Nonparametric Bayes Estimator of Survival Function for Right-Censoring and Left-Truncation Data

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SUMMARY

The nonparametric Bayes estimator of survival function with Dirichlet process prior under squared error loss for right censored data was considered by Susarla and Van Ryzin (1976). Recently, Zhou (2004) investigated the nonparametric Bayes estimator for doubly/interval censored data with the same prior and loss function. In this paper, we obtain the Bayes estimator for right-censoring and left-truncation data with Dirichlet process prior under squared error loss. Explicit formula for the estimator is obtained and a software package for R is also provided.

AMS 2000 Subject Classification: Primary 60E15; secondary 60G30.

Key Words and Phrases: Dirichlet process prior, Truncated data, non-informative prior, NPMLE, square error loss.

1 Introduction

Let T_i, U_i, Y_i $i = 1, \dots n$ be mutually independent random variables, representing the lifetimes, the censoring times and the truncation times respectively. Due to censoring, we observe $X_i = \min(T_i, U_i)$ and $\delta_i = I[T_i \le U_i]$ instead of T_i and U_i. Truncation will further restrict our observations to those data that $X_i > Y_i$. Therefore, the observed data are (X_i, δ_i, Y_i) with $X_i > Y_i$.

Let F be the distribution function of the lifetimes T_i. In Bayesian analysis, $F(\cdot)$ is random. Here we assume that $F(\cdot)$ is distributed as a Dirichlet process with the parameter α, a measure on the real line. The definitions and properties of Dirichlet process prior have been previously reported by Ferguson (1973), Susarla and Van Ryzin (1976) and Ferguson, Phadia and Tiwari (1993) among others. To help understand our discussion later, we give a few basic definitions and properties.

Definition 1 The probability density function of a (n-variate) Dirichlet distribution is

$$\begin{aligned} f(x_1, x_2, ..., x_n | \alpha_1, \alpha_2, ..., \alpha_n) &= \frac{\Gamma(\alpha_1 + ... + \alpha_n)}{\Gamma(\alpha_1) ... \Gamma(\alpha_n)} \prod_{i=1}^{n} x_i^{\alpha_i - 1} \\ &= \frac{\prod_{i=1}^{n} x_i^{\alpha_i - 1}}{D(\alpha_1, \alpha_2, ..., \alpha_n)} \end{aligned}$$

where the domain of the density is $x_i \geq 0$, $\sum_{i=1}^{n} x_i = 1$; and $\alpha_i > 0$ are the parameters. It is an extension of the well known Beta distribution.

Definition 2 (Ferguson) Let α be a non-null finite measure on $(R^+, \mathcal{B})$, where $R^+ = (0, \infty)$ and $\mathcal{B}$ is the Borel σ-field on $(0, \infty)$. The random measure P is said to be a Dirichlet process on $(R^+, \mathcal{B})$ with the parameter α if for every $k = 1, 2, ...$, and for any measurable partition $B_1, ..., B_k$ of R^+, the distribution of $(P(B_1), ..., P(B_k))$ is a random vector with Dirichlet distribution with parameter vector $(\alpha(B_1), ..., \alpha(B_k))$.

Using a squared error loss, Susarla and Van Ryzin (1976) obtained the Bayes estimator for $F(\cdot)$ under the Dirichlet process prior for right-censored data. They also showed that when the weight parameter α on the Dirichlet process prior approaches zero, the nonparametric Bayes estimator reduces to the Kaplan-Meier (1958) estimator. Susarla and Van Ryzin (1978) studied the consistency of the Bayes estimator. Ghosh and Ramaoothi (1995) studied the posterior distribution. Huffer and Doss (1999) used Monte Carlo methods to compute the nonparametric Bayes estimator.

Zhou (2004) obtained the non-parametric Bayes estimator of a survival function when data are right, left, or interval censored. An explicit formula was presented for the Bayes estimator with the Dirichlet process prior. In contrast, there is no explicit formula known for the non-parametric maximum likelihood estimator (NPMLE) for this type of data.

In this part we obtain the Bayes estimator of $1 - F(\cdot)$ when the data are subject to right-censoring and left-truncation. Several illustrative examples are presented.

The non-parametric maximum likelihood estimator (NPMLE) for right-censoring and left-truncation data was considered by Tsai, Jewell and Wang (1987).

