1 Lecture: Integration of rational functions by decomposition into partial fractions

- Recognize and integrate basic rational functions, except when the denominator is a power of an irreducible quadratic.
- Divide to write a rational function as the sum of a polynomial and a proper rational function.
- Write out the form of the partial fractions decomposition.
- Find the partial fractions decomposition of a rational function by solving for the constants.

1.1 Summary.

Theorem 1 *Every rational function has an anti-derivative that can be expressed in terms of polynomials, the logarithm function and the function* $\tan^{-1}$.

The goal of this section is to describe a procedure that will allow us to find this anti-derivative. We will not prove every step of the argument. However, by following this procedure, we will be able to integrate the functions that arise in practice.

We begin by defining rational functions:

Definition. A *rational function* is a function which is the quotient of two polynomials.

A proper rational function is a rational function where the degree of the numerator is strictly less than the degree of the denominator. The division algorithm allows us to write any rational function as the sum of a polynomial and a proper rational function. Thus, we only need to consider how to integrate proper rational functions.

Example. Write

$$\frac{x^3}{x+1}$$

as the sum of a polynomial and a proper rational function.

Solution. Using the division algorithm, we can write

$$x^3 = (x+1)(x^2 - x + 1) + 1$$

thus, we have

$$\frac{x^3}{x+1} = x^2 - x + 1 + \frac{1}{x+1}.$$

Second approach. For simple functions such as this one, one can often to the division by a trick:

$$\begin{aligned}
\frac{x^3}{x+1} &= \frac{x^3-1+1}{x+1} \\
&= \frac{(x+1)(x^2-x+1)}{x+1} + \frac{1}{x+1} \\
&= x^2 - x + 1 + \frac{1}{x+1}.
\end{aligned}$$

where the second step requires us to remember how to factor $b^3 - a^3$. ∎

Notice that in this example, when we divide, we put the function into a form where it is easy to find the anti-derivative:

$$\int \frac{x^3}{x+1}\,dx = \int x^2 - x + 1 + \frac{1}{x+1}\,dx = \frac{x^3}{3} - \frac{x^2}{2} + x + \ln|x+1| + C.$$

In general, we will not be so lucky. However, there are further simplifications that allow us to integrate all rational functions.

1.2 Some elementary rational functions

In this section, we recall several rational functions that we know how to integrate. The main point of this section is that every rational function can be written in terms of these simple functions.

The basic functions we consider are:

$$\int \frac{A}{(ax+b)^k}\,dx \tag{1}$$

where A, $a \neq 0$ and b are constants and k is a natural number. The basic function we consider is

$$\int \frac{Ax+B}{(ax^2+bx+c)^k}\,dx. \tag{2}$$

where A, B, $a \neq 0$, b, and c are constants and the quadratic expression in the denominator is *irreducible.* This means that the equation $ax^2 + bx + c = 0$ has no real roots and happens precisely when the discriminant $b^2 - 4ac < 0$.

We recall the techniques for integrating these functions:

Example. Find

$$\int \frac{1}{(2x+1)^2}\,dx.$$

Solution. Substituting $u = 2x+1$ gives that $du = 2dx$. Thus, we have that

$$\int \frac{1}{(2x+1)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,dx = -\frac{1}{2}u^{-1} + C = -\frac{1}{(4x+2)} + C.$$

∎

Example. Find

$$\int \frac{x+3}{x^2+4x+6}\,dx.$$

Solution. Here we begin by complete the square in the denominator, $x^2+4x+6 = (x+2)^2+2$. Note that if we have an irreducible quadratic, then the expression should be strictly positive. Now, we substitute $u = x+2$, $du = dx$ in the integral and obtain that

$$\int \frac{x+3}{x^2+4x+6}\,dx = \int \frac{u}{u^2+2}\,du + \int \frac{1}{u^2+2}\,du.$$

The first integral on the write can be evaluated by substituting $v = u^2+2$ and we obtain

$$\int \frac{u}{u^2+2}\,du = \frac{1}{2}\ln(u^2+2) + C.$$

The second integral is can be evaluated and leads to a $\tan^{-1}$.

$$\int \frac{1}{u^2+2}\,du = \frac{1}{\sqrt{2}}\tan^{-1}(\frac{u}{\sqrt{2}}) + C.$$

Combining everything, and putting $u = x+2$ gives

$$\int \frac{x+3}{x^2+4x+6}\,dx = \frac{1}{2}\ln((x+1)^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}(\frac{x+2}{\sqrt{2}} + C.$$

∎

The alert reader will observe that we have not discussed how to integrate functions such as

$$\int \frac{1}{(x^2+1)^k}\,dx$$

when $k > 1$. There is a reduction formula which allows us to reduce the value of k, however we will not discuss this case.

1.3 Decomposition into partial fractions.

In this section, we describe how to decompose a general proper rational function into simpler functions that we can integrate as in the previous section.

We begin by factoring the denominator into linear and irreducible quadratic factors. It is a deep theorem of algebra that this is always possible for polynomials with real coefficients. We also assume that all common factors between the numerator and denominator have been canceled.

