Negative q-Stirling numbers

Wachsfest

Margaret Readdy.

Joint with Yue Cai.

Michelle's abilities*

Topological might.

Combinatorial insight.

"Poset topology" she did write

Which we use and cite.

* Not to be confused with "Michellability".

Let's count [~ 50,000 BC*]

$$\sum_{\pi \in \mathfrak{S}_n} 1 = n!$$

$$\sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S| = k}} 1 = \binom{n}{k}.$$

* Source: Wikipedia

Let's q-count [1700's Euler*].

q-analogue of $n \in \mathbb{Z}^+$

$$[n]_q = [n] = 1 + q + \cdots + q^{n-1},$$

q an indeterminate.

$$\lim_{q \to 1} [n]_q = \underbrace{1 + \cdots + 1}_{n} = n.$$

$$[n]! = [n][n-1] \cdots [2][1].$$

* Theta functions

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}}, \quad |ab| < 1$$

Combinatorial interpretation

[MacMahon 1916]

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = [n]!,$$

where

$$\mathrm{inv}(\pi) = \{ (i,j) : \quad i<j \text{ and } \pi_i > \pi_j \}.$$

for $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$.

Gaussian polynomial. (the q-binomial)

$n \in \mathbb{N}$, $k \in \mathbb{Z}$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{[n]!}{[k]!\,[n-k]!} & 0 \le k \le n \\ 0 & k < 0 \text{ or } k > n. \end{cases}$$

Comb'l interpretation.

$$\sum_{\pi \in \mathfrak{S}(1^k, 0^{n-k})} q^{\operatorname{inv} \pi} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

[MacMahon 1916]

ex. $\begin{bmatrix}4\\2\end{bmatrix}$.

π	inv π
0011	0
0101	1
0110	2
1001	2
1010	3
1100	4.

$$\sum_{\pi \in \mathfrak{S}\{1^2,0^2\}} q^{\operatorname{inv} \pi} = q^4 + q^3 + 2q^2 + q + 1.$$

Check $\quad \begin{bmatrix}4\\2\end{bmatrix} = \dfrac{[4][3]}{[2]} = \dfrac{(1+q)(1+q^2)(1+q+q^2)}{(1+q)}.$

The negative q-binomial [Fu - Reiner - Stanton - Thiem, 2012]

def.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q' \triangleq (-1)^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q}$$

ex. $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q' = q^4 - q^3 + 2q^2 - q + 1.$$

Theorem: [Fu - Reiner - Stanton - Thiem].

$$\begin{bmatrix} n \\ k \end{bmatrix}_q' = \sum_{w \in \Omega(n,k)'} wt(w)$$

$$= \sum_{w \in \Omega(n,k)'} q^{a(w)} (q-1)^{p(w)}$$

where $\Omega(n,k)'$ is a certain subset of $\mathfrak{S}\{1^k, 0^{n-k}\}$,

$p(w) =$ number of $\underline{10}$ pairs in w

$a(w) = \mathrm{inv}(w) - p(w)$.

Corollary: [F-R-S-T]

The q-binomial can be expressed as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega(n,k)'} q^{a(w)} (1+q)^{p(w)}.$$

def. Given $w = w_1 \cdots w_n \in \mathfrak{S}\{1^k, 0^{n-k}\}$, pair

i. $n = 1$. Leave letter unpaired.

ii. $n \geq 2$ + k odd: Pair $\underline{w_1 w_2}$

Repeat on $w_3 \cdots w_n$

iii. $n \geq 2$ + k even: Pair w_1.

Repeat on $w_2 \cdots w_n$.

ex. 01 1 00 10 1 01

1 10 0 0 1 00 1

Define

$$\Omega'_{n,k} = \{ w \in \mathfrak{S}\{1^k, 0^{n-k}\} : w \text{ has no paired } \underline{01} \}.$$

ex. $\underline{\Omega'}$

$\underline{0}\ \underline{0}\ \underline{1}\ \underline{1}$

$\underline{0}\ \underline{1}\ \underline{0\,1}$ No.

$\underline{0}\ \underline{1}\ \underline{1\,0}$

$\underline{1}\ \underline{0\,0}\ \underline{1}$

$\underline{1}\ \underline{0\,1}\ \underline{0}$ No.

$\underline{1}\ \underline{1\,0}\ \underline{0}$

ex. (cont'd)

$\Omega(2,2)'$	$q^{\mathrm{inv}(w)}$	$\mathrm{wt}(w)$
0011	1	1
0110	q^2	$q(1+q)$
1001	q^2	q^2
1100	q^4	$q^3(1+q)$

$$\sum = 1 + (q+q^3)(1+q) + q^2$$
$$= q^4 + q^3 + 2q^2 + q + 1.$$

Recall

$$\mathrm{wt}(w) = q^{a(w)} \cdot (1+q)^{p(w)}$$

$p(w) = \#$ $\underline{10}$ pairs in w, $a(w) =$

$a(w) = \mathrm{inv}(w) - p(w)$.

2 : What about other combinatorial objects with q-analogues?

2': Given a q-analogue

$$f[\vec{x}]_q = \sum_{w \in S} wt(w),$$

when can we find a ~~subset~~ $T \subseteq S$ and ~~statistics~~ $A(\cdot)$ + $B(\cdot)$ s.t.

$$f[\vec{x}]_q = \sum_{w \in T} q^{A(w)} \cdot (1+q)^{B(w)} ?$$

The Stirling numbers of the second kind

$S(n,k) =$ # partitions of $\{1, \dots, n\}$ into k blocks.

ex. $S(4,2)$:

1/234	12/34	
134/2	13/24	(written in standard form).
124/3	14/23.	
123/4		

The q-Stirling numbers

$$S_q[n,k] = S_q[n-1,k-1] + [k]\, S_q[n-1,k]$$

with $S_q[n,n] = 1 = S_q[n,1]$.

RG-words [Milne, Rota].

Encode partition π using a restricted growth word w.

$w = w_1 \cdots w_n$ where $w_i = j$ if the elt i is in the jth block of π.

ex. $\pi = 125/36/47 \leftrightarrow 1123123$.

Let $R(n,k)$ = set of all RG-words which encode a set partition of $\{1, -, n\}$ into k parts.

For $w \in \mathcal{B}(n,k)$ let

$$wt(w) = \prod_{i=1}^{n} wt_i(w),$$

where $m_i = \max\{w_1, \ldots, w_i\}$,

$wt_1(w) = 1$ and for $2 \le i \le n$

$$wt_i(w) = \begin{cases} q^{w_i - 1} & \text{if } w_i \le m_{i-1} \\ 1 & \text{if } w_i > m_{i-1} \end{cases}$$

Theorem: [Cai–Readdy]

The q-Stirling number of the second kind is given by

$$S_q[n,k] = \sum_{w \in \mathcal{B}(n,k)} wt(w).$$

ex.

π.	w	$wt(w)$
1/234	1222	$q^1 \cdot q^1 = q^2$
134/2	1211	1
124/3	1121	1
123/4	1112	1
12/34	1122	q^1
13/24	1212	q^1
14/23	1221	q^1

$$\sum = q^2 + 3q + 3 .$$

$$= S_q[4,2]$$

Remark: See Garsia - Remmel, Milne, and especially Wachs - White for a multitude of statistics that generate $S_q[n, k]$.

The wt(·) statistic is related to Wachs - White's ls(·) statistic.

