1 Lecture 17: Improper integrals

1. Consider improper integrals at infinity.

2. Find areas as improper integrals.

3. The comparison theorem.

A question.

Can an region have infinite perimeter and finite area? If so, we can color it in, but we cannot draw its outline.

Example. Find the area bounded by $x = 1$, $y = 0$ and $y = e^{-x}$.

Solution. This is a new problem–until now, we have found the area of regions which were of finite length in every direction.

We attempt to solve this problem, by first finding the area bounded by $x = 1$, $y = 0$, $y = e^{-x}$ and $x = N$. This is

$$\int_1^N e^{-x}\,dx = \frac{1}{e} - e^{-N}.$$

If we take the limit as N goes to infinity, we expect to obtain a reasonable value for the infinite area.

$$\text{area} = \lim_{N\to\infty} \frac{1}{e} - e^{-N} = \frac{1}{e}.$$

∎

Definition of improper integral.

Definition. If a is a real number and f is a function which is Riemann integrable on each interval $[a, N]$ for $N > a$, then we define the *improper integral* as

$$\int_a^\infty f(t)\,dt = \lim_{N\to\infty} \int_a^N f(t)\,dt.$$

If this limit exists and is finite, we say the improper integral is *convergent.* If the limit is infinite or does not exist, we say the improper integral is *divergent.*

Exercise. Write out the corresponding definition for improper integrals at $-\infty$.

Exercise. Re-read the definition of the Riemann integral $\int_a^b f(t)\,dt$ and explain what difficulties would arise if we let $b = \infty$ in this definition.

Example. How should we treat the integral

$$\int_{-\infty}^{\infty} x\,dx?$$

Solution. If we break the integral up as follows:

$$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx,$$

then each of the integrals on the right is divergent. ∎

Example. Evaluate the following improper integrals, or state if the integral diverges.

1. $\displaystyle\int_{-\infty}^{0} e^{2x}\,dx$
2. $\displaystyle\int_{-\infty}^{0} e^{-2x}\,dx$
3. $\displaystyle\int_{1}^{\infty} \frac{1}{1+x^2}\,dx.$

Example. Determine for which p, the integral

$$\int_{1}^{\infty} \frac{1}{x^p}\,dx$$

converges.

Solution. If $p \neq 1$, then

$$\int_{1}^{\infty} \frac{1}{x^p}\,dx = \lim_{N\to\infty} \int_{1}^{N} x^{-p}\,dx = \lim_{N\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_{x=1}^{x=N} = \lim_{N\to\infty} \left(\frac{N^{-p+1}}{-p+1} - \frac{1}{-p+1}\right).$$

This limit will exist and be finite if the exponent of N is negative. That is the integral converges if $p > 1$. The limit will be infinite and the integral diverges if $p < 1$.

The case $p = 1$ is treated separately because it involves the logarithm. If $p = 1$, then the integral is divergent. ∎

The comparison test

Theorem 1 *If f and g are continuous functions on $[a, \infty)$ and $0 \leq f(x) \leq g(x)$, then we have*

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx$$

Hence,

If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.
If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.

Notice that these results do not give us the value, only whether the integral converges or diverges.

Example. Show that

$$\int_0^\infty (1 + \sin x)(e^{-x})\,dx \qquad \int_0^\infty e^{-x^2}\,dx$$

converges.

Solution. We have $(1 + \sin x)e^{-x} \leq 2e^{-x}$ and we know the integral of $\int_0^\infty e^{-x}$ converges and thus the comparison test tells us that the integral of e^{-x^2} converges.

We only need to worry about $\int_1^\infty e^{-x^2}\,dx$ and in this region we have $x \geq x^2$, $-x^2 \leq -x$ and finally, $e^{-x^2} \leq e^{-x}$. Now the comparison test tells us the integral $\int_0^\infty e^{-x^2}$ is convergent. ∎

Example. Determine if the following integrals converge or diverge using the comparison test.

$$\int_0^\infty e^{-x}(2 + \sin x)^{-1}\,dx \quad \int_1^\infty \frac{e^{-x}}{x}\,dx \quad \int_1^\infty \frac{2 + \sin x}{x}\,dx.$$

March 1, 2004