So let us consider a rational function where the denominator is a product of linear factors $L_j(x)$ and quadratic factors $Q_k(x)$ raised to various powers. We claim that we can find constant so that we have

$$\begin{aligned}
\frac{P(x)}{L_1(x)^{k_1}\dots L_m(x)^{k_m}Q_1(x)^{j_1}\dots Q_n(x)^{j_n}} \quad = \quad & \frac{A_{1,1}}{L_1(x)}+\dots+\frac{A_{1,k_1}}{L_1(x)^{k_1}} \\
& \dots \\
& +\frac{A_{m,1}}{L_m(x)}+\dots+\frac{A_{m,k_m}}{L_m(x)^{k_m}} \\
& +\frac{B_{1,1}x+C_{1,1}}{Q_1(x)}+\dots+\frac{B_{1,j_1}x+C_{1,j_1}}{Q_1(x)^{j_1}} \\
& \dots \\
& +\frac{B_{n,1}x+C_{n,1}}{Q_n(x)}+\dots+\frac{B_{n,j_n}x+C_{n,j_n}}{Q_n(x)^{j_n}}.
\end{aligned}$$

I think we can all agree that the right-hand side is a big mess. This mess is called the partial fractions decomposition. Let me summarize the rules for writing the decomposition. For a linear factor raised to a power k, we obtain k terms where the numerator is constant and the denominator is the linear factor raised to the powers $1,\dots,k$. For a quadratic factor raised to the power k, we obtain k terms where the numerator is a linear function and the denominator is a quadratic factor raised to the powers $1,\dots,k$.

An example might help.

Example. Write out the form of the partial fractions decomposition. Do not solve for the constants.

$$f(x)=\frac{x+3}{(x+2)(x-1)^2(x^2+1)^2(x^2-1)}.$$

Solution. Notice that the quadratic expression x^2+1 is irreducible. The quadratic expression x^2-1 factors as $(x-1)(x+1)$. After factoring and collecting like terms in the denominator, we obtain that

$$f(x)=\frac{x+3}{(x-1)^3(x+1)(x+2)(x^2+1)^2}.$$

To obtain the decomposition, we consider each factor separately. A factor raised to the k power will give us k terms. The denominators are the factor raised to the powers $1, \ldots, k$. If the factor is linear, the numerators are constants and if the factor is quadratic, the numerators are linear functions.

Applying these rules give:

$$\begin{aligned} f(x) \quad = \quad & \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \\ & + \frac{D}{x+1} \\ & + \frac{E}{x+2} \\ & + \frac{Fx+G}{x^2+1} + \frac{Hx+I}{(x^2+1)^2}. \end{aligned}$$

∎

1.3.1 Finding the constants

After we have written out the form of the partial fractions decomposition, finding the values of the constants is a straightforward, but often lengthy computation.

- Find a common denominator.
- Collect like terms in the numerator.
- Write a system of equations by comparing coefficients of each power of x.
- If all is going well, this will give a system linear equations where the number of equations is the same as the number of unknowns and one may solve this system by elimination or your favorite method.

We illustrate this method with a simple example.

Example. Find the partial fractions decomposition of the function

$$g(x) = \frac{x+2}{x^2+x}.$$

Solution. The denominator factors as $(x+1)x$, thus the partial fractions decomposition will contain two terms, one for each factor.

$$\frac{3x+2}{x^2+x} = \frac{A}{x+1} + \frac{B}{x}.$$

Obtaining a common denominator on the left gives

$$\frac{3x+2}{x^2+x} = \frac{Ax+B(x+1)}{x(x+1)}$$

Collecting like terms on the left-hand side gives

$$\frac{3x+2}{x^2+x} = \frac{(A+B)x+B}{x(x+1)}$$

We obtain a system of two equations in A and B by comparing the coefficient of x on both sides and the constant term.

$$\begin{aligned} x: \quad & A+B=3 \\ 1: \quad & B=2 \end{aligned}$$

Here, the first entry in each row tells the power of x whose coefficients are equated to obtain the equation. It is easy to see that the solutions of this system are

$$B=2 \quad A=1$$

.

Thus, the partial fractions decomposition is

$$\frac{3x+2}{x^2+x} = \frac{3}{x+1} + \frac{2}{x}.$$

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We were not asked to evaluate an integral in the previous example, but we should observe that the expression on the right is easy to integrate.

We end this lecture by giving one example where we follow the entire procedure.

Example. Find the anti-derivative

$$\int \frac{1}{x^4-1}\,dx$$

Solution. We observe that $x^4 - 1 = (x^2)^2 - 1^2$ is a difference of squares and use this to factor

$$x^4 - 1 = (x^2 - 1)(x^2 + 1).$$

Next, we write out the form of the partial fractions decomposition:

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Obtaining a common denominator on the right hand side gives

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{(x+1)(x^2+1)}{(x+1)(x^2+1)}\frac{A}{x-1} + \frac{(x-1)(x^2+1)}{(x-1)(x^2+1)}\frac{B}{x+1} + \frac{(x^2-1)}{(x^2-1)}\frac{Cx+D}{x^2+1}$$

And if we multiply out each numerator and collect like terms we have

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{(A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)1}{(x-1)(x+1)(x^2+1)}$$

This gives the system of equations:

$$\begin{aligned} x^3 : &\quad A + B + C = 0 \\ x^2 : &\quad A - B + D = 0 \\ x : &\quad A + B - C = 0 \\ 1 : &\quad A - B - D = 1 \end{aligned}$$

Solving this system gives:

$$A = 1/4 \quad B = -1/4 \quad C = -1/2.$$

After all of this, evaluating the integral is easy:

$$\begin{aligned} \int \frac{1}{x^4-1}\,dx &= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx \\ &= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1} x + C \\ &= \frac{1}{4}\ln\frac{|x-1|}{|x+1|} - \frac{1}{2}\tan^{-1} x + C. \end{aligned}$$

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