2 Bayes Estimator of Survival Function for Right-Censoring and Left-Truncation Data

To enhance readability, we first present the Bayes estimator for right-censoring and left-truncation data when all observations are right-censored. Let (X_1, Y_1, δ_1), (X_2, Y_2, δ_2), ..., (X_k, Y_k, δ_k) be independent and identically distributed random vectors. Since we assume all observations are right-censored, $\delta_i = 0$ for $i = 1, 2, ..., k$.

For a given sample, we order the combined sample x_i, y_i, $i = 1, 2, ..., k$ from the smallest to the largest denoted by $a_1 < a_2 < ... < a_{2k}$. The 2k points $a_1, a_2, ..., a_{2k}$ partition R^+ into intervals $(0, a_1)$, $[a_1, a_2)$, ..., $[a_{2k}, \infty)$. The random vector $P(0, a_1)$, $P[a_1, a_2)$, ..., $P[a_{2k}, \infty)$ has a Dirichlet distribution with parameter vector $(\alpha_1, \alpha_2, ..., \alpha_{2k}, \alpha_{2k+1})$ with $\alpha_1 = \alpha(0, a_1)$, $\alpha_2 = \alpha[a_1, a_2)$, ..., $\alpha_{2k+1} = \alpha[a_{2k}, \infty)$. The likelihood function can be written as

$$\prod_{i=1}^{k} \frac{P[x_i, \infty)}{P[y_i, \infty)} = \prod_{i=1}^{2k} (P[a_i, \infty))^{p_i}$$

where $p_i = 1$ if a_i is an observed lifetime while $p_i = -1$ if a_i is a truncation time.

The Bayes estimator of $1 - F(\cdot)$ with Dirichlet process prior under squared error loss is the conditional expectation of $1 - F$ given all the observations. By a similar argument as presented in Susarla and Van Ryzin (1976) Corollary 1, the conditional expectation, E_α, of $1 - F(u)$ given all right-censoring and left-truncation observations is

$$\hat{S}_D(u) = \frac{E_\alpha[P[u,\infty)\prod_{i=1}^{2k}(P[a_i,\infty))^{p_i}]}{E_\alpha[\prod_{i=1}^{2k}(P[a_i,\infty))^{p_i}]}. \tag{1}$$

The numerator and denominator of (1) are of the same type and can be calculated explicitly by the lemmas below.

Lemma 2.1 *With the notations above, we have*

$$\begin{aligned} & E_\alpha\left[\prod_{i=1}^{2k}(P[a_i,\infty))^{p_i}\right] \\ = & \prod_{i=1}^{2k}\left[\alpha[a_i,\infty) + \sum_{r=i+2,\ r\leq 2k+1} p_{r-1} + p_i I[p_i = -1]\right]^{p_i}. \end{aligned} \tag{2}$$

Here, α is a measure satisfying $\alpha(.) > 0$.

Proof: Since the random vector $P(0,a_1)$, $P[a_1,a_2)$, ..., $P[a_{2k},\infty)$ has Dirichlet distribution with parameter vector $(\alpha_1,\alpha_2,...,\alpha_{2k},\alpha_{2k+1})$ with $\alpha_1 = \alpha(0,a_1)$, $\alpha_2 = \alpha[a_1,a_2)$, ..., $\alpha_{2k+1} = \alpha[a_{2k},\infty)$, according to Lemma 2(a) of Susarla and Van Ryzin (1976) we have

$$E_\alpha\left[\prod_{i=1}^{2k}(P[a_i,\infty))^{p_i}\right] = \frac{1}{D(\alpha_1,...,\alpha_{2k+1})}\prod_{i=1}^{2k} B\left(\alpha_i, \sum_{r=i+1}^{2k+1}(\alpha_r + p_{r-1})\right).$$

We notice that for any i, $\sum_{r=i+1}^{2k+1} p_{r-1} \geq 0$ due to features of the left truncation data. Therefore $\alpha(.)$ should be larger than 0 to make Beta functions well defined. Moreover, we recall that

$$\begin{aligned} D(\alpha_1,...,\alpha_{2k+1}) &= \frac{\prod_{i=1}^{2k+1}\Gamma(\alpha_i)}{\Gamma(\alpha(R^+))} \\ B(\gamma,\eta) &= \frac{\Gamma(\gamma)\Gamma(\eta)}{\Gamma(\gamma+\eta)}, \end{aligned}$$

and notice that

$$\sum_{r=i}^{2k+1}\alpha_r = \alpha[a_i,\infty).$$

After algebraic calculation, we can simplify

$$\frac{1}{D(\alpha_1,...,\alpha_{2k+1})}\prod_{i=1}^{2k} B\left(\alpha_i, \sum_{r=i+1}^{2k+1}(\alpha_r + p_{r-1})\right)$$

into

$$\prod_{i=1}^{2k}\left[\alpha[a_i,\infty)+\sum_{r=i+2,\ r\leq 2k+1}p_{r-1}+p_iI[p_i=-1]\right]^{p_i}.$$

