Note 1, Sta624

1. Suppose we have a MC with finite states, some of them transient, some of them recurrent (or absorbing).

2. Without loss of generality, assume the transient states are in the front (upper left corner of the transition matrix), the recurrent states are in the back.

3 The transition probability matrix then will have a "block-wise upper triangle" shape. (once the MC goes to the recurrent states, it will not come back to the transient block).

4. Denote the upper-left block of the transition probability matrix, corresponding to the transient states, as P_t. (notice this is only part of a transition probability matrix, so the row sum of P_t may not be 1. In fact at least one row do not sum to 1, for otherwise this part will be recurrent)

5. Suppose the initial state of the MC is in some transient state, with probability π_0. (π_0 may be a distribution. Its length same as the number of transient states)

6. Then the expected number of visits to a transient state j (before this MC is absorbed to recurrent states) starting from the initial distribution π_0, is the jth element of the following vector

$$\pi_0 + \pi_0 P_t + \pi_0 P_t^2 + \cdots = \pi_0[I + P_t + P_t^2 + P_t^3 + \cdots]$$

note: If there is difficult seeing this: write the number of visits to j as $\sum_i \eta_{ij}$, and then taking the expectation, inside the summation, and then the expectation of η_{ij} is the probability.

7. Use matrix result to get (we can check the convergence since $P_t^n \to 0$ as $n \to \infty$).

$$I + P_t + P_t^2 + P_t^3 + \cdots = (I - P_t)^{-1}$$

8. Finally, the expectation we want is the jth element of

$$\pi_0(I - P_t)^{-1}$$

9. The expectation above is usually denoted by s_j or s_{ij} when the π_0 is all zero but a 1 at i^{th} place.

- Chapman-Kolmogorov equations.

- P_{ij}: the probability of going from state i to state j (in one step). Elements of Transition probability matrix.

- π_i: stationary distribution; Limiting distribution. $\pi P = \pi$.

- f_i or f_{ij}: start at state i, probability of ever going into state j. For recurrent state j, $f_j = f_{jj} = 1$. For transient state j, $f_{jj} < 1$.

- s_j or s_{ij}: start at state i, the expected number of visits to state j (in the whole life of the MC). For transient state j, $s_j < \infty$ and $s_{ii} = 1/(1 - f_{ii})$.

- $m_i = m_{ii}$: expected number of steps before MC returns to i, starting in state i. $\pi_i = 1/m_i$ for recurrent state i.

- One way to check if a state is recurrent/transient: if $\sum_n P_{ii}^n < \infty$ then $f_i < 1$. If $\sum_n P_{ii}^n = \infty$ then $f_i = 1$.

Brownian Motion etc.

Suppose $X(t)$ is a (standard) Brownian Motion process.

Define $Y(t) = \sigma X(t) + \mu t$, (a BM with drift μt and volitility σ). It is not hard to verify that $Y(t)$ also have independent increment property.

Easy to verify, $M_1(t) = Y(t) - \mu t = \sigma X(t)$ is a martingale, with mean zero. Just check (for $0 < s < t$; that $E[M_1(t)|M_1(u); 0 \le u \le s] = M_1(s)$).

For any stopping time T, we have $E[M_1(T)] = E[M_1(0)] = 0$.

For $0 < s < t$, we compute

$$E[e^{-2\frac{\mu}{\sigma}Y(t)}|e^{-2\frac{\mu}{\sigma}Y(u)}; 0 \le u \le s] = e^{-2\frac{\mu}{\sigma}Y(s)}$$

in other words, the process

$$M_2(t) = e^{-2\frac{\mu}{\sigma}Y(t)}$$

is also a Martingale process, which has mean $= E[M_2(0)] = 1$ always.

Define a first hitting time of boundary AB as

$$T_{ab} = \min\{t | Y(t) = A \text{ or } -B\}$$

Apply the Optional Sampling Theorem (or Optional Stopping Theorem), to either the martingale $M_1(t)$ or $M_2(t)$ and hitting time T_{ab} above. We have

$$0 = E[Y(T_{ab})] - \mu E T_{ab} \quad \text{or} \quad pA - (1-p)B = \mu E T_{ab}$$

and

$$1 = E M_2(T_{ab}) = p(e^{-2\frac{\mu}{\sigma}A}) + (1-p)(e^{2\frac{\mu}{\sigma}B})$$

where p is the probability that $Y(t)$ hit A before hit $-B$. Also $p = P(Y(T_{ab} = A)$.

Solving the above two equations, you get p and $E(T_{ab})$.

$$p = \frac{1 - e^{2\frac{\mu}{\sigma}B}}{e^{-2\frac{\mu}{\sigma}A} - e^{2\frac{\mu}{\sigma}B}} \quad \text{and} \quad E(T_{ab}) = \frac{1}{\mu}[pA - (1-p)B]$$

This is the result for $Y(t)$, a BM with a drift and volatility σ. If you want the similar result for a BM WITHOUT a drift, simply take the limit as $\mu \to 0$. After some calculation [using $e^x = 1 + x + x^2/2 + ...$] we have

$$p = \frac{B}{A+B} \quad \text{and} \quad E(T_{ab}) = \frac{AB}{\sigma} \ .$$