1 IEOR 6712: Notes on Brownian Motion I

We present an introduction to *Brownian motion*, an important continuous-time stochastic process that serves as a continuous-time analog to the simple symmetric random walk on the one hand, and shares fundamental properties with the Poisson counting process on the other hand.

Throughout, we use the following notation for the real numbers, the non-negative real numbers, the integers, and the non-negative integers respectively:

$$\mathbb{R} \stackrel{\text{def}}{=} (-\infty, \infty) \tag{1}$$
$$\mathbb{R}_+ \stackrel{\text{def}}{=} [0, \infty) \tag{2}$$
$$\mathbb{Z} \stackrel{\text{def}}{=} \{\cdots, -2, -1, 0, 1, 2, \cdots\} \tag{3}$$
$$\mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, \cdots\}. \tag{4}$$

1.1 Normal distribution

Of particular importance in our study is the normal distribution, $N(\mu, \sigma^2)$, with mean $-\infty < \mu < \infty$ and variance $0 < \sigma^2 < \infty$; the density and cdf are given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \ x \in \mathbb{R}, \tag{5}$$
$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy, \ x \in \mathbb{R}. \tag{6}$$

When $\mu = 0$ and $\sigma^2 = 1$ we obtain the *standard* (or *unit*) normal distribution, $N(0, 1)$, and the density and cdf reduce to

$$\theta(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \tag{7}$$
$$\Theta(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}} dy. \tag{8}$$

As we shall see over and over again in our study of Brownian motion, one of its nice features is that many computations involving it are based on evaluating $\Theta(x)$, and hence are computationally elementary.

If $Y \sim N(0, 1)$, then $X = \sigma Y + \mu$ has the $N(\mu, \sigma^2)$ distribution. Conversely, if $X \sim N(\mu, \sigma^2)$, then $Y = (X - \mu)/\sigma$ has the standard normal distribution.

It thus follows that that if $X \sim N(\mu, \sigma^2)$, then $F(x) = P(X \le x) = \Theta((x - \mu)/\sigma)$.

Letting $X \sim N(\mu, \sigma^2)$, the moment generating function of the normal distribution is given by

$$\begin{aligned} M(s) &= E(e^{sX}) \\ &= \int_{-\infty}^{\infty} e^{sx} f(x) dx \\ &= e^{s\mu + s^2\sigma^2/2}, \ -\infty < s < \infty. \end{aligned} \tag{9}$$

The normal distribution is also called the *Gaussian* distribution after the famous German mathematician and physicist Carl Friedrich Gauss (1777 - 1855).

1.1.1 Central limit theorem (CLT)

Theorem 1.1 *If $\{X_i : i \geq 1\}$ are iid with finite mean $E(X) = \mu$ and finite non-zero variance $\sigma^2 = Var(X)$, then*

$$Z_n \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{n}}\Big(\sum_{i=1}^{n} X_i - n\mu\Big) \Longrightarrow N(0,1),\ n \to \infty,\ \textit{in distribution};$$

$\lim_{n\to\infty} P(Z_n \leq x) = \Theta(x),\ x \in \mathbb{R}$.

If $\mu = 0$ and $\sigma^2 = 1$, then the CLT becomes

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \Longrightarrow N(0,1).$$

1.2 Construction of Brownian motion from the simple symmetric random walk

Recall the simple symmetric random walk, $R_0 = 0$,

$$R_n = \Delta_1 + \cdots + \Delta_n,\ n \geq 1,$$

where the Δ_i are iid with $P(\Delta = -1) = P(\Delta = 1) = 0.5$.

We view time n in minutes, and R_n as the position at time n of a particle, moving on $\mathbb{R}$, which every minute takes a step, of size 1, equally likley to be forwards or backwards. Because $E(\Delta) = 0$ and $Var(\Delta) = 1$, it follows that $E(R_n) = 0$ and $Var(R_n) = n,\ n \geq 0$.