$\diamondsuit$

Further, we can compute the numerator of (1) by adding an additional term $P[u,\infty)$ to the likelihood function.

Lemma 2.2 *Similarly to Lemma* 2.1*, we have*

$$\begin{aligned}
&E_\alpha\left[P[u,\infty)\prod_{i=1}^{2k}(P[a_i,\infty))^{p_i}\right]\\
=\ &E_\alpha\left[\prod_{i=1}^{2k+1}(P[a_i{}',\infty))^{p_i{}'}\right]\\
=\ &\prod_{i=1}^{2k+1}\left[\alpha[a_i{}',\infty)+\sum_{r=i+2,\ r\leq 2k+2}p'_{r-1}+p'_iI[p'_i=-1]\right]^{p'_i}\frac{1}{\alpha[0,\infty)}.
\end{aligned} \tag{3}$$

Here, we order x_i, y_i, u, $i=1,2,...,2k,2k+1$ from the smallest to the largest denoted by $a'_1,a'_2,...,a'_{2k},a'_{2k+1}$. And $p'_i=1$ if a'_i is an observed lifetime or u while $p'_i=-1$ if a'_i is a truncation time. We notice that when $a_i<u$, $a_i=a'_i$ and $p_i=p'_i$.

PROOF: The proof is very similar to that of Lemma 2.1, except that when we add $P[u,\infty)$ to the product the sum of p'_is is 1 instead of 0. This explains why there is one more term $\frac{1}{\alpha[0,\infty)}$ on the right hand side of the equation (3).
$\diamondsuit$

Next, we derive the Bayes estimator for right-censoring and left-truncation data when there are k right-censored observations and m uncensored observations. First, we notice that the likelihood for an uncensored and left-truncated data point is

$$\frac{P(\{x_j\})}{P[y_j,\infty)}.$$

Following our earlier discussion on the case without uncensored observations, the Bayes estimator of $1-F(.)$ with Dirichlet Process prior under squared error loss for right-censoring and left-truncation data is the ratio of two expectations, i.e.,

$$\hat{S}_D(u)=\frac{E_\alpha\left[P[u,\infty)\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\prod_{j=1}^{m}\frac{P(\{x_j\})}{P[y_j,\infty)}\right]}{E_\alpha\left[\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\prod_{j=1}^{m}\frac{P(\{x_j\})}{P[y_j,\infty)}\right]}. \tag{4}$$

The following lemma gives the explicit formula for the denominator.

Lemma 2.3 *With notations introduced before, we have*

$$
\begin{aligned}
& E_\alpha\left[\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\prod_{j=1}^{m}\frac{P(\{x_j\})}{P[y_j,\infty)}\right] \\
= & \prod_{l=1,\ l\in L_1}^{2(k+m)}\left[\alpha[a_l,\infty)+\sum_{r=l+2,\ r\le 2(k+m)+1}^{2(k+m)+1} p_{r-1}+p_l I[p_l=-1]\right]^{p_l}\cdot \\
& \prod_{l=1,\ l\in L_2}^{2(k+m)}\alpha(\{a_l\}), \qquad (5)
\end{aligned}
$$

where L_1 denotes the set of all ordered time points excluding uncensored lifetime points while L_2 denotes the uncensored lifetime points.

Proof: Part of the calculations in the proof are omitted for simplification.

If $m=0$, i.e., no uncensored observations, it is the case in Lemma 2.1.