Let $wt'(w) = \prod_{i=1}^{n} wt'_i(w)$, $m_i = \max\{w_1, \dots, w_i\}$,

and

$$wt'_i(w) = \begin{cases} q^{w_i - 1}(1+q) & \text{if } w_i < m_{i-1} \\ q^{w_i-1} & \text{if } w_i = m_{i-1} \\ 1 & \text{if } w_i > m_{i-1} \text{ or } i = 1, \end{cases}$$

Write $A(w) = \sum_{i=1}^{n} A_i(w)$ and $B(w) = \sum_{i=1}^{n} B_i(w)$

where

$$A_i(w) = \begin{cases} w_i - 1 & \text{if } w_i \le m_{i-1} \\ 0 & \text{if } w_i > m_{i-1} \\ & \text{or } i = 1 \end{cases} \qquad B_i(w) = \begin{cases} 1 & \text{if } w_i < m_{i-1} \\ 0 & \text{otherwise.} \end{cases}$$

Allowable RG-Words

def. An RG-word $w \in \mathcal{R}(n,k)$ is allowable if it is of the form

$$\underbrace{1 \cdots 1}_{1\text{'s}} \quad 2 \quad \underbrace{\cdots}_{1\text{'s},\, 3\text{'s}} \quad 4 \quad \underbrace{}_{1\text{'s},\, 3\text{'s},\, 5\text{'s}} \quad 6 \cdots$$

ex. $w = 1\,1\,2\,1\,3\,3\,1\,4\,3\,5 \in \mathcal{A}(10,5)$.

$$wt(w) = 1\cdot 1\cdot 1\cdot (1+q)\cdot 1 \cdot q^2 \cdot \quad \cdot (1+q)\cdot 1\cdot q^2(1+q)\cdot 1$$

Allowable words are denoted by $\mathcal{A}(n,k)$

ex.

$\underline{w}$	$\underline{wt'(w)}$
1222	–
1211	$(1+q)^2$
1121	$(1+q)$
1112	$\underline{1}$
1122	–
1212	–
1221.	–

$$\sum = (1+q)^2 + (1+q) + \underline{1}$$

$$= q^2 + 3q + 3$$

$$\overset{\shortparallel}{S_q[4,2]}$$

ex. $S_q[5,3]$.

w	$wt'(w)$
12311	$(1+q)^2$
12131	$(1+q)^2$
12113	$(1+q)^2$
12133	$(1+q)\cdot q^2$
12313	$(1+q)\cdot q^2$
12331	$q^2\cdot(1+q)$
12333	$q^2\cdot q^2$
11213	$(1+q)$
11231	$(1+q)$
11233	q^2
11123	1

$$\sum = q^4 + 3q^2(1+q) + q^2 + 3\cdot(1+q)^2 + 2(1+q) + 1.$$

$$S_q[5,3] = q^4 + 3q^3 + 7q^2 + 8q + 6$$

Theorem: [Cai - Readdy]

$$S_q[n,k] = \sum_{w \in \mathcal{A}(n,k)} wt'(w)$$

$$= \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot (1+q)^{B(w)},$$

Stembridge's $q=-1$ phenomenon

B finite set

$$X(q) = \sum_{b \in B} q^{wt(b)}.$$

Set $q=-1$ to count fixed pts in an involution.

Corollary: [Cai - Readdy]
$S_q[n,k]$ when $q=-1$
counts the # of weakly increasing allowable words in $\mathcal{A}(n,k)$.

Form: $\underline{1} \cdots \underline{1}\ 2\ 3 \cdots 3\ 4\ 5 \cdots 5\ 6\ \cdots$
(No $(1+q)$ terms).

The Stirling poset of the second kind $\Pi(n,k)$

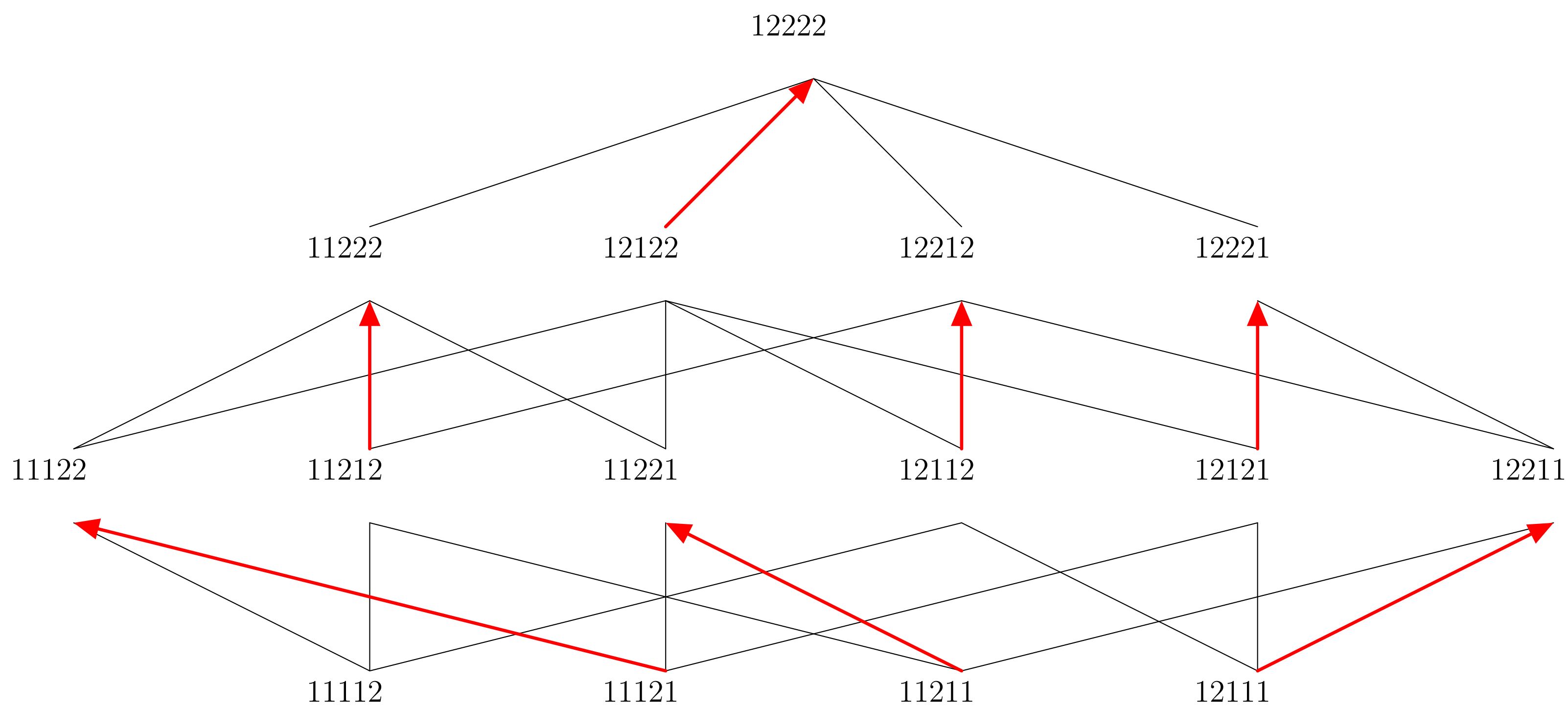
For $\pi, \sigma \in \mathcal{R}(n,k)$ let $\pi \lessdot \sigma$ if

