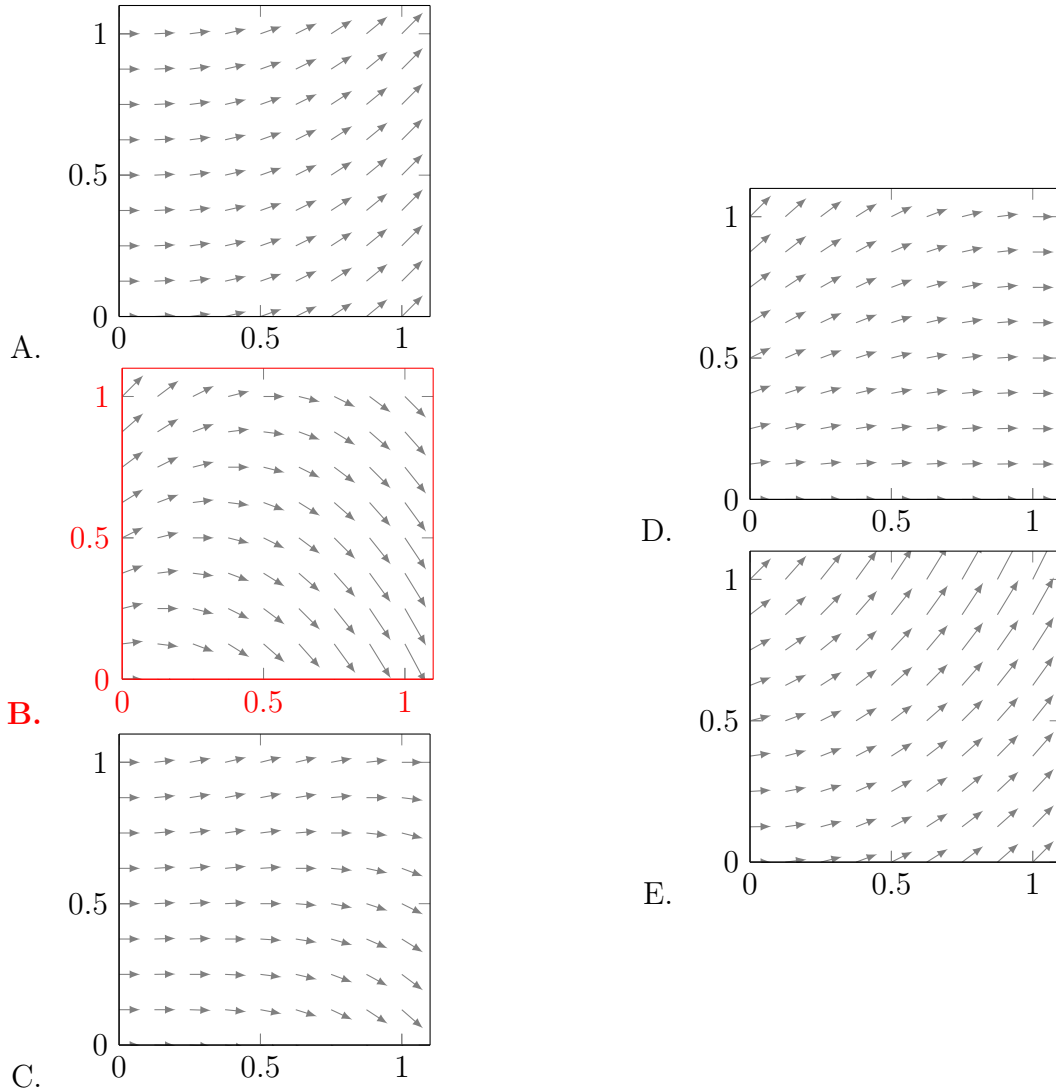
Free Response Questions

11. Compute the following integrals.

(a) (5 points) $\int x \sin x \cos x \, dx$

> **Solution:** Let $u = x$ and $dv = \sin x \cos x \, dx$. Then $du = dx$ and $v = \frac{1}{2} \sin^2 x$
>
> $$\int x \sin x \cos x \, dx = \frac{1}{2} x \sin^2 x - \int \frac{1}{2} \sin^2 x \, dx$$
> $$= \frac{1}{2} x \sin^2 x - \frac{1}{4} \int (1 - \cos(2x)) \, dx$$
> $$= \frac{1}{2} x \sin^2 x - \frac{1}{4} \left(x - \frac{1}{2} \sin(2x) \right) + C$$

