Exam 4

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. The wise student will show work for the multiple choice problems as well.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

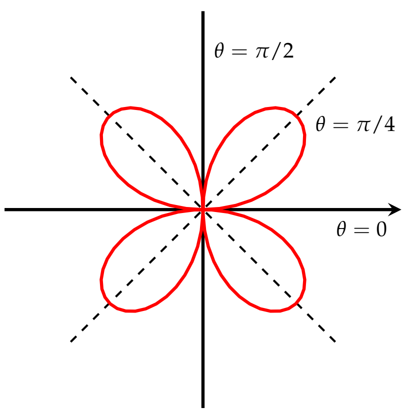
1. (5 points) Identify the type of conic section given by

$$4x^2 = y^2 + 4$$

and give its vertices and foci.

   A. Ellipse, vertices $(\pm 1, 0)$ and $(0, \pm 2)$, foci $(\pm\sqrt{3}, 0)$
   B. Hyperbola, vertices $(\pm 2, 0)$, foci $(\sqrt{5}/2, 0)$
   C. Hyperbola, vertices $(\pm 1, 0)$, foci $(\pm\sqrt{5}/2, 0)$
   D. Hyperbola, vertices $(\pm 1, 0)$, foci $(\pm\sqrt{5}, 0)$
   E. Ellipse, vertices $(\pm 2, 0)$ and $(0, \pm 1)$, foci $(0, \pm\sqrt{3})$

2. (5 points) Which of the following correctly computes the area of one leaf of the curve $r = \sin(2\theta)$ shown below?



   A. $\displaystyle\int_0^{\pi/2} \frac{1}{2}\sin(2\theta)\, d\theta$
   B. $\displaystyle\int_0^{\pi/2} \frac{1}{2}\sin^2(2\theta)\, d\theta$
   C. $\displaystyle\int_0^{\pi/4} \frac{1}{2}\sin(2\theta)\, d\theta$
   D. $\displaystyle\int_0^{\pi/4} \frac{1}{2}\sin^2(2\theta)\, d\theta$
   E. $\displaystyle\int_0^{\pi} \frac{1}{2}\sin^2(2\theta)\, d\theta$

3. (5 points) Which of the following differential equations has *both* $y = e^{-x}$ *and* $y = e^{2x}$ as solutions?

   A. $y'' - y' - 2y = 0$

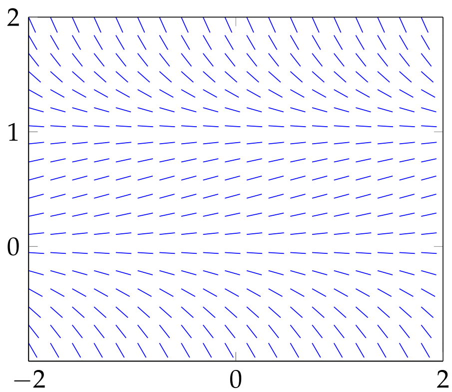
   B. $y'' + y' - 2y = 0$

   C. $y'' = 4y$

   D. $y'' + y = 0$

   E. $y' = y$

4. (5 points) Which of the following equations has the direction field shown?



   A. $y' = y$

   B. $y' = \sin(y)$

   C. $y' = xy$

   D. $y' = y(1-y)$

   E. $y' = y(2-y)$

5. (5 points) Which of the following is equal to $\int_0^x t f'(t)\,dt$?

   A. $f(x) - \int_0^x t f(t)\,dt$

   B. $x f(x) - \int_0^x f(t)\,dt$

   C. $x f'(x) - \int_0^x f(t)\,dt$

   D. $f'(x) - \int_0^x t f(t)\,dt$

   E. $f'(x) - \int_0^x t f'(t)\,dt$

6. (5 points) The curve

$$x^2 - 2x + 2y^2 - 8y + 7 = 0$$

is correctly described by which of the following?

A. Ellipse, center $(2, 1)$, $a = 2$, $b = 1$

B. Circle, center $(1, 2)$, radius $\sqrt{2}$

C. Ellipse, center $(1, 2)$, $a = \sqrt{2}$, $b = 1$

D. Hyperbola, center $(1, 2)$, foci $(1/2, 2)$ and $(3/2, 2)$

E. Ellipse, center $(2, 1)$, $a = \sqrt{2}$, $b = 1$

7. (5 points) State the partial fraction decomposition of

$$\frac{x^3}{(x^2 + 2x - 3)(x^2 + x + 1)}.$$

A. $\dfrac{A}{x+1} + \dfrac{B}{x-3} + \dfrac{Cx+D}{x^2+x+1}$

B. $\dfrac{A}{x-1} + \dfrac{B}{x+3} + \dfrac{Cx+D}{x^2+x+1}$

C. $\dfrac{A}{x-1} + \dfrac{B}{x+3} + \dfrac{C}{x^2+x+1}$

D. $\dfrac{A}{x+1} + \dfrac{B}{x-3} + \dfrac{C}{x^2+x+1}$

E. $\dfrac{A}{x-1} + \dfrac{B}{x-3} + \dfrac{Cx+D}{x^2+x+1} + \dfrac{Ex+F}{(x^2+x+1)^2}$

8. (5 points) A function has Maclaurin series (about $a = 0$)

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$$

What is $f'''(0)$?