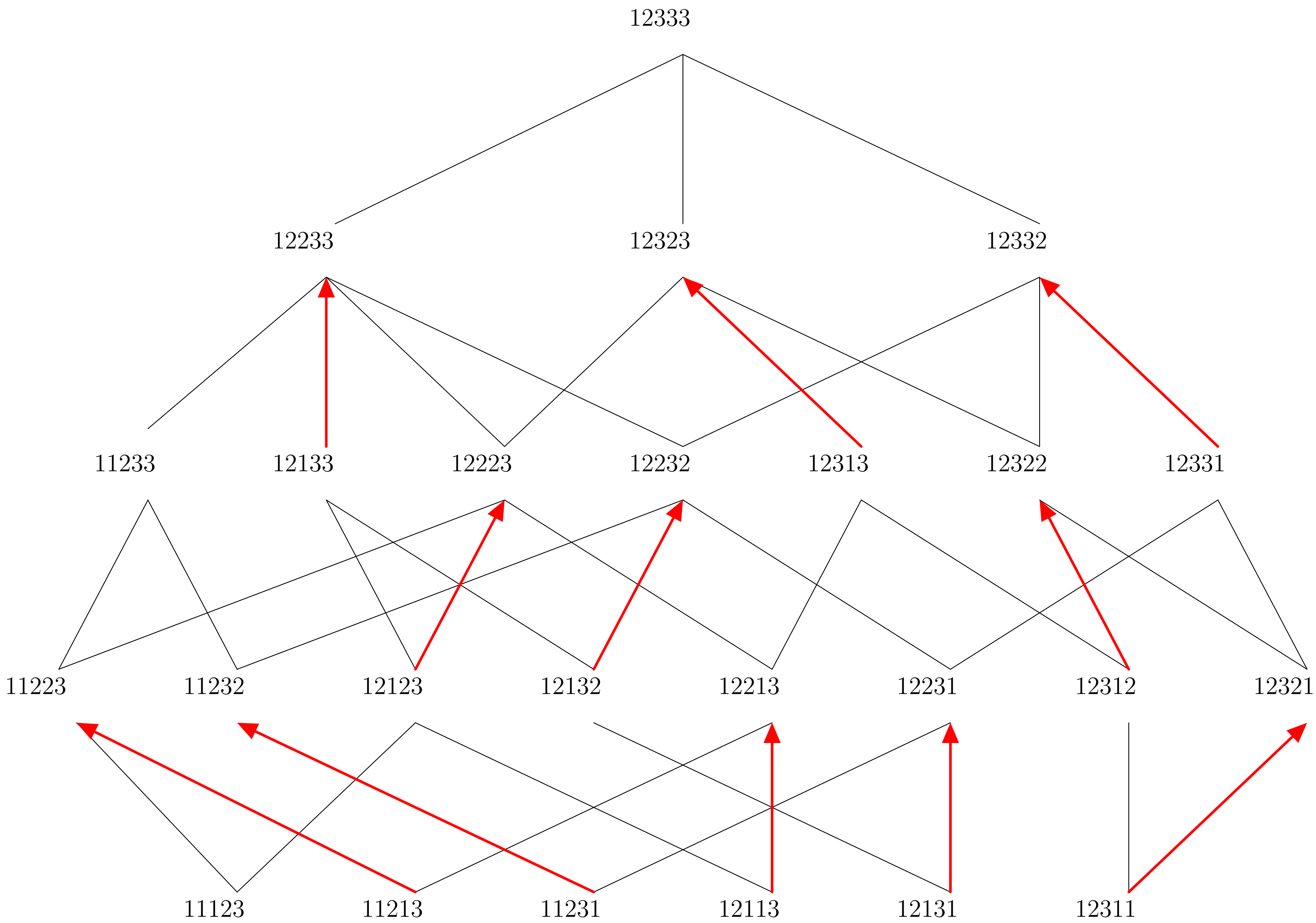
$$\sigma = \pi_1 \pi_2 \cdots (\pi_i + 1) \cdots \pi_n .$$

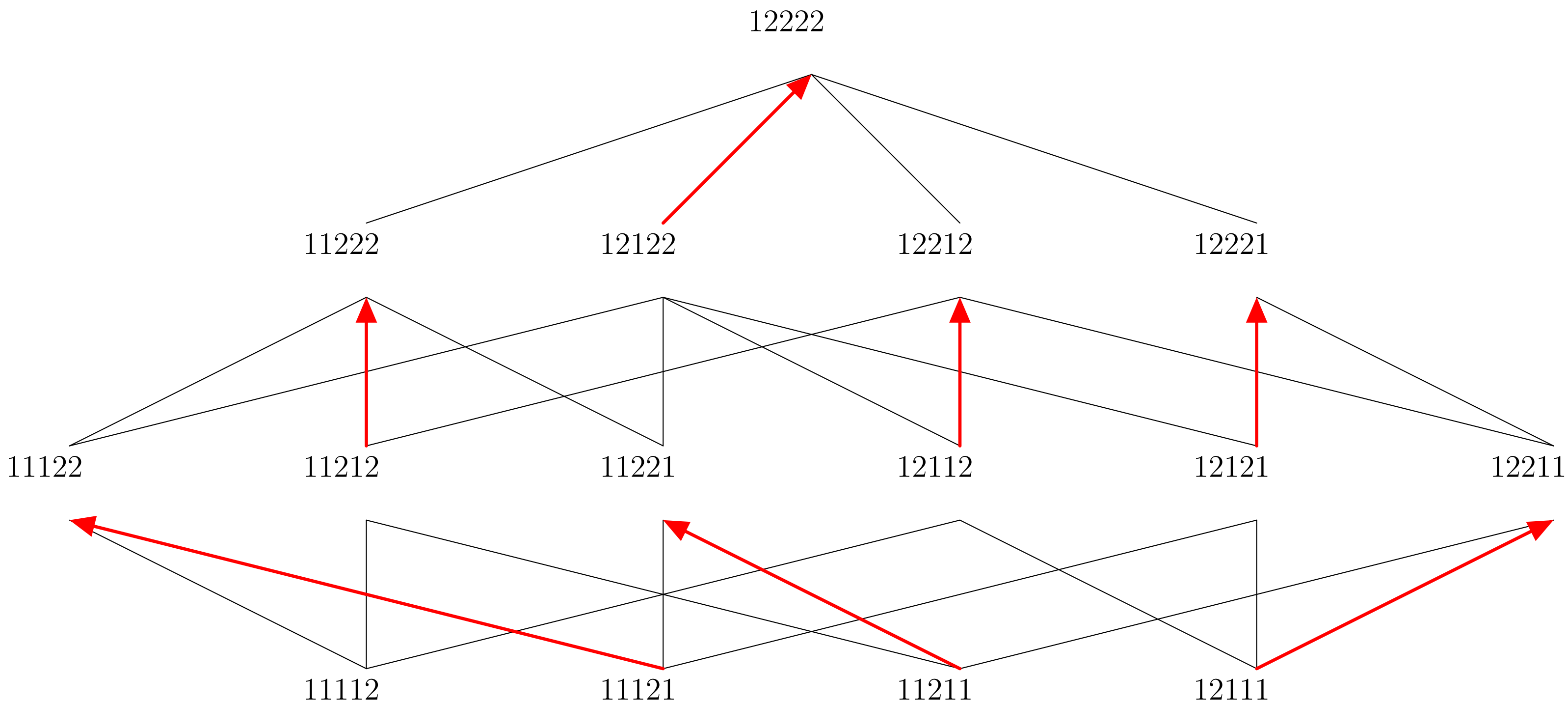
for some index i.