Choosing a large integer $k > 1$, if we instead make the particle take a step every $1/k$ minutes and make the step size $1/\sqrt{k}$, then by time t the particle will have taken a large number, $n = tk$, of steps and its position will be

$$B_k(t) = \frac{1}{\sqrt{k}}\sum_{i=1}^{tk} \Delta_i. \tag{10}$$

(By convention if tk is not an integer then we replace it by the largest integer less than or equal to it; $[tk]$.) This leads to the particle taking many many iid steps, but each of small magnitude, in any given interval of time. We expect that as $k \to \infty$, these small steps become a continuum and the process $\{B_k(t) : t \geq 0\}$ should converge to a process $\{B(t) : t \geq 0\}$ with continuous sample paths. We call this process Brownian motion (BM) after the Scottish botanist Robert Brown.[1] Its properties will be derived next.

Notice that for fixed k, any increment

$$B_k(t) - B_k(s) = \frac{1}{\sqrt{k}}\sum_{i=sk}^{tk} \Delta_i,\ 0 \leq s < t,$$

[1]Brown himself noticed in 1827, while carrying out some experiments, the unusual "motion" of particles within pollen grains suspended in water, under his microscope. The physical cause of such motion (bombardment of the particles by water molecules undergoing thermal motion) was not formalized via kinetic theory until Einstein in 1905. The rigorous mathematical construction of a stochastic process as a model for such motion is due to the mathematician Norbert Weiner; that is why it is sometimes called a Weiner process.

has a distribution that only depends on the length, $t-s$, of the time interval $(s,t]$ because it only depends on the number, $k(t-s)$, of iid Δ_i making up its construction. Thus we deduce that the limiting process will possess *stationary increments*.

Notice further that given two non-overlapping time intervals, $(t_1,t_2]$ and $(t_3,t_4]$, the corresponding increments

$$B_k(t_4) - B_k(t_3) = \frac{1}{\sqrt{k}} \sum_{i=t_3k}^{t_4k} \Delta_i, \tag{11}$$

$$B_k(t_2) - B_k(t_1) = \frac{1}{\sqrt{k}} \sum_{i=t_1k}^{t_2k} \Delta_i, \tag{12}$$

are independent because they are constructed from different Δ_i. Thus we deduce that the limiting process will also possess *independent increments*.

Observing that $E(B_k(t)) = 0$ and $Var(B_k(t)) = [tk]/k \to t,\ k \to \infty$, we infer that the limiting process will satisfy $E(B(t)) = 0,\ Var(B(t)) = t$ just like the random walk $\{R_n\}$ does in discrete-time n $(E(R_n) = 0,\ Var(R_n) = n)$.

Finally, a direct application of the CLT yields (via setting $n = tk$)

$$B_k(t) = \sqrt{t}\Big(\frac{1}{\sqrt{kt}}\sum_{i=1}^{tk}\Delta_i\Big) \Longrightarrow N(0,t),\ k \to \infty,\ \textit{in distribution},$$

and we conclude that for each fixed $t > 0$, $B(t)$ has a normal distribution with mean 0 and variance t. Similarly, using the stationary and independent increments property, we conclude that $B(t) - B(s)$ has a normal distribution with mean 0 and variance $t - s$, and more generally:

> *the limiting BM process is a process with continuous sample paths that has both stationary and independent normally distributed (Gaussian) increments: If* $t_0 = 0 < t_1 < t_2 < \cdots < t_n$, *then the rvs.* $B(t_i) - B(t_{i-1}),\ i \in \{1, \dots n\}$, *are independent with* $B(t_i) - B(t_{i-1}) \sim N(0, t_i - t_{i-1})$.

If we define $X(t) = \sigma B(t) + \mu t$, then $X(t) \sim N(\mu t, \sigma^2 t)$, $\sigma \in \mathbb{R}_+,\ \mu \in \mathbb{R}$, and we obtain, by such scaling and translation, more generally, a process with stationary and independent increments in which $X(t) - X(s)$ has a normal distribution with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$.

When $\sigma^2 = 1$ and $\mu = 0$ (as in our construction) the process is called *standard Brownian motion*, and denoted by $\{B(t) : t \geq 0\}$. Otherwise, it is called Brownian motion with variance σ^2 and *drift* μ.