(b) (5 points) $\int \dfrac{dx}{x^2 \sqrt{x^2 - 4}}$

> **Solution:** Take $x = \sqrt{2} \sec \theta$. Then $dx = \sqrt{2} \sec \theta \tan \theta \, d\theta$.
>
> $$\int \frac{dx}{x^2 \sqrt{x^2 - 2}} = \int \frac{\sqrt{2} \sec \theta \tan \theta \, d\theta}{(\sqrt{2} \sec \theta)^2 \sqrt{(\sqrt{2} \sec \theta)^2 - 2}}$$
> $$= \int \frac{\sqrt{2} \sec \theta \tan \theta \, d\theta}{2 \sec^2 \theta \sqrt{2 \sec^2 \theta - 2}} = \int \frac{\tan \theta \, d\theta}{2 \sec \theta \tan \theta} = \frac{1}{2} \int \cos \theta \, d\theta$$
> $$= \frac{1}{2} \sin \theta + C = \frac{1}{2} \frac{\sqrt{x^2 - 2}}{x} + C$$

12. Determine if the following series converge or diverge. Justify your answer!

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{\frac{3}{2}}}$.

> **Solution:** Compute
>
> $$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{1+n^{\frac{3}{2}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+n^{\frac{3}{2}}}n = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{1+n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{\frac{3}{2}n^{\frac{1}{2}}}{\frac{3}{2}n^{\frac{1}{2}}} = 1.$$
>
> Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges this series diverges by the limit comparison test.

(b) (5 points) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^{k}$.

> **Solution:**
>
> $$\lim_{k\to\infty} \left(\left(\frac{k}{3k+1}\right)^{k}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{3k+1} = \frac{1}{3}$$
>
> So the series converges by the root test.

13. Let S be the solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ around the line $x = 3$.

(a) (3 points) Set up the integral that computes the volume of S using the disk/washer method.

Solution:
$$\int_0^1 \pi((3 - y^2)^2 - (3 - \sqrt{y})^2)\, dy$$

(b) (3 points) Set up the integral that computes the volume of S using the cylindrical shells method.

Solution:
$$\int_0^1 2\pi(3 - x)(\sqrt{x} - x^2)\, dx$$

(c) (4 points) Choose one of these integrals and find the volume of S.

Solution: With the integral in (a)

$$\int_0^1 \pi((3 - y^2)^2 - (3 - \sqrt{y})^2)\, dy = \int_0^1 \pi(9 - 6y^2 + y^4 - 9 + 6\sqrt{y} - y)\, dy$$

$$= \int_0^1 \pi(-6y^2 + y^4 + 6\sqrt{y} - y)\, dy$$

$$= \pi\left(-2y^3 + \frac{1}{5}y^5 + 4y^{\frac{3}{2}} - \frac{1}{2}y^2\right) dy = \pi\left(-2 + \frac{1}{5} + 4 - \frac{1}{2}\right)$$

$$= \pi\frac{20 + 2 - 5}{10} = \pi\frac{17}{10}$$

With the integral in (b)

$$\int_0^1 2\pi(3 - x)(\sqrt{x} - x^2)\, dx = \int_0^1 2\pi(3\sqrt{x} - 3x^2 - x^{\frac{3}{2}} + x^3)\, dx$$

$$= 2\pi\left(2x^{\frac{3}{2}} - x^3 - \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{4}x^4\right) \Big|_0^1$$

$$= 2\pi\left(2 - 1 - \frac{2}{5} + \frac{1}{4}\right) = 2\pi\frac{20 - 8 + 5}{20} = \pi\frac{17}{10}$$