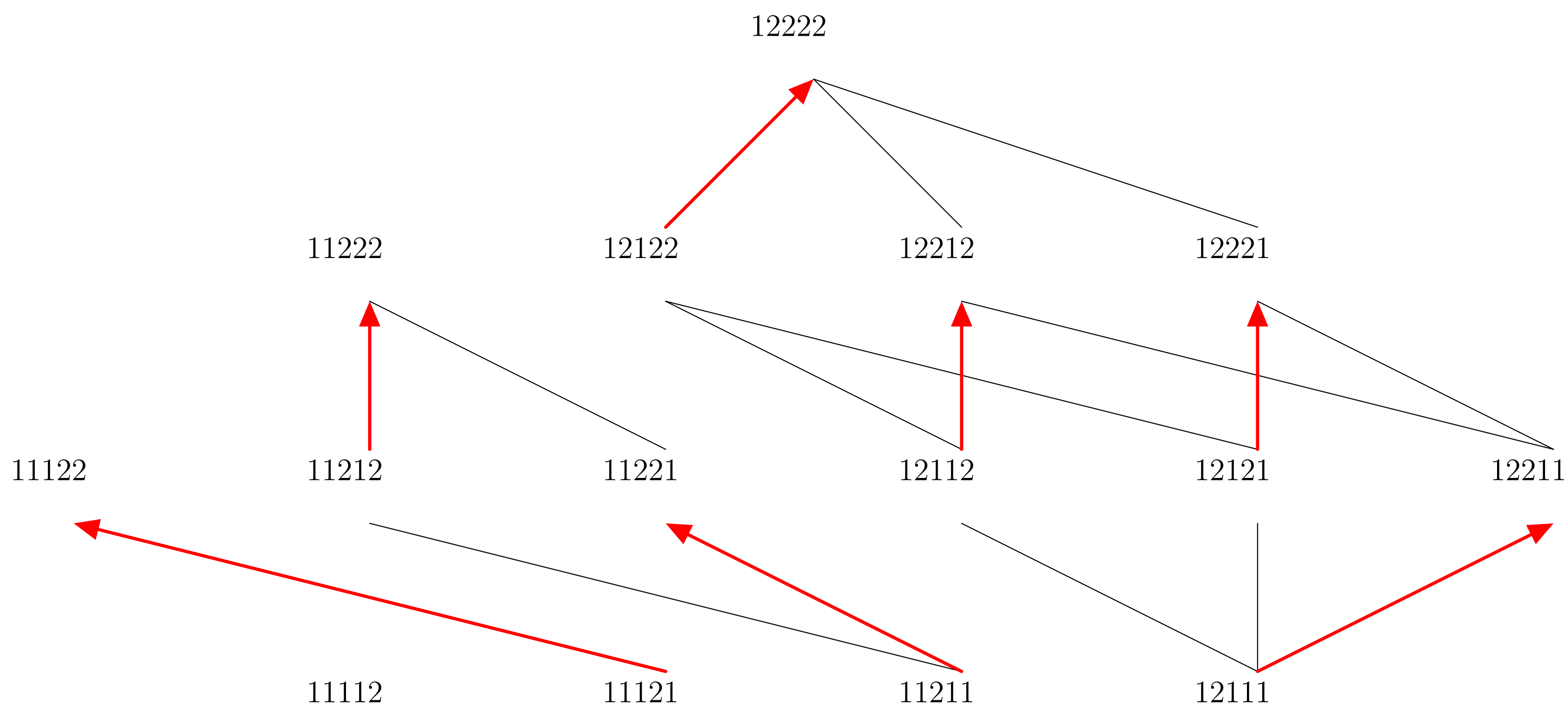
Clearly, $\pi \lessdot \sigma \Rightarrow wt(\sigma) = q \cdot wt(\pi)$.