Definition 1.1 *A stochastic process* $\mathbf{B} = \{B(t) : t \geq 0\}$ *possessing (wp1) continuous sample paths is called standard Brownian motion if*

1. $B(0) = 0$.
2. $\mathbf{B}$ *has both stationary and independent increments.*
3. $B(t) - B(s)$ *has a normal distribution with mean* 0 *and variance* $t - s,\ 0 \leq s < t$.

For Brownian motion with variance σ^2 and drift μ, $X(t) = \sigma B(t) + \mu t$, the definition is the same except that *3* must be modified;

> $X(t) - X(s)$ *has a normal distribution with mean* $\mu(t-s)$ *and variance* $\sigma^2(t-s)$.

Remark 1.1 It can in fact be proved that Condition *3* above is redundant: a stochastic process with stationary and independent increments that possesses (wp1) continuous sample paths must be Brownian motion, that is, the increments must be normally distributed. This is analogous to the Poisson counting process which is the unique simple *counting process* that has both stationary and independent increments: the stationary and independent increments property forces the increments to be Poisson distributed. (*Simple* means that the arrival times of the underlying point process are strictly increasing; no batches.)

Donsker's theorem

Our construction of Brownian motion as a limit is in fact a rigorous one, but requires more advanced mathetical tools (beyond the scope of these lecture notes) in order to state it precisely and to prove it. Suffice to say, the stochastic process $\{B_k(t) : t \geq 0\}$ as defined by (10) converges in distribution (weak convergence in path (function) space), as $k \to \infty$, to Brownian motion $\{B(t) : t \geq 0\}$. This is known as *Donsker's theorem* or the *functional central limit theorem.* The point is that it is a generalization of the central limit theorem, because it involves an entire stochastic process (with all its multi-dimensional joint distributions, for example) as opposed to just a one-dimensional limit such as (for fixed $t > 0$) $B_k(t) \to N(0, t)$ in distribution.

1.3 BM as a Gaussian process

We observe that the vector $(B(t_1), \ldots, B(t_n))$ has a multivariate normal distribution because the event

$$\{B(t_1) = x_1, \ldots, B(t_n) = x_n\}$$

can be re-written in terms of independent increment events

$$\{B(t_1) = x_1, B(t_2) - B(t_1) = x_2 - x_1, \ldots, B(t_n) - B(t_{n-1}) = x_n - x_n\},$$

yielding the joint density of $(B(t_1), \ldots, B(t_n))$ as

$$f(x_1, \ldots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \cdots f_{t_n - t_{n-1}}(x_n),$$

where

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$

is the density for the $N(0, t)$ distribution.

The finite dimensional distributions of BM are thus multivariate normal, or Gaussian, and BM is an example of a *Gaussian process*, that is, a process with continuous sample paths in which the finite dimensional distributions are multivariate normal, that is, for any fixed choice of n time points $0 \leq t_1 < t_2 < \cdots < t_n,\ n \geq 1$, the joint distribution of the vector $(X(t_1), \ldots, X(t_n))$ is multivariate normal.

Since a multivariate normal distribution is completely determined by its mean and covariance parameters, we conclude that a Gaussian process is completely determined by its mean and covariance function $m(t) \stackrel{\text{def}}{=} E(X(t)),\ a(s, t) \stackrel{\text{def}}{=} cov(X(s), X(t)),\ 0 \leq s \leq t.$

For standard BM, $m(t) = 0$ and $a(s, t) = s$:

$$\begin{aligned}
cov(B(s), B(t)) &= cov(B(s), B(s) + B(t) - B(s)) \\
&= cov(B(s), B(s)) + cov(B(s), B(t) - B(s)) \\
&= var(B(s)) + 0 \textit{ (via independent increments)} \\
&= s.
\end{aligned}$$

Thus standard BM is the unique Gaussian process with $m(t) = 0$ and $a(s,t) = \min\{s,t\}$. Similarly, BM with variance σ^2 and drift μ, $X(t) = \sigma B(t) + \mu t$, is the unique Gaussian process with $m(t) = \mu t$ and $a(s,t) = \sigma^2 \min\{s,t\}$.

