1 Lecture 20: Sequences

1. Find limits of sequences using sum, product, and squeeze theorem.
2. Use the convergence of monotone sequences to find limits of recursively defined sequences.

Sequence

A *sequence* is a list of real numbers $\{a_1, a_2, \ldots\}$. (More formally, we might define a sequence as a function whose domain is the natural numbers $1, 2, 3, \ldots$ and whose range is the real numbers. However, we always write a_n rather than use our standard notation for functions $a(n)$).

We say that a sequence $\{a_n\}$ converges to L if for each $\epsilon > 0$, there is an N so that if $n > N$, then $|a_n - L| < \epsilon$. We write

$$\lim_{n\to\infty} a_n = L$$

If the limit of a sequence exists and is finite, the sequence is a convergent sequence. Else, it is divergent.

If one recalls the definition of limits at infinity, then it is reasonable to expect that there is a connection between limits at infinity and limits of sequence. The connection is:

Proposition 1 *If f is a function with*

$$\lim_{x\to\infty} f(x) = L$$

and $a_n = f(n)$, then

$$\lim_{n\to\infty} a_n = L.$$

Example. If $a_n = (n^2+1)/(2n^2-1)$, find $\lim_{n\to\infty} a_n$.

Solution. We already understand the limit of $(x^2+1)/(2x^2-1)$. ∎

Example. Can you find a function where

$$\lim_{n\to\infty} f(n) \text{exists} \qquad \lim_{x\to\infty} f(x) \text{does not exist.}$$

Solution. $f(x) = \sin x$. ∎

Sum, product and squeeze theorems.

As with limits of functions, we have the following rules about limits.

Theorem 1 *If we have two convergent sequences* $\{a_n\}$ *with* $\lim_{n\to\infty} a_n = L$ *and* $\{b_n\}$ *with* $\lim_{n\to\infty} b_n = M$, *then*

$$\begin{aligned} \lim_{n\to\infty} a_n + b_n &= L + M \\ \lim_{n\to\infty} a_n b_n &= LM \end{aligned}$$

If in addition, c is a real number, then

$$\lim_{n\to\infty} ca_n = cL.$$

If, in addition, $M \neq 0$, *then*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}.$$

Also, we have a squeeze theorem for sequences.

Theorem 2 *If* $a_n \leq b_n \leq c_n$ *and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then the sequence b_n *is convergent and*

$$\lim_{n\to\infty} b_n = L.$$

Example. Find the limits of the following sequences:

$$a_n = \cos(n\pi),\ b_n = \frac{2n+1}{\sqrt{n^2+1}},\ c_n = n^100/e^n,\ d_n = \frac{\cos n}{n}.$$

Monotone sequences

This section states an important theoretical result that gives us conditions when a sequence will converge. We will see that this can be useful because there are sequences where we can use convergence to help us compute the limit.

Some definitions: A sequence $\{a_n\}$ is monotone increasing if we have

$$a_1 \leq a_2 \leq a_3 \leq \cdots.$$

A sequence is monotone decreasing if....

A sequence is *monotone* if it is monotone increasing or monotone decreasing.

Example.

$$1/n, -1/n^2, (-1)^n/n^2.$$

A sequence $\{a_n\}$ is bounded below if there is a number M so that

$$a_n \geq M, \qquad \text{for all } n.$$

Exercise. Complete the following definition. A sequence is bounded above if ....

A sequence is *bounded* if it is bounded above or below.

Example. $1/n$, n.

Theorem 3 *If a sequence is bounded and monotone, then it is convergent.*

–Draw picture.

Example. r^n, $0 < r < 1$

Example. Define $a_1 = 1$ and $a_n = 3/(5 - a_n)$. Show the sequence is convergent and find its limit.

Solution. 1. Try a few examples. 1, 3/4, .235294...
2. Prove by induction that $0 < a_n \leq 1$.
3. Prove by induction that $a_{n+1} < a_n$.
4. We know $\alpha = \lim a_n$ exists. The limit, α satisfies

$$\alpha = 3/(5 - \alpha).$$

Solving this equation gives

$$\alpha = \frac{5 \pm \sqrt{13}}{2}.$$

Since $0 < \alpha < 1$, then $\alpha = \frac{1}{2}(5 - \sqrt{13})$. ∎

March 11, 2004