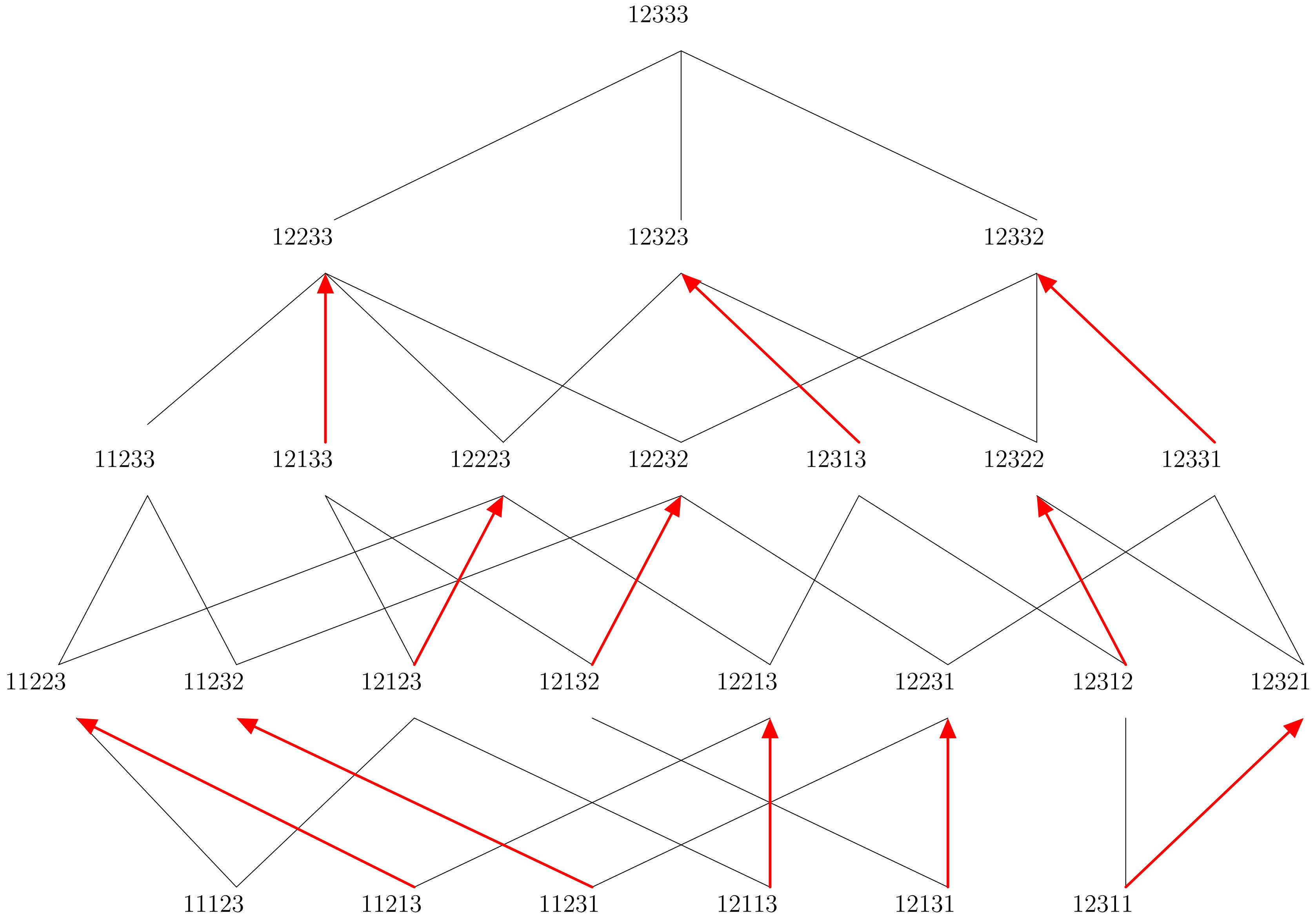
Thus $\Pi(n,k)$ is a graded poset

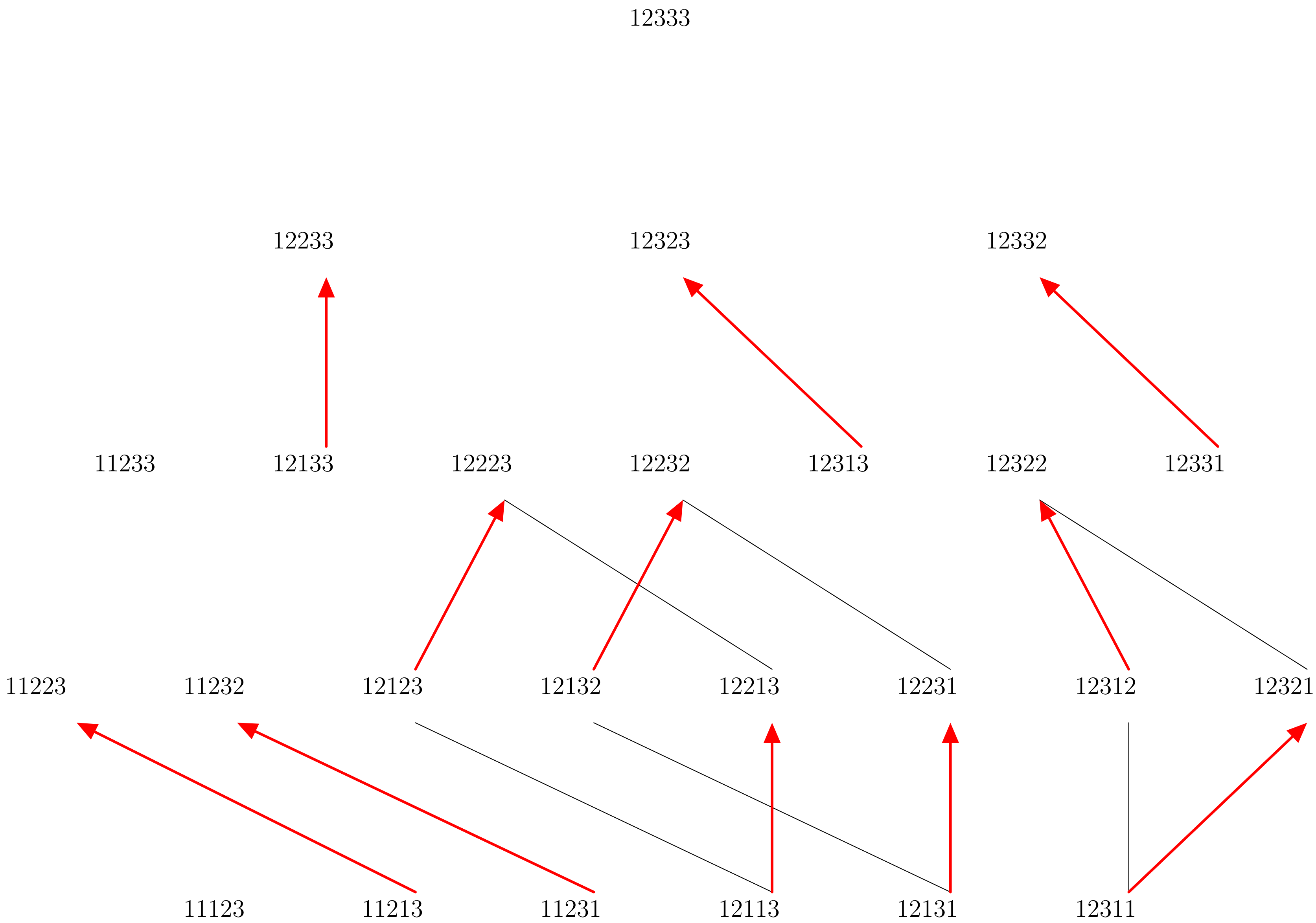

12222
11222
12122
12212
12221
11122
11212
11221
12112
12121
12211
11112
11121
11211
12111


12333
12233
12323
12332
11233
12133
12223
12232
12313
12322
12331
11223
11232
12123
12132
12213
12231
12312
12321
11123
11213
11231
12113
12131
12311


12222
11222
12122
12212
12221
11122
11212
11221
12112
12121
12211
11112
11121
11211
12111


12222
11222
12122
12212
12221
11122
11212
11221
12112
12121
12211
11112
11121
11211
12111


12333
12233
12323
12332
11233
12133
12223
12232
12313
12322
12331
11223
11232
12123
12132
12213
12231
12312
12321
11123
11213
11231
12113
12131
12311


12333
12233
12323
12332
11233
12133
12223
12232
12313
12322
12331
11223
11232
12123
12132
12213
12231
12312
12321
11123
11213
11231
12113
12131
12311

Theorem: [Cai - Readdy].

The Stirling poset of the second kind has the decomposition.

$$\Pi(n,k) \cong \dot{\bigcup_{w \in \mathcal{A}(n,k)}} B_{|\mathrm{Inv}(w)|}$$

where B_j is the Boolean algebra on j elts, $\mathrm{Inv}(w) = \{ w_i : w_j > w_i \text{ for some } j < i \}$ is the set of all entries in w that contribute to an inversion, and $\mathcal{A}(n,k)$ are allowable RG-words in $\Pi(n,k)$.

Homological $q=-1$ phenomenon [Hersh - Shareshian - Stanton]

Claim: Stembridge's $q=-1$ phenomenon is same Euler characteristic computation.

Idea: Define a chain complex $(\mathcal{C}, \partial)$.

Ranks of chain groups are coeffs in the polynomial $X(q)$.

Euler characteristic is $X(-1)$

Also, Euler characteristic = alternating sum of ranks of homology groups.

Best scenario: $(\mathcal{C}, \partial)$ has homology concentrated in ranks of same parity + has basis indexed by fixed points of involution $= X(-1)$.

def. P graded poset

W_i = rank i elts of P

The poset P supports a chain complex $(\mathcal{C}, \partial)$

of $\mathbb{F}$-vector spaces C_i if:

C_i has basis indexed by the elts W_i

$C_i \neq 0 \iff W_i \neq \emptyset$

∂ boundary map.

For $x \in W_{i-1}$, $y \in W_i$ the coeff of

$\partial_{y,x}$ of x in $\partial_i(y)$ is zero unless $x < y$.

ex. The algebraic complex $(\mathscr{C}, \partial)$ supported by the poset $\Pi(n,k)$.

For $w \in \mathcal{R}(n,k)$ let

$E(w) = \{ w_{i_1}, \dots, w_{i_j} :$ $i_1 < \dots < i_j$, the elt $w_{i_k} \in 2\mathbb{Z}$ with $w_r \geq w_{i_k}$ for some $r < i_k \}$

be the set of all repeated entries in w arranged by index.

($w = 122344 \implies E(w) = \{ w_3, w_6 \} = \{2, 4\}$).

The boundary map ∂ on $\mathcal{R}(n,k)$:

$$\partial(w) = \begin{cases} \displaystyle\sum_{w_{i_r} \in E(w)} (-1)^{r-1} w_1 \cdots w_{i_r - 1} (w_{i_r} - 1) w_{i_r + 1} \cdots w_n & \text{if } w \notin \mathcal{A}(n,k) \\ 0 & \text{if } w \in \mathcal{A}(n,k). \end{cases}$$

ex. $w = 122344$

$\mathbb{E}(w) = \{w_3, w_6\} = \{2, 4\}$

$\partial(w) = 121344 - 122343.$

Lemma: $\partial^2 = 0.$

Algebraic Morse Theory.

See [Kozlov 2005, Sköldberg 2006,
Jöllenbeck – Welker 2009].

P poset
Orient edges in Hasse diagram downwards.

A partial matching is a subset $M \subseteq P \times P$ s.t.

i. $(a,b) \in M \Rightarrow a \lessdot b$

ii. Each elt $a \in P$ belongs to at most one elt in M.

For $(a,b) \in M$ write $b = u(a)$, $a = d(b)$ "up" "down".

A partial matching is acyclic if there are no cycles in the directed Hasse diagram.

Matching M on $\Pi(n,k)$:

Let π_i be ~~first~~ entry in $\pi = \pi_1 \cdots \pi_n \in R(n,k)$ s.t. π ~~is~~ weakly decreasing:

$$\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{i-1} \geq \pi_i \cdots$$

and $\pi_{i-1} \geq \pi_i$ ~~is~~ strict ~~unless~~ both π_{i-1} + π_i even.