1.4 Levy Processes

BM shares something special with the Poisson counting process $\{N(t)\}$; both have stationary and independent increments. But they differ significantly in other ways: while BM has continuous sample paths, the sample paths of $N(t)$ have discontinuities (jumps) of size 1. A *Levy process* is a stochastic process with both stationary and independent increments. It turns out that BM is the unique Levy process with continuous sample paths, and the Poisson counting process is the unique Levy process which is a counting process with jumps of size 1. There exist many other Levy processes too, and the key to understanding them involves *infinitely divisible* distributions.

A distribution F is called *infinitely divisible* if for any $n \geq 1$ (no matter how large), F can be expressed as the n^{th}-fold convolution $G^{*n} = G * \cdots * G$ of some distribution G (that depends on n). In terms of random variables this means that if X has an infinitely divisible distribution, then for each $n \geq 1$, X has representation $X = Y_1 + \cdots + Y_n$ for some iid rvs. $\{Y_i : 1 \leq i \leq n\}$.

Both the Poisson and Gaussian distributions are infinitely divisible:

$$\begin{aligned} N(\mu, \sigma^2) &= N(\mu/n, \sigma^2/n) * \cdots * N(\mu/n, \sigma^2/n), \\ Poisson(\lambda) &= Poisson(\lambda/n) * \cdots * Poisson(\lambda/n). \end{aligned}$$

Notice how G is the same kind of distribution as the original but with scaled down parameters.

If $\{X(t) : t \geq 0\}$ is a Levy process, then it follows that $X(t)$ (for each $t > 0$) has an infinitely divisible distribution because $X(t) = Y_1 + \cdots + Y_n$, where $Y_i = X(it/n) - X((i-1)t/n),\ 1 \leq i \leq n$. Conversely, if F is an infinitely divisible distribution, then it can be proved that there exists a Levy process for which $X(1) \sim F$. An easy way of constructing infinitely divisible distributions is by taking the convolution of two known ones. For example, $Poisson(\lambda) * N(\mu, \sigma^2)$ is infinitely divisible. Another important Levy process is the *compound Poisson process*, in which starting with a Posson counting process $N(t)$ at rate λ, the jump sizes, instead of being of size 1, are allowed to be iid with a general distrubution. Letting $\{J_i\}$ be iid with distribution $H(x) = P(J \leq x),\ x \in \mathbb{R}$, and independent of $\{N(t)\}$,

$$X(t) \stackrel{\text{def}}{=} \sum_{i=1}^{N(t)} J_i$$

yields such a Levy process, and $E(X(t)) = \lambda E(J)$. Compound Poisson processes are very useful in the modeling of queues and insurance risk businesses; $X(t)$ can denote the total amount of work that arrived to the queue by time t, or the total claim damages incurred by the insurance business by time t. The underlying infinitely divisible distribution of any Levy process is given by the moment generating function $M_1(s) = E(e^{sX(1)})$, if it exists (otherwise one uses the characteristic function, $E(e^{isX(1)})$ where $i = \sqrt{-1}$). A Levy process then has the nice characterizing property that

$$M_t(s) \stackrel{\text{def}}{=} E(e^{sX(t)}) = (M_1(s))^t,\ t \geq 0.$$

Examples

1. *Poisson counting process at rate λ:*

Recalling that $E(e^{sX}) = e^{\alpha(e^s-1)}$ if X is Poisson with mean α; we have
$M_t(s) = E(e^{sN(t)})$
$= e^{\lambda t(e^s-1)}$
$= \left(e^{\lambda(e^s-1)}\right)^t$
$= (M_1(s))^t$.

2. *Compound Poisson process:*

   Letting $\tilde{H}(s) = E(e^{sJ})$ denote the moment generating function of the jump size distribution $H(x) = P(J \leq x)$, conditioning on $N(t) = n$ yields
   $M_t(s) = E\left(e^{s\left(\sum_{i=1}^{N(t)} J_i\right)}\right)$
   $= e^{\lambda t(\tilde{H}(s)-1)}$
   $= \left(e^{\lambda(\tilde{H}(s)-1)}\right)^t = (M_1(s))^t$.