If $m=1$, we notice that

$$P(\{x_j\}) = P[x_j,\infty) - \lim_{\varepsilon\longrightarrow 0} P[x_j+\varepsilon,\infty).$$

As a result, we have

$$
\begin{aligned}
& E_\alpha\left[\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\cdot\frac{P(\{x_j\})}{P[y_j,\infty)}\right] \\
= & E_\alpha\left[\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\cdot\frac{P[x_j,\infty)-\lim_{\varepsilon\longrightarrow 0}P[x_j+\varepsilon,\infty)}{P[y_j,\infty)}\right] \\
= & \underbrace{E_\alpha\left[\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\cdot\frac{P[x_j,\infty)}{P[y_j,\infty)}\right]}_{(a)} - \underbrace{E_\alpha\left[\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\cdot\frac{\lim_{\varepsilon\longrightarrow 0}P[x_j+\varepsilon,\infty)}{P[y_j,\infty)}\right]}_{(b)}.
\end{aligned}
$$

For (a), x_i, y_i $(i=1,2,...,k)$ and x_j, y_j partition R^+ into $(0,a_1)$, $[a_1,a_2)$, ..., $[a_j,a_{j+1})$, ..., $[a_{2k+2},\infty)$. For (b), the partition of R^+ is the same as (a) except $[a_j,a_{j+1})$ is replaced by $[a_j+\varepsilon,a_{j+1})$. Also, we observe that the $p_i's$ remain same for (a) and (b). Then, by applying Lemma 2.1 we can calculate that $(a)-(b)$ =right hand side of equation (5).

Suppose equation (5) is true for $m=m'$, then for $m=m'+1$, we can express the added uncensored observation as the difference of two censored observations. Consequently, similar to the way we have proved for the case of $m=1$, we can easily show that this is true for $m=m'+1$.

$\diamondsuit$.

Lemma 2.4 *In the similar way, we can calculate the numerator of the Bayes estimator:*

$$E_\alpha\left[P[u,\infty)\prod_{i=1}^{k}\frac{P[x_i,\infty)}{P[y_i,\infty)}\prod_{j=1}^{m}\frac{P(\{x_j\})}{P[y_j,\infty)}\right]$$
$$=\prod_{l=1,\ l\in L_1'}^{2(k+m)+1}\left[\alpha[a_l',\infty)+\sum_{r=l+2,\ r\leq 2(k+m)+2}^{2(k+m)+2}p_{r-1}'+p_l'I[p_l'=-1]\right]^{p_l'}\cdot$$
$$\prod_{l=1,\ l\in L_2'}^{2(k+m)+1}\alpha(\{a_l'\})\cdot\frac{1}{\alpha[0,\infty)}.$$

where $L_1^{'}$ denotes the set of all ordered time points and u excluding uncensored lifetime points while $L_2^{'}$ denotes the uncensored lifetime points.

Finally, we are ready to state the following theorem.

Theorem 2.1 *With the notations defined above, the non-parametric Bayes estimator of survival function $S(u)=1-F(u)$ with Dirichlet prior under squared error loss for right-censoring and left-truncation data is*

$$\hat{S}_D(u)$$
$$=\frac{\prod_{l=1,\ l\in L_1',a_l'\leq u}^{2n+1}\left[\alpha[a_l',\infty)+\sum_{r=l+2,\ r\leq 2n+2}^{2n+2}p_{r-1}'+p_l'I[p_l'=-1]\right]^{p_l'}}{\prod_{l=1,\ l\in L_1,a_l\leq u}^{2n}\left[\alpha[a_l,\infty)+\sum_{r=l+2,\ r\leq 2n+1}^{2n+1}p_{r-1}+p_lI[p_l=-1]\right]^{p_l}\alpha[0,\infty)},\tag{6}$$

Here, n is the total number of observations in the sample and α is a measure satisfying $\alpha(.)>0$ as explained in the proof of Lemma 2.1.

Proof: According to the lemmas we have developed in this chapter, the Bayes estimator can be written as

$$\hat{S}_D(u)=\frac{(A)}{(B)}$$

where

$$(A)=\prod_{l=1,\ l\in L_1'}^{2n+1}\left[\alpha[a_l',\infty)+\sum_{r=l+2,\ r\leq 2n+2}^{2n+2}p_{r-1}'+p_l'I[p_l'=-1]\right]^{p_l'}\cdot$$
$$\prod_{l=1,\ l\in L_2'}^{2n+1}\alpha(\{a_l'\})\cdot\frac{1}{\alpha[0,\infty)};$$
$$(B)=\prod_{l=1,\ l\in L_1}^{2n}\left[\alpha[a_l,\infty)+\sum_{r=l+2,\ r\leq 2n+1}^{2n+1}p_{r-1}+p_lI[p_l=-1]\right]^{p_l}\cdot$$
$$\prod_{l=1,\ l\in L_2}^{2n}\alpha(\{a_l\}).$$

Note that there are some common terms in the numerator and denominator. After cancellation, (6) can be obtained.