For π_i even:

$$d(\pi) = \pi_1 \pi_2 \cdots \pi_{i-1} (\pi_i - 1) \pi_{i+1} \cdots \pi_n.$$

For π_i odd.

$$u(\pi) = \pi_1 \pi_2 \cdots \pi_{i-1} (\pi_i + 1) \pi_{i+1} \cdots \pi_n.$$

Lemma: The unmatched words in $\Pi'(n,\omega)$ are of the form.

$$\underline{1}\cdots\underline{1}\ 2\ 3\cdots 3\ 4\ 5\cdots 5\ 6\cdots$$

Theorem: [Kozlov]
A partial matching on P is acyclic $\Longleftrightarrow$ There exists a linear extension L of P such that the elts a and $w(a)$ follow consecutively in L

Theorem: [Cai - Rearddy]
The matching described for $\Pi(n,\omega)$ is an acyclic matching.

Lemma: [Hersh - Shareshian - Stanton].

P graded poset supporting an algebraic complex $(\mathcal{C}, \partial)$.

Assume P has a Morse matching M s.t. for all $q = M(p)$ with $q < p$ one has $\partial_{p,q} \in \mathbb{F}^{\times}$.

If all unmatched elts occur in ranks of the same parity then.

$\dim H_i(\mathcal{C}, \partial) = |P_i^{\text{un} M}|$, that is, the # of unmatched elts of rank i.

Lemma: The weighted generating function of the unmatched words in $\Pi(n,k)$ is given by the q^2-binomial coefficient

$$\sum_{w\in U(n,k)}{}' \; wt(w) = \begin{bmatrix} n-1-\lfloor k/2 \rfloor \\ \lfloor k-1/2 \rfloor \end{bmatrix}_{q^2}.$$

Theorem: [Cai- Readdy]
The algebraic complex $(\mathcal{E}, \partial)$ supported by $\Pi(n,k)$ has basis for homology given by the increasing allowable RG-words in $\mathcal{A}(n,k)$.
Furthermore

$$\sum_{i\geq 0}{}' \dim(H_i)\, q^i = \begin{bmatrix} n-1-\lfloor k/2 \rfloor \\ \lfloor k-1/2 \rfloor \end{bmatrix}_{q^2}.$$

q-Stirling number of the first kind

$$c[n,k] = c[n-1,k-1] + [n-1]\, c[n-1,k]$$

with $c[n,0] = \delta_{n,0}$.

Recall Stirling number $c(n,k)$ counts # $\pi \in \mathfrak{S}_n$ with k disjoint cycles.

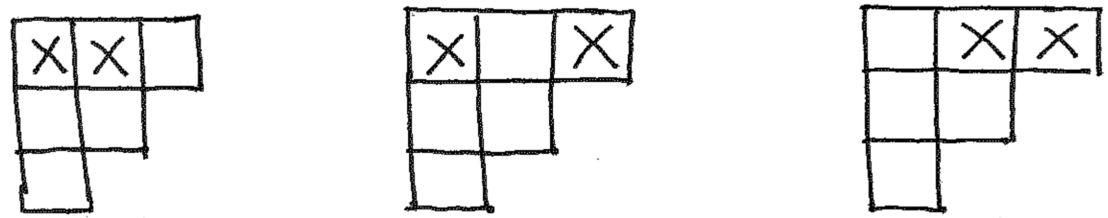
Theorem: [de Médicis - Leroux].

$$c[n,k] = \sum_{T \in \mathcal{P}(n-1,\, n-k)} q^{s(T)}$$

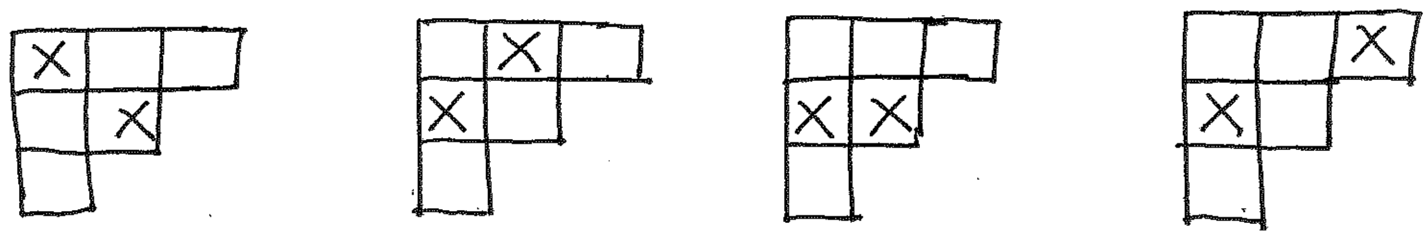
$\mathcal{P}(m,n)$ = set of ways to place n rocks on a length m staircase board with no two rocks in same column.

For $T \in \mathcal{P}(m,n)$, $s(T)$ = # of squares to the south of the rocks in T.