3. *BM:*

   For $X(t) = \sigma B(t) + \mu t$,
   $M_t(s) = E(e^{sX(t)})$
   $= e^{\mu st + \frac{\sigma^2 s^2 t}{2}}$
   $= \left(e^{\mu s + \frac{\sigma^2 s^2}{2}}\right)^t$
   $= (M_1(s))^t$.
   For standard BM this reduces to

$$M_t(s) = e^{\frac{s^2 t}{2}}.$$

Another nice example of an infinitely divisible distribution is the gamma distribution.

1.5 BM as a Markov Processes

If B is standard BM, then the independent increments property implies that $B(s+t) = B(s) + (B(s+t) - B(s))$, in which $B(s)$ and $(B(s+t) - B(s))$ are independent. The independent increments property implies further that $(B(s+t) - B(s))$ is also independent of the past before time s, $\{B(u) : 0 \leq u < s\}$.

Thus the future, $B(s+t)$, given the present state, $B(s)$, only depends on a rv, $B(s+t) - B(s)$, that is independent of the past. Thus we conclude that BM satisfies the Markov property. Since the increments are also stationary, we conclude that BM is a time-homogenous Markov process.

Letting $p(x,t,y)$ denote the probability density function for $B(s+t)$ given $B(s) = x$, we see, from $B(s+t) = x + (B(s+t) - B(s))$, that $p(x,t,y)$ is the density for $x + B(s+t) - B(s)$. But $x + B(s+t) - B(s) = y$ if and only if $(B(s+t) - B(s)) = y - x$, yielding

$$p(x,t,y) = f_t(y-x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(y-x)^2}{2t}}. \tag{13}$$

More generally, $X(t) = \sigma B(t) + \mu t$ is a Markov process with

$$p(x,t,y) = \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{-((y-x-\mu t)^2}{2\sigma^2 t}}. \tag{14}$$

1.6 BM as a martingale

Standard BM is a *martingale*:

$$E(B(t+s)|B(u): 0 \le u \le s) = B(s),\ t \ge 0,\ s \ge 0,$$

which asserts that the conditional expectation of BM at any time in the future after time s equals the original value at time s. This of course follows from the independent increments property and using $B(s+t) = B(s) + (B(s+t) - B(s))$:

$$\begin{aligned} E(B(t+s)|B(u): 0 \le u \le s) &= E(B(t+s)|B(s)),\ \textit{via the Markov property} \\ &= E(B(s) + (B(s+t) - B(s))|B(s)) \\ &= B(s) + E(B(s+t) - B(s)|B(s)) \\ &= B(s) + E(B(s+t) - B(s)),\ \textit{via independent increments} \\ &= B(s) + 0 \\ &= B(s). \end{aligned}$$

A martingale captures the notion of a fair game, in that regardless of your current and past fortunes, your expected fortune at any time in the future is the same as your current fortune: on average, you neither win nor lose any money.

The simple symetric random walk is a martingale (and a Markov chain) in discrete time;

$$E(R_{n+k}|R_n, \ldots, R_0) = E(R_{n+k}|R_n) = R_n,\ k \ge 0,\ n \ge 0,$$

because

$$R_{n+k} = R_n + \sum_{i=1}^{k} \Delta_{n+i},$$

and $\sum_{i=1}^{k} \Delta_{n+i}$ is independent of R_n (and the past before time n) and has mean 0.

1.7 Hitting times for standard BM

Consider

$$\tau = \min\{t \ge 0 : B(t) \notin (a, -b)\},$$

the first time that BM hits either a or $-b$.[2]

Recall from the gambler's ruin problem that for the simple *symmetric* random walk $\{R_n\}$, $p_a = \frac{b}{a+b}$, where $a > 0$, $b > 0$ (integers), and p_a denotes the probability that the random walk starting at $R_0 = 0$ first hits a before hitting $-b$;

$$\tau = \min\{n \ge 0 : R_n \in \{a, -b\} | R_0 = 0\},$$

and $p_a = P(R_\tau = a)$. Thus p_a denotes the probability that R_n goes up a steps before going down b steps.

