Exam 4

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Find $\int x^2 \ln(x)\, dx$.

   A. $x^2 \ln(x) - \frac{1}{3}x^3 + C$

   B. $\frac{1}{6}x^3 \ln(x) + C$

   C. $\frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$

   D. $x + 2x \ln(x) + C$

   E. $x^3 \ln(x) + \frac{1}{3}x^3 + C$

2. (5 points) Find the center of the ellipse with equation $y^2 + 4y + x^2 + 3x = 1$.

   A. $(-\frac{3}{2}, -2)$

   B. $(\frac{3}{2}, -1)$

   C. $(3, -1)$

   D. $(\frac{9}{4}, 2)$

   E. $(-\frac{9}{4}, -3)$

3. (5 points) Which of the following **sequences** converge?

   A. $b_n = \frac{3^n}{5^n}$

   B. $c_n = \frac{16 + (-1)^n n}{n^2}$

   C. $a_n = \sin\left(\frac{1}{n}\right)$

   D. None of the above.

   E. All of the above.

4. (5 points) Which of the following **series** converge?

A. $\sum_{n=10}^{\infty} \frac{n+1}{\sqrt{n^2-1}}$

B. $\sum_{n=1}^{\infty} \frac{n}{(n+2)^{\frac{3}{2}}}$

C. $\sum_{n=1}^{\infty} \frac{2n-1}{2n+1}$

D. $\sum_{n=1}^{\infty} \frac{1}{(n^2+3n)^{\frac{5}{2}}}$

E. None of the above series converge.

5. (5 points) Consider the curve C parametrized by $x(t) = t^3 + 1$ and $y(t) = t^2 + t - 6$. Find the slope of the tangent line to C at $(2, -4)$.

A. 6

B. 1

C. $\frac{2}{3}$

D. $\frac{2}{9}$

E. $\frac{4}{3}$

6. (5 points) Evaluate $\int_0^{\infty} \frac{1}{(x+2)^3}\, dx$

A. $\frac{1}{8}$

B. $\frac{1}{3}$

C. 0

D. $-\frac{1}{4}$

E. This integral diverges

7. (5 points) Find the sum of the series $\sum_{n=1}^{\infty}\left[\left(\frac{2}{3}\right)^n - \left(\frac{1}{4}\right)^n\right]$

A. $\frac{1}{3}$

B. $\frac{5}{7}$

C. 0

D. This series is divergent.

E. $\frac{5}{3}$

8. (5 points) Find the center of mass of the system of particles given by a mass of 2 grams at $(-2, 0)$, a mass of 5 grams at $(7, 1)$, and a mass of 3 grams at $(1, 5)$.

A. $(4, 2)$

B. $(2, 4)$

C. $(\frac{34}{10}, 2)$

D. $(2, \frac{37}{12})$

E. $(0, \frac{27}{12})$

9. (5 points) Which of the following is the equation of a circle with center $(1, 2)$ and radius 2?

   A. $x^2 - 2x + y^2 - 4y + 1 = 0$

   B. $x^2 + 2x + y^2 - 4y + 1 = 0$

   C. $x^2 - 2x - y^2 - 4y + 1 = 0$

   D. $9x^2 - 2x + 4y^2 - 4y + 1 = 0$

   E. $2y + 4x^2 - 4x + 1 = 0$

10. (5 points) For any constant C, the function $y(x) = Ce^{\frac{1}{2}x^2} + 1$ is a solution to the differential equation $y' = x(y - 1)$. The unique solution satisfying $y(2) = 2$ is

   A. $y(x) = 2e^{\frac{1}{2}x^2}$

   B. $y(x) = e^{\frac{1}{2}x^2} + 1$

   C. $y(x) = e^{\frac{1}{2}x^2} + 2$

   D. $y(x) = e^{-2}e^{\frac{1}{2}x^2} + 1$

   E. $y(x) = 2e^{\frac{1}{2}x^2} - 2$

Free Response Questions

11. A parametric curve C is given by $x(t) = t^2 + 1$ and $y(t) = t^4 + t^2$ for $0 \le t \le 2$.

(a) (4 points) Set up an integral which computes the arc length of C.

Solution: Compute the derivatives:

$$x'(t) = 2t, \qquad y'(t) = 4t^3 + 2t,$$

and use the formula

$$L = \int_0^2 [x'(t)^2 + y'(t)^2]^{\frac{1}{2}}\, dt = \int_0^2 [4t^2 + (4t^3 + 2t)^2]^{\frac{1}{2}}\, dt,$$

that reduces to

$$L = 2\sqrt{2} \int_0^2 t[2t^4 + 2t^2 + 1]^{\frac{1}{2}}\, dt.$$

GRADING: 1 point for the derivatives, 2 points for the formula, and 1 point for the final result. It isn't necessary to obtain the last expression above for L.

(b) (6 points) Eliminate the t parameter to find a function $f(x)$ with the property that points on C satisfy $y = f(x)$.

Solution: Solve the x-coordinate equation to obtain $t^2 = x - 1$, and substitute this into the y-coordinate equation to obtain

$$y(x) = (x-1)^2 + (x-1) = x(x-1).$$

GRADING: 3 points for solving the equation and substituting, 3 points for the solution.

12. (a) (5 points) Find the Taylor series of the function $\frac{x}{1-\frac{2}{3}x^2}$ centered at 0.