$\diamondsuit$.

Remark 1: *In case of ties, the p_i would be any integer number. We can still use a similar technique to develop Bayes estimator.*

Remark 2: *Following the similar technique, we can also obtain the Bayes estimator for left-truncation and interval censored data.*

3 Example

We have wrote computational codes for the computing the Bayes estimator. These codes are packaged as an R software package. Available for downloading at

`http://ms.uky.edu/`

You need first to install R on your computer:

The examples below are performed with these codes.

Example 1 The following example contains one uncensored observation and two right censored observations. Suppose the three observations with the format (X_i, Y_i, δ_i) are $(9, 0.2, 1)$, $(13, 4, 0)$, $(15, 10, 0)$. Then we order all time points (lifetimes and truncation times).

Table 1: Ordered Data Points for Example 1

Time Points	y_1	y_2	x_1	y_3	x_2	x_3
Time	0.2	4	9	10	13	15
p_i	-1	-1	1	-1	1	1

For $u = 14$, after simplification based on equation (6), we obtain

$$\hat{S}_D(u) = \frac{\frac{(\alpha[14,\infty)+1)(\alpha[13,\infty)+2)}{(\alpha[10,\infty)+2)(\alpha[4,\infty)+2)(\alpha[0.2,\infty)+1)}}{\frac{(\alpha[13,\infty)+1)\alpha[0,\infty)}{(\alpha[10,\infty)+1)(\alpha[4,\infty)+1)(\alpha[0.2,\infty)+0)}}.$$

Let

$$\alpha[\omega, \infty) = B \exp(-\theta\omega). \tag{7}$$

The following table illustrates the values of $\hat{S}_D(14)$ for $\theta = 0.12$ and different values of B.

Moreover, the product limit estimator is

$$\hat{S}_{PL}(u) = \prod_{x_i \le u} \left(1 - \frac{D_i}{R_i}\right) = (1 - \frac{1}{2}) = 0.5$$

Table 2: Nonparametric Bayes Estimator of $\hat{S}_D(14)$ for Example 1

B	θ	$\hat{S}_D(14)$
8	0.12	0.2509
1	0.12	0.3741
0.1	0.12	0.4687
0.001	0.12	0.4879

We can observe that when $\alpha(.)$ approaches 0, the Bayes estimator is very close to the product limit estimator developed in Part I.

The figure below presents the nonparametric Bayes estimator of survival function for $B = 8$ and $\theta = 0.12$.

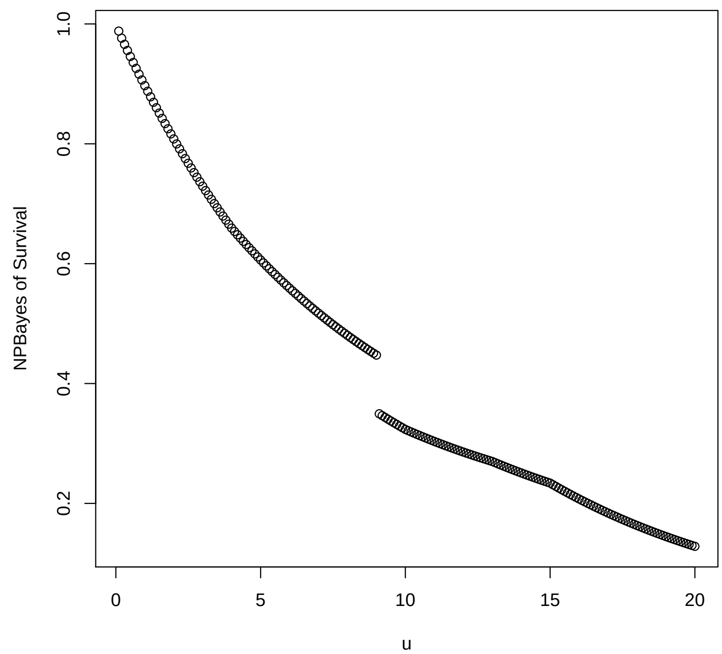

Figure 1: Nonparametric Bayes Estimator of Survival Function for Example 1

Example 2 Presented below is an example with two uncensored observations and four right censored observations. Suppose the six observations with the format of (X_i, Y_i, δ_i) are $(0.6, 0.1, 1)$, $(1.5, 0.3, 1)$, $(2.9, 0.5, 0)$, $(3.1, 0.9, 0)$, $(3.7, 3.2, 0)$ and $(4.3, 4.2, 0)$. Then we order all time points (lifetimes and truncation times).