For the process $\{B_k(t) : t \ge 0\}$ to hit a before $-b$ would require the random walk

$$R_n^{(k)} = \frac{1}{\sqrt{k}} \sum_{i=1}^{n} \Delta_i,\ n \ge 0$$

[2]The contiuity of sample paths implies that $B(t)$ either hits a or hits $-b$ (e.g, there is no overshoot).

to hit a before $-b$. This would require (approximately) that $R_n^{(k)}$ goes up $a\sqrt{k}$ steps before going down $b\sqrt{k}$ steps; yielding the same answer

$$p_a = \frac{b\sqrt{k}}{a\sqrt{k} + b\sqrt{k}} = \frac{b}{a+b}.$$

We thus deduce (via letting $k \to \infty$) that the same holds for standard BM (where a and b need not be integers now). Letting $p_a = P(B(\tau) = a)$, where
$\tau = \min\{t \geq 0 : B(t) \in \{a, -b\} \mid B(0) = 0\}$, we state this as

Proposition 1.1 *For standard BM, the probability that a is first hit before hitting $-b$ is given by*

$$p_a = \frac{b}{a+b},\ a > 0, b > 0.$$

Proof : A rigorous proof is provided by the optional stopping theorem for martingales: $E(B(\tau)) = E(B(0)) = 0$ implies that $aP_a - b(1 - p_a) = 0$; solving yields $p_a = \frac{b}{a+b}$.

UI of the stopped MG $B(t \wedge \tau)$ follows since it is bounded between $-b$ and a, thus justifying the use of the optional stopping theorem. ■

Proposition 1.2 *For standard BM, if $\tau = \min\{t \geq 0 : B(t) \in \{a, -b\} | B(0) = 0\}$, then*

$$E(\tau) = ab.$$

Note that if a variance term is introduced, $\sigma B(t)$, $\sigma > 0$, then $\sigma B(t) \in \{a, -b\}$ if and only if $B(t) \in \{a/\sigma, -b/\sigma\}$ yielding $E(\tau) = \frac{ab}{\sigma^2}$.
Proof : Here we use optional stopping on the the MG $B^2(t) - t$ yielding $E(B^2(\tau)) = E(\tau)$, or $E(\tau) = a^2 p_a + b^2(1 - p_a) = ab$, where we use the fact that $p_a = \frac{b}{a+b}$. UI holds via: The stopped process $\overline{X}(t) = B^2(\tau \wedge t) - \tau \wedge t$ is a MG, so $E(\overline{X}(t)) = 0$ for all t; $E(\tau \wedge t) = E(B^2(\tau \wedge t)) \leq (a+b)^2 < \infty$. But $\tau \wedge t$ is monotone increasing to τ and non-negative, so by the monotone convergence theorem we conclude that $E(\tau) \leq (a+b)^2 < \infty$; $\tau \in L^1$. Thus $|\overline{X}(t)| \leq B^2(\tau \wedge t) + \tau \wedge t \leq (a+b)^2 + \tau$, and since we have just shown that $\tau \in L^1$, we conclude that $\{\overline{X}(t)\}$ is UI.

■

Now let $T_x = \min\{t \geq 0 : B(t) = x \mid B(0) = 0\}$, the hitting time to $x > 0$. From our study of the simple symmetric random walk, we expect $P(T_x < \infty) = 1$, but $E(T_x) = \infty$: although any level x will be hit with certainty, the mean length of time required is infinite. We will prove this directly and derive the cdf $P(T_x \leq t),\ t \geq 0$ along the way.