Solution: Use the Geometric series formula to get

$$\frac{1}{1-\frac{2}{3}x^2} = \sum_{n=0}^{\infty}\left(\frac{2}{3}x^2\right)^n = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n x^{2n+1}.$$

Multiply by x to get

$$\frac{x}{1-\frac{2}{3}x^2} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n x^{2n}.$$

GRADING: 3 points for the geometric series and 2 points for the correct result.

(b) (5 points) Find the radius of convergence for the series $\sum_{n=1}^{\infty}\frac{5^n(n+1)(x-3)^n}{n+7}$.

Solution: Ratio test (the coefficients are positive) applied to the coefficients

$$a_n = \frac{5^n(n+1)}{n+7},$$

gives

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{5^{n+1}}{5^n}\right)\left(\frac{n+7}{n+8}\right)\left(\frac{n+2}{n+1}\right) = 5.$$

The radius of convergence is $R = \frac{1}{5}$.

GRADING: 3 points for setting up the ratio test, and 2 points for the correct result. Note: They can include $(x-3)^n$ in the coefficients a_n. They will then arrive at the condition

$$5|x-3| < 1,$$

from which one concludes $R = \frac{1}{5}$.

13. Consider the polar curve C defined by the equation $r = 1 + \cos(2\theta)$.

(a) (8 points) Find an equation for the tangent line to C at the point defined by the angle $\theta = \frac{\pi}{4}$.

Solution: 1. (6 points) Slope at $\theta = \frac{\pi}{4}$. Recall

$$y(\theta) = r(\theta)\sin\theta, \quad x(\theta) = r(\theta)\cos\theta,$$

and that

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}.$$

At $\theta = \frac{\pi}{4}$, we get $r(\pi/4) = 1$, and $r'(\pi/4) = -2$ so that the slope at $\theta = \frac{\pi}{4}$ is

$$\frac{y'(\frac{\pi}{4})}{x'(\frac{\pi}{4})} = \frac{(-2)(\frac{\sqrt{2}}{2}) + (1)(\frac{\sqrt{2}}{2})}{(-2)(\frac{\sqrt{2}}{2}) - (1)(\frac{\sqrt{2}}{2})} = \frac{1}{3}.$$

2. (2 points) For the equation for the tangent line, use $y = mx + b$ and the point $(x, y) = (x(\pi/4), y(\pi/4) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, to get

$$y = \frac{1}{3}x + \frac{\sqrt{2}}{3}.$$

GRADING: Part 1: 2 points for the slope formula, 2 points for evaluating the coordinates correctly, 2 points for the correct answer.

(b) (2 points) Set up an integral which computes the area between C and the origin for $0 \leq \theta \leq \frac{\pi}{2}$.

Solution: The formula for the area C bounded by a polar curve $r(\theta)$ is

$$A_C = \frac{1}{2}\int_0^{\frac{\pi}{2}} r(\theta)^2 \, d\theta.$$

In our case, $r(\theta) = 1 + \cos(2\theta)$, so we get

$$A_C = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos(2\theta))^2 \, d\theta.$$

GRADING: 1 point for the first integral and 1 point for the second (if the first is missing but the second is written, award 2 points.)

14. (a) (6 points) Set up the integral for the volume of a solid obtained by revolving the region between the graph of $f(x) = 3x^2 - x^3$ and the x-axis **around the y-axis.** (Hint: use the shell method.)

Solution: The shell method gives a basic volume element:

$$\Delta V = (2\pi x)(3x^2 - x^3)\Delta x,$$

since we are rotating around the y-axis. Since $f(x) = x^2(3 - x)$, the curve satisfied $f(0) = 0$ and $f(3) = 0$. The integral expression for the volume is

$$\int_0^3 2\pi x(3x^2 - x^3)\ dx.$$

GRADING: 3 points for the set-up: correct choice of integration variable, length, and height of the shell; 3 points for writing the correct formula for V. You can also use the washer method slicing perpendicular to the y-axis. (Using the shell method is only a hint.) The max of the curve is at $x = 2$ and $f(2) = 4$. Then, the volume is

$$V = \int_0^4 \pi(r_o(y)^2 - r_i(y)^2)\ dy,$$

where $r_0(y)$ is the outer radius, and $r_i(y)$ is the inner radius.

(b) (4 points) Evaluate the integral in part (a) to find the volume of the solid of revolution.

Solution: The integral is

$$V = 2\pi\left[\frac{3}{4}x^4 - \frac{1}{5}x^5\right]_0^3 = 2\pi\left[3^5\left(\frac{1}{4} - \frac{1}{5}\right)\right] = \frac{243}{10}\pi = 24.3\pi.$$

GRADING: 2 points for correct integration and 2 points for the correct numerical result.

15. (10 points) Using the method of partial fractions, compute

$$\int \frac{x}{(x-1)(x^2+1)}\ dx.$$

Solution: 1. The partial fractions decomposition is:

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

2. Solving this, cross multiply to get

$$x = A(x^2+1) + (Bx+C)(x-1) = (A+B)x^2 + (C-B)x + (A-C).$$

Equating coefficients of powers of x gives: $A = -B$, $C - B = 1$, and $A = C$. As a result, $A = C = \frac{1}{2}$, and $B = -\frac{1}{2}$.
3. Integrate: The integral I equals:

$$I = \frac{1}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int \frac{x}{x^2+1} + \frac{1}{2}\int \frac{dx}{x^2+1},$$

giving

$$I = \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C.$$

GRADING: Part 1: 3 points, part 2: 4 points, and part 3: 3 points.