#3.

(a) indep. increment $\Rightarrow [N(t)-N(s)]$ indep $N(s)$

$$\Rightarrow \quad Cov(N(t)-N(s),\, N(s)) = 0$$

$$\Rightarrow \quad Cov(N(t),\, N(s)) - Cov(N(s),\, N(s)) = 0$$

Recall $Cov(x, x) = Var(X)$.

$$\Rightarrow \quad Cov(N(t),\, N(s)) = Var(N(s)) = \lambda s.$$

(b)

$$P(N(s)=0,\, N(t)=3) = P(N(s)=0,\, N(t)-N(s) = 3-0)$$

$$= P(N(s)=0) \cdot P(N(t)-N(s)=3) \qquad \text{[indep. increments]}$$

$$= e^{-\lambda s}\frac{(\lambda s)^0}{0!} \cdot P(N(t-s)=3) \qquad \text{[stationary]}$$

$$= e^{-\lambda s} \cdot e^{-(t-s)\lambda}\frac{[\lambda(t-s)]^3}{3!} = e^{-t\lambda}\frac{\lambda^3(t-s)^3}{6}.$$

(c) $\mathbb{E}[N(t) \mid N(s)=4] = \mathbb{E}[N(t)-N(s)+N(s) \mid N(s)=4]$

$$= \mathbb{E}[N(t)-N(s) \mid N(s)=4] + \mathbb{E}[N(s) \mid N(s)=4] \underset{\substack{\uparrow \\ \text{indep. Increment}}}{=} \mathbb{E}[N(t)-N(s)] + 4$$

$$= \lambda t - \lambda s + 4$$

(d) By a theorem, given $N(t)=4$, the 4 arrival times in $[0, t]$ behave like 4 indep uniformly distributed points. The # of points [arrival times] that land before s is a binomial r.v. with success prob $\left(\frac{s}{t}\right)$, $(n=4)$.

Therefore $\mathbb{E}(N(s) \mid N(t)=4) = \mathbb{E}\left(\text{binomial}\left(n=4,\, p=\frac{s}{t}\right)\right) = 4 \cdot \frac{s}{t}\left(1-\frac{s}{t}\right).$

1.

$$P(N(t+h)-N(t)=1) = e^{-\lambda h}\cdot\frac{(\lambda h)^1}{1!} \quad , \quad \text{By using } e^{-\varepsilon} = 1-\varepsilon+o(\varepsilon)$$

$$= [1-\lambda h + o(\lambda h)]\cdot \lambda h$$

$$= \lambda h - (\lambda h)^2 + o(\lambda^2 h^2) = \lambda h + o(h).$$ Since $\lambda > 0$ is fixed here.

$$P(N(t+h)-N(t)\geq 2) = 1 - P(N(t+h)-N(t)=1) - P(N(t+h)-N(t)=0)$$

$$= 1-\lambda h - o(h) - e^{-\lambda h} = 1-\lambda h - o(h) - 1 + \lambda h - o(\lambda h)$$

$$= o(\lambda h) = o(h)$$

4.

let $N(3t) = X(t)$.

(i) $X(0) = N(0) = 0$ obvious

(ii) $X(t+h) - X(t) = N(3(t+h)) - N(3t)$

$$= N(3t+3h) - N(3t)$$

this is indep of

$X(t) = N(3t)$ ✓ indep. increments

Since $N(3t+3h)-N(3t)$ indep $N(3t)$

(iii) stationary: the distribution of $X(t+h)-X(t)$ (should not depend on t).

$= N(3t+3h) - N(3t) \sim \text{Poisson}(\lambda\cdot 3h)$ ✓ yes!