The key to our analysis is based on a simple observation involving the symmetry of standard BM: If $T_x < t$, then $B(s) = x$ for some $s < t$. Thus the value of $B(t)$ is determined by where the BM went in the remaining $t - s$ units of time after hitting x. But BM, having stationary and independent Gaussian increments, will continue having them after hitting x. So by symmetry (about x), the path of BM during the time interval $(s, t]$ with $B(s) = x$ is just as likely to lead to $B(t) > x$ as to $B(t) < x$. So the events $\{B(t) > x|\ T_x \leq t\}$ and $\{B(t) < x|\ T_x \leq t\}$ are equally likely; both have probability 1/2. ($P(B(t) = x) = 0$ since $B(t)$ has a continuous distribution.) To be precise, if $T_x = s < t$, then $B(t) = x + B(t) - B(s)$ which has the $N(x, t-s)$ distribution (which is symmetric about x). Thus $P(B(t) > x \mid T_x \leq t) = 1/2$. On the other hand $P(B(t) > x \mid T_x > t) = 0$ because BM (having continuous sample paths) can not be above x at time t if it never hit x prior to t. Summarizing yields

$$\begin{aligned} P(B(t) > x) &= P(B(t) > x \mid T_x \le t)P(T_x \le t) + P(B(t) > x \mid T_x > t)P(T_x > t) \\ &= P(B(t) > x \mid T_x \le t)P(T_x \le t) + 0 \\ &= \frac{1}{2}P(T_x \le t), \end{aligned}$$

or

$$P(T_x \le t) = 2P(B(t) > x) = \frac{2}{\sqrt{2\pi t}} \int_x^\infty e^{\frac{-y^2}{2t}} dy,$$

because $B(t) \sim N(0,t)$. Changing variables $u = y/\sqrt{t}$ then yields

Proposition 1.3 *For standard BM, for any fixed* $x \neq 0$

$$P(T_x \le t) = 2P(B(t) > x) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t}}}^\infty e^{\frac{-y^2}{2}} dy = 2(1 - \Theta(x/\sqrt{t})),\ \ t \ge 0.$$

In particular T_x *is a proper random variable;* $P(T_x < \infty) = 1$.

$P(T_x < \infty) = 1$ because taking the limit as $t \to \infty$ yields $P(T_x < \infty) = 2(1 - \Theta(0)) = 2(1 - 0.5) = 1$.

Corollary 1.1 *For standard BM, for any fixed* $x \neq 0$, $E(T_x) = \infty$.

Proof : We shall proceed by computing $E(T_x) = \infty$ by integrating the tail $P(T_x > t)$;

$$E(T_x) = \int_0^\infty P(T_x > t)dt.$$

To this end, $P(T_x > t) = 1 - P(T_x \le t) = \frac{2}{\sqrt{2\pi}} \int_0^{\frac{x}{\sqrt{t}}} e^{\frac{-y^2}{2}} dy$. Since the constant factor $\frac{2}{\sqrt{2\pi}}$ plays no role in wether the integrated tail is infinite or finite, we leave it out for simplicity. It thus suffices to show that

$$\int_0^\infty \int_0^{\frac{x}{\sqrt{t}}} e^{\frac{-y^2}{2}} dydt = \infty.$$

Changing the order of integration, we re-write as

$$\begin{aligned} \int_0^\infty \int_0^{\frac{x^2}{y^2}} e^{\frac{-y^2}{2}} dtdy &= x^2 \int_0^\infty \frac{1}{y^2} e^{\frac{-y^2}{2}} dy \\ &\ge x^2 \int_0^1 \frac{1}{y^2} e^{\frac{-y^2}{2}} dy \\ &\ge x^2 e^{-1/2} \int_0^1 \frac{1}{y^2} dy \\ &= \infty. \end{aligned}$$

The second inequality is due to the fact that the decreasing function $e^{\frac{-y^2}{2}}$ is minimized over the interval $(0,1]$ at the end point $y = 1$.