Table 3: Ordered Data Points for Example 2

Time Points	y_1	y_2	y_3	x_1	y_4	x_2	x_3	y_5	x_4	x_5	y_6	x_6
Time	0.1	0.3	0.5	0.6	0.9	1.5	2.9	3.1	3.2	3.7	4.2	4.3
p_i	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1

For $u = 3.9$, after simplification based on equation (6), we have

$$\hat{S}_D(u) = \frac{\frac{(\alpha[3.9,\infty)+1)(\alpha[3.7,\infty)+2)(\alpha[3.2,\infty)+3)(\alpha[2.9,\infty)+3)}{(\alpha[3.1,\infty)+2)(\alpha[0.9,\infty)+3)(\alpha[0.5,\infty)+3)(\alpha[0.3,\infty)+2)(\alpha[0.1,\infty)+1)}}{\frac{(\alpha[3.7,\infty)+1)(\alpha[3.2,\infty)+2)(\alpha[2.9,\infty)+2)(\alpha[0,\infty))}{(\alpha[3.1,\infty)+1)(\alpha[0.9,\infty)+2)(\alpha[0.5,\infty)+2)(\alpha[0.3,\infty)+1)(\alpha[0.1,\infty)+0)}}.$$

Again, let

$$\alpha[\omega, \infty) = B \exp(-\theta\omega). \tag{8}$$

The $\hat{S}_D(3.9)$ for $\theta = 0.12$ and different values of B are presented in the table below.

Table 4: Nonparametric Bayes Estimator of $\hat{S}_D(3.9)$ for Example 2

B	θ	$\hat{S}_D(3.9)$
8	0.12	0.5585
1	0.12	0.4612
0.1	0.12	0.4277

Moreover, the product limit estimator is

$$\hat{S}_{PL}(u) = \prod_{x_i \leq u} \left(1 - \frac{D_i}{R_i}\right) = (1 - \frac{1}{3})(1 - \frac{1}{3}) = 0.44.$$

We can see that when $\alpha(.)$ approaches 0, the Bayes estimator is very close to product limit estimator developed in Part I.

The nonparametric Bayes estimator of survival function is plotted below for $B = 8$ and $\theta = 0.12$.

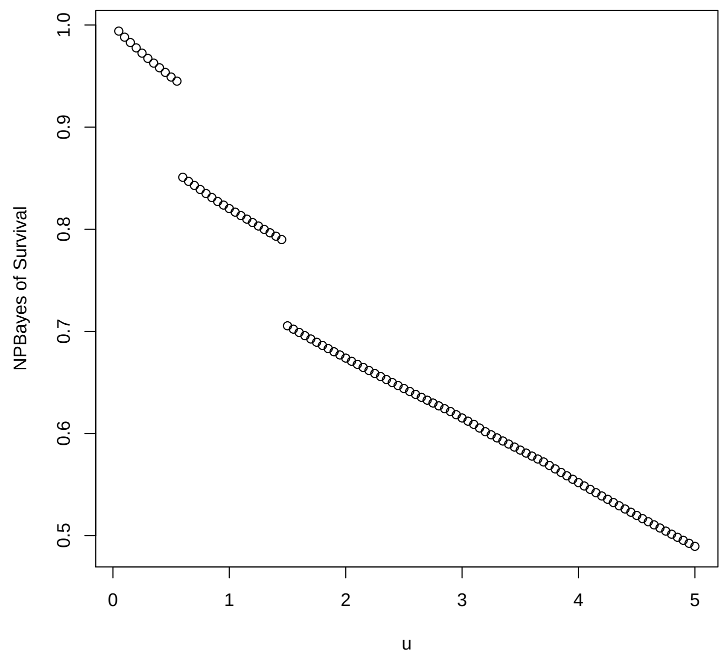

Figure 2: Nonparametric Bayes Estimator of Survival Function for Example 2

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