A. 0

B. 2

C. -2

D. $-\dfrac{1}{3}$

E. $\dfrac{1}{3}$

9. (5 points) The substitution $x = 2\sin\theta$ in the integral

$$\int \frac{dx}{x^2\sqrt{4-x^2}}$$

leads to which one of the following?

A. $-\int \frac{d\theta}{\cos^2\theta}$

B. $\int \frac{\cot\theta\, d\theta}{2\sin^2\theta}$

C. $\int \frac{d\theta}{\sin^2\theta}$

D. $\int \frac{d\theta}{4\sin^2\theta}$

E. $-\int \frac{4\,d\theta}{\sin^2\theta}$

10. (5 points) The Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{3^n(n+1)}$$

has what radius and interval of convergence?

A. Radius of convergence 3 and interval of convergence $(-1,5)$

B. Radius of convergence 3 and interval of convergence $(-2,4)$

C. Radius of convergence 3 **and interval of convergence** $(-1,5]$

D. Radius of convergence 3 and interval of convergence $[-2,4]$

E. Radius of convergence 3 and interval of convergence $[-1,5)$

Free Response Questions

11. (10 points) Find the equation of the tangent line to the polar curve $r = 2\cos\theta$ at $\theta = \pi/3$.

Solution: Since

$$x(\theta) = 2\cos^2\theta, \quad y(\theta) = 2\cos\theta\sin\theta = \sin(2\theta)$$

we get

$$x'(\theta) = -4\cos\theta\sin\theta, \quad y'(\theta) = 2\cos(2\theta).$$

At $\theta = \pi/3$ we have

$$x'(\pi/3) = (-4)(1/2)(\sqrt{3}/2) = -\sqrt{3}, \quad y'(\pi/3) = 1$$

so

$$\frac{dy}{dx} = -\frac{1}{\sqrt{3}}.$$

Since $(x(\pi/3), y(\pi/3)) = (1/2, \sqrt{3}/2)$ we get

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x - 1/2).$$

Suggested grading rubric:

(2 points) Convert to Cartesian coordinates

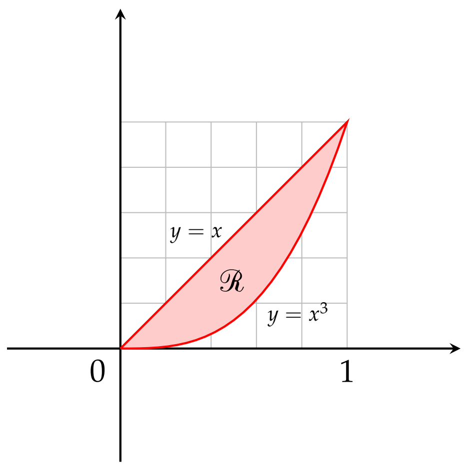
(2 points) Obtain correct derivatives x' and y'

(2 points) Obtain correct slope

(2 points) Obtain correct (x, y) coordinates of point

(2 points) Obtain correct equation in point-slope or slope-intercept form

12. (a) (5 points) Let $\mathscr{R}$ be the region bounded by the curves $y = x^3$ and $y = x$ for $x \geq 0$. Set up but do not evaluate an integral which gives the volume of the solid obtained by rotating $\mathscr{R}$ about the x-axis. Be sure to state which method (disc, washer, shell) you are using. Be sure to state the radius and height of the shell (shell method) or the inner and outer radius of the washer (washer method).



Solution:
Washer method: Integrate along the x axis from 0 to 1. A washer at x has inner radius x^3 and outer radius x. Hence

$$V = \int_0^1 \pi \left(x^2 - x^6\right)\, dx.$$

Shell method: integrate y from 0 to 1. In this case the integral is

$$V = \int_0^1 2\pi y \left(y^{1/3} - y\right)\, dy.$$

Suggested grading rubric for parts (a) and (b):

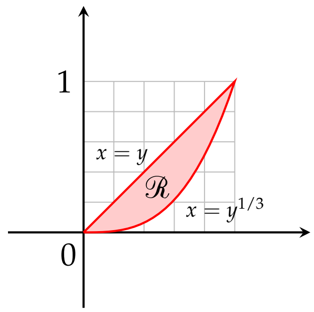
(1 point) correct limits of integration
(2 points) correct radius and height (shell) or inner/outer radii (washer)
(2 points) correct form of integrand

(b) (5 points) Set up but do not evaluate an integral which gives the volume of the solid obtained by rotating the region $\mathscr{R}$ from part (a) about the y-axis. Be sure to state which method (disc, washer, shell) you are using. Be sure to state the radius and height of the shell (shell method) or the inner and outer radius of the washer (washer method).