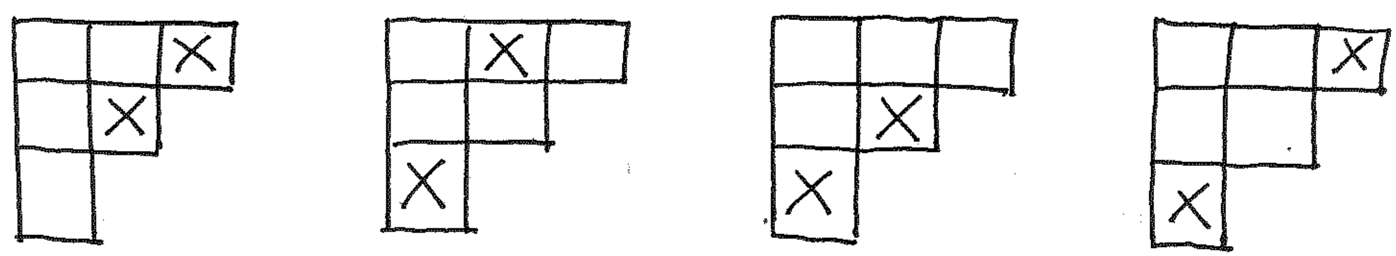
ex. $c[4,2]. = q^3 + 3q^2 + 4q + 3$



q^3 q^2 q^1

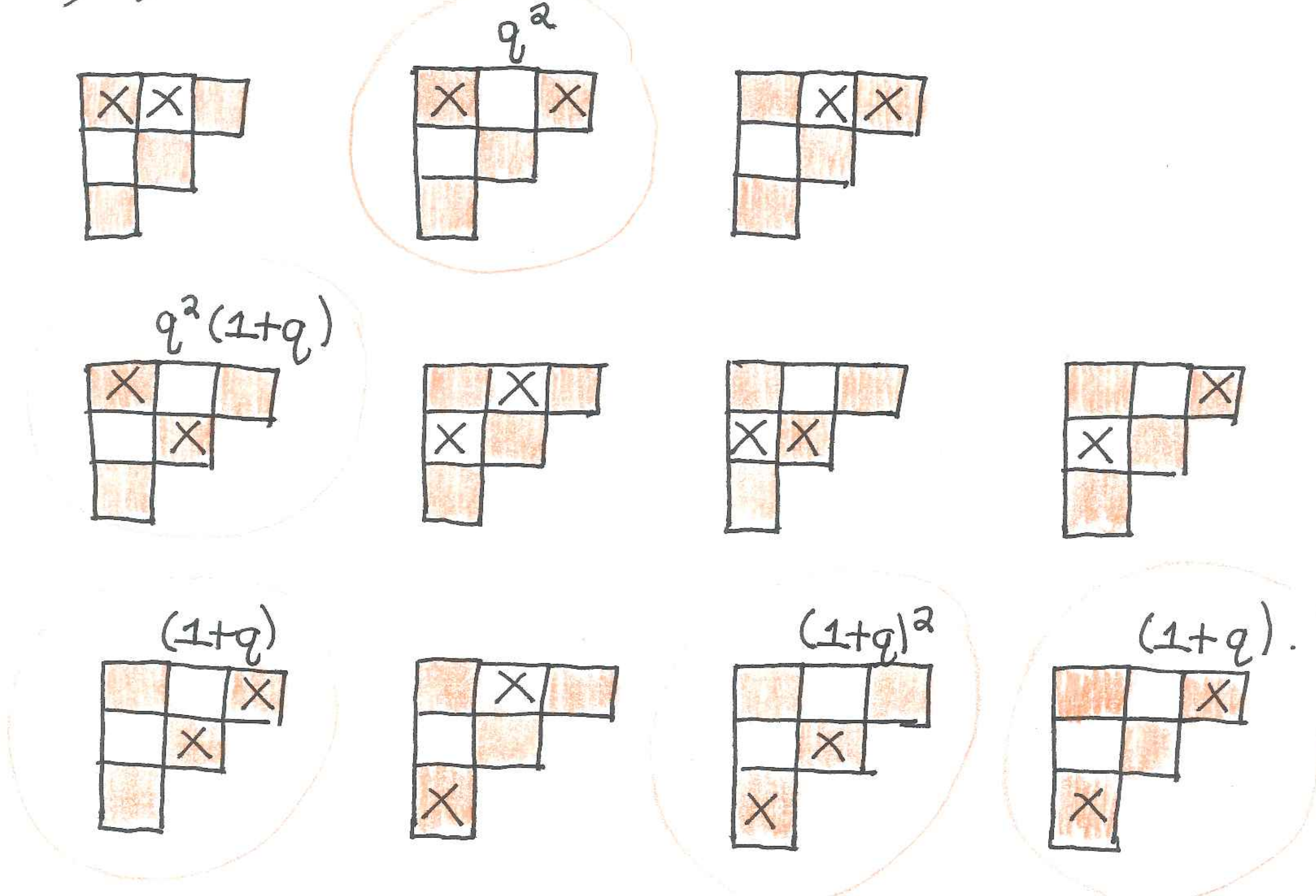


q^2 q^2 q^1 q^1



$1 = q^0$ q^1 1 1

To find a ~~subset~~ $Q(n-1, n-k)$ of $P(n-1, n-k)$:



$$C[4,2] = q^2(1+q) + (1+q)^2 + q^2 + 2\cdot(1+q)$$

$$\stackrel{\checkmark}{=} q^3 + 3q^2 + 4q + 3.$$

Theorem: [Cai- Readdy].

$$c[n,k] = \sum_{T \in Q(n-1,n-k)} q^{s(T)} (1+q)^{r(T)}$$

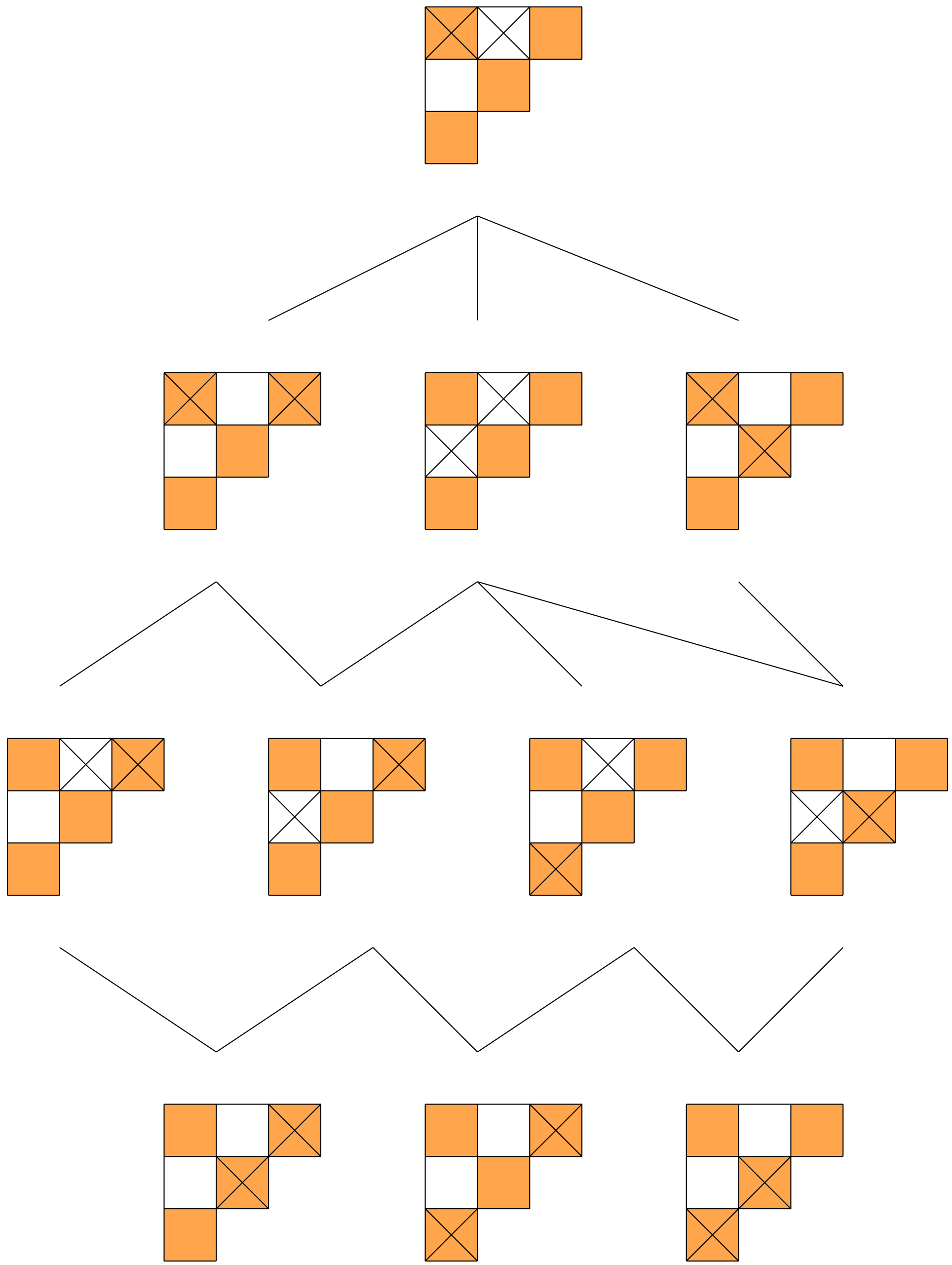
where $Q(n-1, n-k) \subseteq \mathcal{P}(n-1, n-k)$ are rook placements on the alternating shaded staircase board (shaded alternatingly starting from lowest diagonal),

$s(T)$ = # squares to the south of the rooks in T

$r(T)$ = # rooks not in first row.