■

Let $M_t \stackrel{\text{def}}{=} \max_{0 \le s \le t} B(t)$ denote the maximum value of BM up to time t. Noting that $M_t \ge x$ if and only if $T_x \le t$, we conclude that $P(M_t \ge x) = P(T_x \le t)$ yielding (from Proposition 1.3) a formula for the distribution of M_t:

Corollary 1.2 *For standard BM, for any fixed $t \geq 0$,*

$$P(M_t > x) = \frac{2}{\sqrt{2\pi}} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{\frac{-y^2}{2}} dy = 2(1 - \Theta(x/\sqrt{t})),\ x \geq 0.$$

1.8 Hitting times for BM with drift

For $X(t) = \sigma B(t) + \mu t$, let's assume that $\mu < 0$ so that the BM has *negative drift*. This is analogous to the simple random walk with negative drift, that is, $\{R_n\}$ when the increments have distribution $P(\Delta = 1) = p,\ P(\Delta = 1) = q = 1 - p$ and $q > p$. Recall from the gambler's ruin problem that in this case

$$p_a = \frac{1 - (p/q)^b}{(p/q)^{-a} - (p/q)^b},$$

and thus by letting $b \to \infty$ we obtain the probability that the random walk will ever exceed level a;

$$P(\max_{n \geq 0} R_n \geq a) = \lim_{b \to \infty} p_b = (p/q)^a.$$

We conclude that the maximum of the random walk has a geometric distribution with "success" probability $1 - p/q$. The point is that the negative drift random walk will eventually drift off to $-\infty$, but before it does there is a positive probability, $(p/q)^a$, that it will first reach level $a > 0$.

$X(t)$ is similar. We let $M = \max_{t \geq 0} X(t)$ denote the maximum of the BM:

Proposition 1.4 *For BM with negative drift, $X(t) = \sigma B(t) + \mu t$, $\mu < 0$,*

$$p_a = \frac{1 - e^{-\alpha b}}{e^{\alpha a} - e^{-\alpha b}},$$

where $\alpha = 2|\mu|/\sigma^2$; thus (letting $b \to \infty$)

$$P(M > a) = e^{-\alpha a},\ a \geq 0,$$

and we conclude that M has an exponential distribution with mean $\alpha^{-1} = \sigma^2/2|\mu|$.

Note how α^{-1} decreases in $|\mu|$ and increases in σ^2.
Proof :
Here we use an exponential martingale of the form

$$e^{\lambda X(t) - (\lambda \mu + \frac{1}{2}\lambda^2 \sigma^2)t}.$$

This is a MG for any value of λ. Choosing $\lambda = \alpha = -2\mu/\sigma^2$, so that the second term in the exponent vanishes, we have the MG

$$U(t) = e^{\alpha X(t)}.$$

Then for $\tau = \min\{t \geq 0 : X(t) \in \{a, -b\} | X(0) = 0\}$, we use optional sampling to obtain $E(Y(\tau)) = 1$ or $e^{\alpha a} p_a + e^{-\alpha b}(1 - p_a)$; solving for p_a yields the result. ($U(t \wedge \tau)$ is bounded hence UI.)

■

We also have (proof will be given as a hmwk problem):

Proposition 1.5 *For BM with drift, $X(t) = \sigma B(t) + \mu t$, $\mu \neq 0$, if $\tau = \min\{t \geq 0 : X(t) \in \{a, -b\} | X(0) = 0\}$, then*

$$E(\tau) = \frac{a(1 - e^{\frac{2\mu b}{\sigma^2}}) + b(1 - e^{\frac{-2\mu a}{\sigma^2}})}{\mu(e^{\frac{-2\mu a}{\sigma^2}} - e^{\frac{2\mu b}{\sigma^2}})}$$

What about T_x? If the drift is negative, then we already know that for $x > 0$, the BM might not ever hit x; $P(T_x = \infty) = P(M < x) > 0$. But if the drift is positive, x will be hit with certainty (because this is so even when $\mu = 0$; Proposition 1.3). In this case the mean is finite (proof given as hmwk):

Proposition 1.6 *For BM with positive drift,$X(t) = \sigma B(t) + \mu t$, $\mu > 0$, if $T_x = \min\{t \geq 0 : X(t) = x | X(0) = 0\}$, then*

$$E(T_x) = \frac{x}{\mu},\ x > 0.$$

Note how, as $\mu \to 0$, $E(T_x) \to \infty$, and this agrees with our previous calculation (Corollary 1.1) that $E(T_x) = \infty$ when $\mu = 0$ (even though $P(T_x < \infty) = 1$).