Solution:
Shell Method: Interate along the x axis from 0 to 1. One gets

$$V = 2\pi \int_0^1 x(x - x^3)\, dx.$$

Washer method: Integrate along the y axis from 0 to 1. We need to express the curves as functions of y as shown in the figure below.



A washer at y has inner radius y and outer radius $y^{1/3}$. Hence

$$V = \int_0^1 \pi \left(y^{2/3} - y^2\right)\, dy.$$

13. (a) (5 points) Use Euler's method with stepsize 0.25 to estimate $y(1)$ if $y(x)$ solves the initial value problem

$$y' = x - y, \quad y(0) = 1.$$

Show your steps in tabular form, and give results correct to three decimal places.

Solution: With $h = 0.25$ we can make a table:

n	x_n	y_n	$x_n - y_n$
0	0	1	-1
1	0.25	0.75	-0.5
2	0.5	0.625	-0.125
3	0.75	0.594	0.156
4	1.0	**0.633**	

and estimate $y(1) = 0.633$ to three decimal places.

Suggested grading rubric:

(1 point) correct values of x_n
(1 point) table set-up
(2 points) correct values of y_n
(1 point) correct answer for $y(1)$

(b) (5 points) Using the method of separation of variables, find the general solution of the differential equation

$$\frac{dy}{dx} = 3x^2y^2$$

Then find the unique solution y with $y(0) = 1/2$.

Solution:

$$\frac{dy}{y^2} = 3x^2\,dx$$

$$-\frac{1}{y} = x^3 + C$$

$$y = \frac{1}{-C - x^3}$$

This gives the general solution. To solve the initial value problem we solve

$$\frac{1}{2} = -\frac{1}{C}$$

and conclude that $C = -2$. Hence

$$y(x) = \frac{1}{2 - x^3}.$$

Suggested grading rubric:

(1 point) separate variables in ODE
(1 point) integrate
(1 point) correct general solution
(2 points) evaluate C and obtain correct solution

14. (10 points) Using the method of partial fractions, compute

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx.$$

Solution: First we find the partial fraction decomposition of the integrand. Writing

$$\frac{A}{x-2}+\frac{B}{x-3}=\frac{x-4}{(x-2)(x-3)}$$

we see that

$$A(x-3)+B(x-2)=x-4.$$

Substituting $x=2$ we get $-A=-2$ or $A=2$. Substituting $x=3$ we get $B=-1$. Hence

$$\frac{x-4}{(x-2)(x-3)}=\frac{2}{x-2}-\frac{1}{x-3}.$$

Now we can compute

$$\begin{aligned}\int_0^1 \frac{x-4}{(x-2)(x-3)}\,dx &= \int_0^1 \frac{2}{x-2}\,dx-\int_0^1 \frac{1}{x-3}\,dx\\ &= [2\ln|x-2|]_0^1-[\ln|x-3|]_0^1\\ &= -2\ln 2-(\ln 2-\ln 3)\\ &= \ln 3-3\ln 2.\end{aligned}$$

Suggested grading rubric:

(2 points) Correct form of partial fraction decomposition

(4 points) Solve for A and B

(4 points) Correct computation of the integral

15. (a) (5 points) State the ratio test for a series $\sum_{n=1}^{\infty} a_n$. Be sure to include each of the three cases.

Solution:

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent)

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Suggested grading rubric:

2 points for a correct and complete statement of (i)
2 points for a correct and complete statement of (ii)
1 point for a correct and complete statement of (iii)

(b) (5 points) Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{9^n}{(n+1)4^{2n+1}}$$

is convergent or divergent.

Solution: Let $a_n = \frac{9^n}{(n+1)4^{2n+1}}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{9^{n+1}}{9^n} \frac{(n+1)4^{2n+1}}{(n+2)4^{2n+3}} = \frac{9}{16}\left(\frac{n+1}{n+2}\right).$$

Hence

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{9}{16}\left(\frac{n+1}{n+2}\right)\right) = \frac{9}{16} < 1$$

and the series converges.

Suggested grading rubric:

2 points for correctly computing the ratio
2 points for correctly computing its limit
1 point for the conclusion