Probability Exam 2017

May 26, 2017

Please begin the answer to each question on a new sheet of paper.
Write your name on each sheet and use only one side of the sheet to record your answer.

1. (Chose 3 out of 4 parts below. Worth 9 points each)

(a) Let $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ be two sequences of random variables with $\sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty$. Suppose X is a random variable and $X_n \xrightarrow{a.s.} X$. Prove that $Y_n \xrightarrow{a.s.} X$ also.

(b) Suppose $X_n \xrightarrow{pr} X$ and $Y_n \xrightarrow{pr} Y$. Prove that $X_n + Y_n \xrightarrow{pr} X + Y$.

(c) Suppose $X_n \xrightarrow{pr} X$ and $Y_n \xrightarrow{pr} Y$. Prove that $X_n Y_n \xrightarrow{pr} XY$.

(d) Suppose $X_n \xrightarrow{pr} X$ and $Y_n \xrightarrow{D} Y$. Either prove or give a counterexample to the statement that: $X_n + Y_n \xrightarrow{D} X + Y$.

2. Do one of the following parts (worth 10 points each):

(a) Consider a sequence of distribution functions $\{F_n\}_{n=1}^{\infty}$ and let $C > 0$. Prove that there exists $K > C$ such that both K and $-K$ are continuity points of all of the F_n's.

(b) Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of distribution functions. Helly's Compactness Theorem tells us that there exists a subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ and a sub-distribution function G, (that is, G is nondecreasing, right continuous, $\lim_{x\to\infty} G(x) \leq 1$ and $\lim_{x\to-\infty} G(x) \geq 0$) such that $F_{n_k}(x) \to G(x)$ as $k \to \infty$ for every x which is a continuity point of G.

Now suppose the sequence $\{F_n\}_{n=1}^{\infty}$ is tight. (that is, suppose that for every $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $F_n(C_\epsilon) - F_n(-C_\epsilon) > 1 - \epsilon$ for all n). Prove that G is a distribution function by showing that $\lim_{x\to\infty} G(x) = 1$ and $\lim_{x\to-\infty} G(x) = 0$. [Do not prove Helly's Compactness Theorem]

3. (You must do part (a) for 7 points, and then chose two parts for 9 points each.)

(a) State the Lindeberg Central Limit Theorem.

(b) Prove that the Lindeberg condition holds if the random variables are uniformly bounded (i.e. there exist M such that $|X_n| \leq M$ for all n) and $\sum_{k=1}^{n} Var(X_k) \to \infty$ as $n \to \infty$.

(c) Prove that the Lindeberg condition holds if the random variables are independent and identically distributed with finite, positive variance.

(d) Prove that the Lyapunov condition imply Lindeberg condition.

The Lyapunov Condition: for some $\beta > 2$ we have, as $n \to \infty$

$$\frac{1}{s_n^\beta} \sum_{i=1}^{n} E|X_i - \mu_i|^\beta \longrightarrow 0$$

where μ_i is the mean of X_i, and $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$ with σ_i^2 the variance of X_i.

4. (worth 10+7 points)

Let F be a nondecreasing right continuous function with $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$. Define $F^{-1}(x) = \inf\{t : F(t) \geq x\}$ for $x \in (0,1)$.

(a) Prove: for every s and every $x \in (0,1)$, $F^{-1}(x) \leq s$ if and only if $x \leq F(s)$.

(b) Let U be a random variable having a uniform distribution on $(0,1)$. Prove that the distribution of $F^{-1}(U)$ is F.

5. (worth 7+14 points)

(a) State the Submartingale Convergence Theorem

(b) Now prove it.