The Stirling poset of the first kind $\Gamma(m,n)$

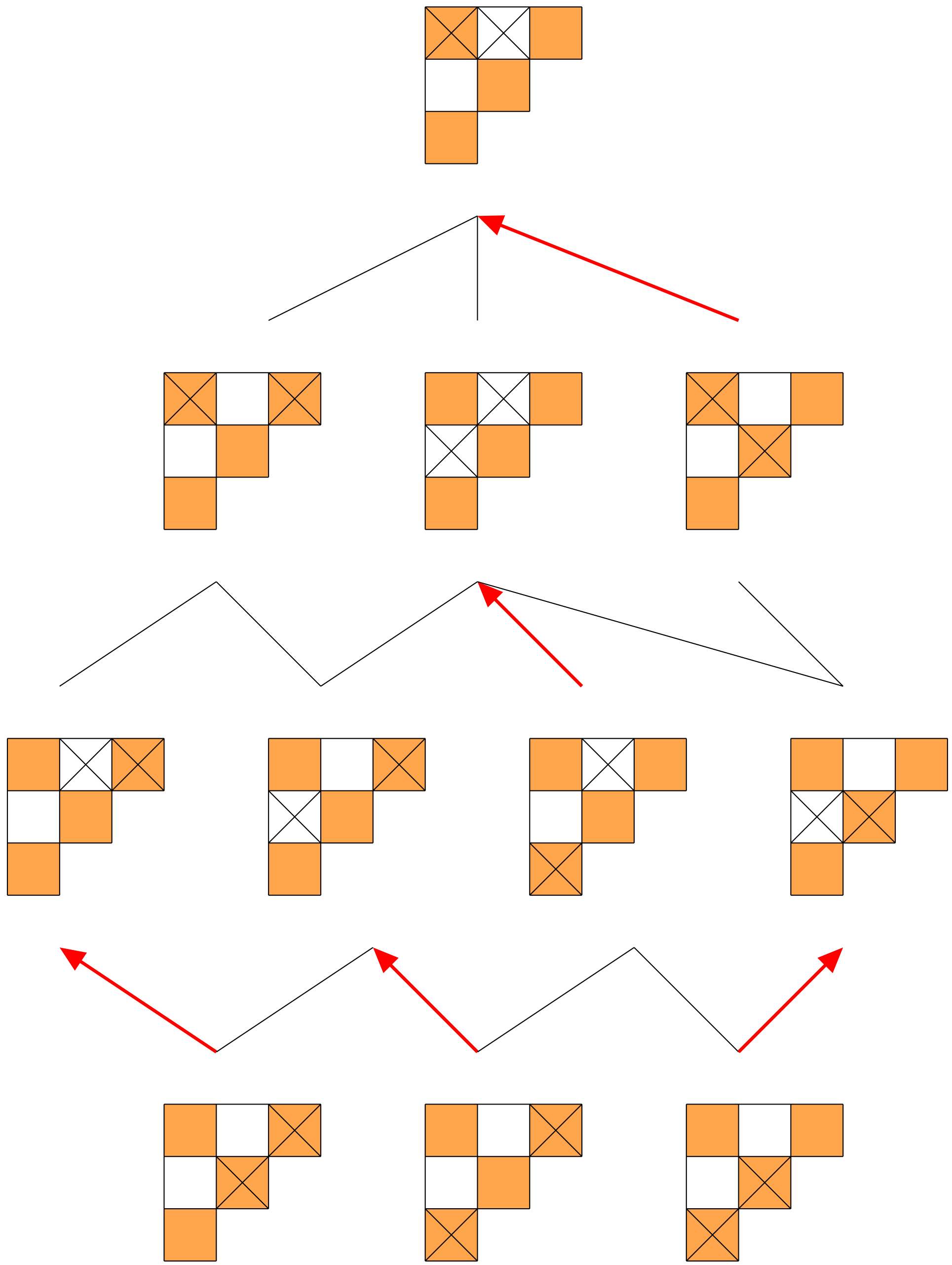
For $T, T' \in \mathcal{P}(m,n)$ let $T \lessdot T'$ if T' can be obtained from T by moving one rook to the left (west) or up (north)



Define a matching m:

For $T \in P(m,n)$, let r be the first rook (reading left to right) that is not in a shaded square in first row.

Match T to T' where T' is obtained from T by moving r one square down if r is not in a shaded square, or one square up if r is in a shaded square but not in first row.



Lemma: The unmatched rook placements in $\mathcal{P}(m,n)$ have all of the rooks occur in shaded squares in the first row.

Theorem: [Cai - Readdy].

i. The matching described for $\mathcal{P}(m,n)$ is acyclic.

ii.
$$\sum_{\substack{T \in \mathcal{P}(m,n) \\ T \text{ unmatched}}} wt(T) = q^{n(n-1)} \begin{bmatrix} \lfloor m+1/2 \rfloor \\ n \end{bmatrix}_{q^2}.$$

For $T \in \Gamma(m,n)$, let

$$N(T) = \{r_i : \text{the rook } r_i \text{ in } T \text{ is not in a shaded square}\}.$$

$$I(T) = \{i_j : r_{i_j} \in N(T) \text{ and } i_1 < i_2 < \cdots < i_{|N(T)|}\}.$$

The boundary map ∂ on $\Gamma(m,n)$:

$$\partial(T) = \sum_{r_{i_j} \in N(T)} (-1)^{j-1} \, T_{r_{i_j}}.$$

where $T_{r_{i_j}}$ is obtained by moving the rook r_{i_j} in T down by one square.

Theorem: [Cai-Readdy]

The algebraic complex $(\mathcal{C}, \partial)$ supported by $\Gamma(m,n)$ has basis for homology given by the rook placements in $\Phi(m,n)$ having all of the rooks occur in shaded squares in the first row.

Furthermore,

$$\sum_{i \geq 0} \dim(H_i)\, q^i = q^{n(n-1)} \begin{bmatrix} \lfloor m+1/2 \rfloor \\ n \end{bmatrix}_{q^2}.$$

Orthogonality

Recall the signed q-Stirling numbers of the first kind.

$$s_q[n,k] = (-1)^{n-k}\, c[n,k].$$

Known generating polynomials.

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n,k]\, x^k$$

$$x^n = \sum_{k=0}^{n} S_q[n,k]\, (x)_{k,q}$$

where

$$(x)_{n,q} = \prod_{m=0}^{n-1} (x - [m]_q).$$

def. Define the (q,t) Stirling numbers of the first and second kind by

$$s_{q,t}[n,k] = (-1)^{n-k} \sum_{T \in Q(n-1,n-k)} q^{s(T)} t^{r(T)}$$

$$S_{q,t}[n,k] = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot t^{B(w)},$$

respectively, where $t = q+1$.

Let

$$[k]_{q,t} = \begin{cases} (q^{k-2} + q^{k-4} + \dots + 1)\cdot t & \text{for } k \text{ even} \\ q^{k-1} + (q^{k-3} + q^{k-5} + \dots + 1)t & \text{for } k \text{ add.} \end{cases}$$

Theorem: [Cai- Readdy].
The generating polynomials for the (q,t)- Stirling numbers are

$$(x)_{n,q,t} = \sum_{k=0}^{n} s_{q,t}[n,k]\cdot x^k$$

$$x^n = \sum_{k=0}^{n} S_{q,t}[n,k]\,(x)_{k,q,t}$$

where

$$(x)_{n,q,t} = \prod_{m=0}^{n-1} (x - [m]_{q,t}).$$

Theorem: [de Médicis - Leroux].

The signed q-Stirling numbers $s_q[n,k]$ and the q-Stirling numbers $S_q[n,k]$ are orthogonal, that is,

$$\sum_{k=m}^{n} s_q[n,k]\, S_q[k,m] = \delta_{m,n}$$

and.

$$\sum_{k=m}^{n} S_q[n,k]\, s_q[k,m] = \delta_{m,n}$$

Furthermore, this orthogonality holds bijectively.

Theorem: [Cai - Readdy].

The (q,t)-Stirling numbers are orthogonal, that is,

$$\sum_{k=m}^{n} s_{q,t}[n,k] \cdot S_{q,t}[k,m] = \delta_{m,n}$$

and

$$\sum_{k=m}^{n} S_{q,t}[n,k]\, s_{q,t}[k,m] = \delta_{m,n}.$$

Furthermore, this orthogonality holds bijectively.

Thank you!