14. (a) (5 points) Find the slope of the tangent line to the parametric curve given by $x = t \sin t$, $y = t \cos t$ at the point $(0, -\pi)$.
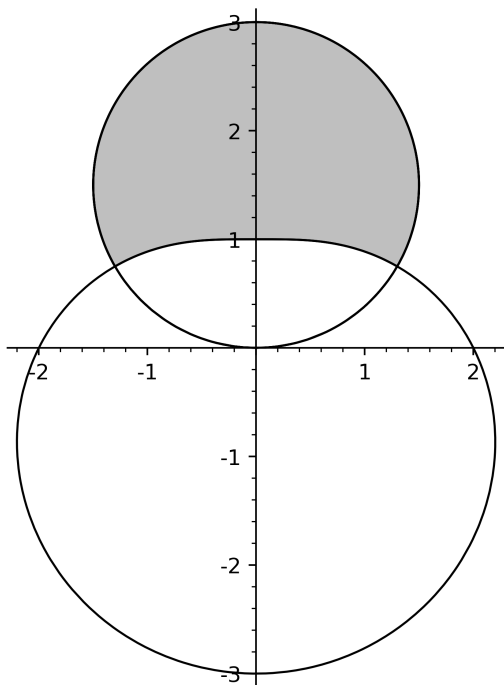
> **Solution:** $\frac{dx}{dt} = \sin t + t \cos t$ and $\frac{dy}{dt} = \cos t - t \sin t$ so
>
> $$\frac{dy}{dx} = \frac{\cos t - t \sin t}{\sin t + t \cos t}$$
>
> For $t \sin t = 0$ we need $t = 0$ or $\sin t = 0$ so $t = k\pi$ for some integer k. For $t \cos t = -\pi$ and $t = k\pi$, we have $k\pi \cos(k\pi) = -\pi$ or $k \cos(k\pi) = -1$, Then $k = 1$. So the slope of the tangent line at $t = \pi$ is
>
> $$\frac{dy}{dx} = \frac{\cos \pi - \pi \sin \pi}{\sin \pi + \pi \cos \pi} = \frac{-1}{-\pi} = \frac{1}{\pi}$$

(b) (5 points) Write an integral that computes the area of the region (shown below) that is inside the polar curve $r = 3 \sin \theta$ and outside the polar curve $r = 2 - \sin \theta$.

> **Solution:** The curves intersect when
>
> $$3 \sin \theta = 2 - \sin \theta$$
> $$4 \sin \theta = 2$$
> $$\sin \theta = \frac{1}{2}$$
>
> so $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. Then the area is given by the integral
>
> $$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}((3 \sin \theta)^2 - (2 - \sin \theta)^2) d\theta$$

15. (a) (5 points) For what values of k does $y = e^{kt}$ satisfy the differential equation

$$y'' - y' - 2y = 0?$$

Solution: $y' = ke^{kt}$ and $y'' = k^2 e^{kt}$ and so we need

$$0 = k^2 e^{kt} - ke^{kt} - 2e^{kt} = e^{kt}(k^2 - k - 2) = e^{kt}(k+1)(k-2)$$

This holds if $k = -1$ or $k = 2$.

(b) (5 points) Find the solution to the differential equation $y' = y^2 x$ that satisfies the initial condition $y(0) = 1$.

Solution: Let $f(y) = y^2$ and $g(x) = x$ then

$$\int \frac{1}{y^2} dy = \int y^{-2} dy = -y^{-1} + C_1$$

$$\int x\, dx = \frac{1}{2} x^2 + C_2$$

$$\frac{-1}{y} = \frac{x^2 + C}{2}$$

$$y = \frac{-2}{x^2 + C}$$

Then

$$1 = \frac{-2}{0^2 + C} = \frac{-2}{C}$$

So $C = -2$ and $y = \frac{-2}{x^2 - 2}$.