1.

prior $$\pi(\lambda) = \theta e^{-\theta\lambda}, \quad \text{for } \lambda > 0.$$

joint density
$$f(x|\lambda)\cdot\pi(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \cdot \theta e^{-\theta\lambda}.$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \cdot \theta e^{-\theta\lambda}$$

$$= \lambda^n e^{-\lambda[\sum x_i + \theta]} \cdot \theta .$$

therefore the posterior (marginal for λ) is

$$\pi(\lambda|x_1 \cdots x_n) \sim c \cdot \lambda^n e^{-\lambda[\sum x_i + \theta]} ;$$

where c is a constant. x_i also consider as given.

you recognize it is a Gamma density for λ.

$$\text{Gamma}\left(\alpha = n+1,\ \beta = \frac{1}{\sum x_i + \theta}\right) \text{ with density } \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} ; \quad \lambda > 0$$

ⓐ posterior is $\text{Gamma}\left(n+1, \frac{1}{\sum x_i + \theta}\right)$

ⓑ mean of posterior, $\Rightarrow$ mean of Gamma $= \alpha \cdot \beta = (n+1) \times \frac{1}{\sum x_i + \theta} = \frac{n+1}{\sum x_i + \theta}$ ✱

2. Similar to one of our homeworks. [$\sum X_i$ is suff, and Complete statistics]

(a) joint p.m.f. is

$$\prod_{i=1}^{n} \frac{(c_i\theta)^{x_i} e^{-c_i\theta}}{x_i!} = \prod \frac{1}{x_i!} \prod c_i^{x_i} \cdot \theta^{\sum x_i} \cdot e^{-\theta \sum c_i};$$

$$\log Lik(\theta) = Const. + \sum x_i \log\theta + (-\theta \sum c_i) = Const + \log\theta \sum x_i - \theta \sum c_i$$

$$\frac{\partial}{\partial\theta}(\ \ ) = \frac{\sum x_i}{\theta} - \sum c_i;$$

$$\frac{\partial^2}{\partial\theta^2}(\ \ ) = -\frac{\sum x_i}{\theta^2}, \quad I(\theta) = -\mathbb{E}\frac{\partial^2}{\partial\theta}[\log Lik] = \mathbb{E}\frac{\sum X_i}{\theta^2}$$

since $\mathbb{E}X_i = c_i\theta$, we have

$$I(\theta) = \frac{\sum c_i\theta}{\theta^2} = \frac{\sum c_i}{\theta} \quad ※.$$

(b) MLE.

$$\frac{\partial}{\partial\theta}(\ \ ) = 0 \Rightarrow \frac{\sum x_i}{\theta} - \sum c_i = 0 \Rightarrow \theta = \frac{\sum x_i}{\sum c_i}$$

and the second derivative $-\frac{\sum x_i}{\theta^2}$ is always negative, so this is a maximum.

$$\hat{\theta}_{MLE} = \frac{\sum X_i}{\sum c_i}.$$

(c) We can easily check that the MLE is unbiased for est. θ.

and its var $= Var\left(\frac{\sum X_i}{\sum c_i}\right) = \frac{1}{[\sum c_i]^2} Var(\sum X_i) = \frac{\sum c_i\theta}{[\sum c_i]^2} = \frac{\theta}{\sum c_i} \left(= \frac{1}{I(\theta)}\right)$

We notice this is = the CR lower bound. So it is best unbiased estimator. OR we could try to use Lehmann-Scheffee Theorem.

3.

(a) Just put down the log Lik function and take one derivative of the log Lik wrt θ. Setting the derivative to zero gives a quadratic eq. [as the solution suggest]

$$-\left\{\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} - \frac{\sum x_i^2}{\theta^3}\right\} = 0 \quad ;$$

since $\theta > 0$ we multiply $-\theta^3$ through out.

$$n\theta^2 + \sum x_i \theta - \sum x_i^2 = 0$$

The quadratic eq has two roots, but clearly one root is negative so not in the parameter space. We can only take the positive root

$$\frac{-\sum x_i + \sqrt{(\sum x_i)^2 + 4n\sum x_i^2}}{2n}.$$

notice the $\sqrt{\ }$ is larger than $|\sum x_i|$. therefore the root is positive.

We also need to check it is a max.

one way to check is to show the derivative (1st) is positive to the left of the root, and negative to the right of th root.

notice the 1st derivative has the same sign as

$$-\{n\theta^2 + \sum x_i \theta - \sum x_i^2\}$$

for θ small ($\to 0$) this is $\approx \sum x_i^2$ positive.

also same sign as $-\left\{n + \frac{\sum x_i}{\theta} - \frac{\sum x_i^2}{\theta^2}\right\}$

for θ large. ($\to \infty$) this is negative $\approx -n\theta$.

So, the positive root is max of Log Lik and thus the MLE.

B

(b) Notice

$$\hat{\theta}_{MLE} = \frac{1}{2}\sqrt{\frac{(\sum X_i)^2 + 4n\sum X_i^2}{n^2}} - \frac{\sum X_i}{2n}$$

$$= \frac{1}{2}\sqrt{(\bar{X})^2 + 4\frac{\sum X_i^2}{n}} - \frac{1}{2}\bar{X}$$

We recall from WLLN

$$\bar{X} \xrightarrow{P} \mathbb{E}X_1 = \theta$$

$$\frac{1}{n}\sum X_i^2 \xrightarrow{P} \mathbb{E}X_1^2 = (\text{mean})^2 + \text{Var} = \theta^2 + \theta^2 = 2\theta^2$$

and by Cont. mapping th.

$$(\bar{X})^2 \xrightarrow{P} \theta^2$$

therefore

$$\hat{\theta}_{MLE} = \frac{1}{2}\sqrt{(\bar{X})^2 + 4\frac{\sum X_i^2}{n}} - \frac{1}{2}\bar{X}$$

$$\xrightarrow{P} \frac{1}{2}\sqrt{\theta^2 + 4(2\theta^2)} - \frac{1}{2}\theta \quad \text{again by Cont. mapping}$$

$$= \frac{1}{2}\sqrt{9\theta^2} - \frac{1}{2}\theta = \frac{3\theta}{2} - \frac{1}{2}\theta, \quad (\text{since } \theta > 0)$$

$$= \theta$$

※

[We do not need Normality. Just $\mathbb{E}X = \theta$, $\text{Var}X = \theta^2$]

4 The statement is false.

One possible counter example [there are many]

is $X_1 \cdots X_n \overset{iid}{\sim} \exp(\theta)$ and take $\eta = \frac{1}{\theta}$.

[~~since~~ you supply the details]

An after thought: ① how is the 2 Fisher information related?

②