Affine and toric hyperplane arrangements

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Abstract

We extend the Billera–Ehrenborg–Readdy map between the intersection lattice and face lattice of a central hyperplane arrangement to affine and toric hyperplane arrangements. For toric arrangements, we also generalize Zaslavsky's fundamental results on the number of regions.

1 Introduction

Traditionally combinatorialists have studied topological objects that are spherical, such as polytopes, or which are homeomorphic to a wedge of spheres, such as those obtained from shellable complexes. In this paper we break from this practice and study hyperplane arrangements on the n-dimensional torus.

It is classical that the convex hull of a finite collection of points in Euclidean space is a polytope and its boundary is a sphere. The key ingredient in this construction is convexity. At the moment there is no natural analogue of this process to obtain a complex whose geometric realization is a torus.

In this paper we are taking a zonotopal approach to working with arrangements on the torus. Recall that every central hyperplane arrangement gives rise to a zonotope, that is, a spherical object. By considering an arrangement on the torus, we are able to obtain a subdivision whose geometric realization is indeed the torus. This amounts to restricting ourselves to arrangements whose subspaces in the Euclidean space $\mathbb{R}^n$ have coefficient matrices with rational entries. Under the quotient map $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n = T^n$ these subspaces are sent to subtori of the n-dimensional torus T^n.

Zaslavsky initiated the modern study of hyperplane arrangements in his fundamental treatise [45]. For early work in the field, see the references given in Grünbaum's text [27, Chapter 18]. Zaslavsky showed that evaluating the characteristic polynomial of a central hyperplane arrangement at -1 gives the number of regions in the complement of the arrangement. For central hyperplane arrangements, Bayer and Sturmfels proved the flag f-vector of the arrangement can be determined from the intersection lattice [6]; see Theorem 2.3. Billera, Ehrenborg and Readdy proved that the flag f-vector of the arrangement can be determined from the flag f-vector of the intersection lattice. Recall that the **cd**-index of a regular cell complex is an efficient tool to encode its flag f-vector without linear redundancies [5]. The Billera–Ehrenborg–Readdy theorem gives an explicit way to compute the **cd**-index of the arrangement [8].

The first step is to generalize Zaslavsky's theorem on the number of regions of a hyperplane arrangement to the toric case. Although there is no intersection lattice, one works with the intersection poset. From the Zaslavsky result we obtain a toric version of the Bayer–Sturmfels result for hyperplane arrangements, that is, there is a natural poset map from the face poset to the intersection poset and the cardinality of the inverse image of a chain under this map is described.

As in the case of a central hyperplane arrangement, our toric version of the Bayer–Sturmfels result determines the flag f-vector of the face poset of a toric arrangement in terms of its intersection poset. However, this is far from being explicit. Using the coalgebraic techniques from [18], we are able to determine the flag f-vector explicitly in terms of the flag f-vector of the intersection poset. Moreover, the answer is given by a **cd** type of polynomial. The flag f-vector of a regular spherical complex is encoded by the **cd**-index, a non-commutative polynomial in the variables **c** and **d**, whereas the n-dimensional toric analogue is a **cd**-polynomial plus the **ab**-polynomial $(\mathbf{a} - \mathbf{b})^{n+1}$.

Zaslavsky also showed that evaluating the characteristic polynomial of an affine arrangement at 1 gives the number of bounded regions in the complement of the arrangement. Thus we return to affine arrangements in Euclidean space with the twist that we study the *unbounded* regions. The unbounded regions form a spherical complex. In the case of central arrangements, this complex is exactly what was studied previously by Billera, Ehrenborg and Readdy [8]. For non-central arrangements, we determine the **cd**-index of this complex in terms of the lattice of unbounded intersections of the arrangement.

Interestingly, the techniques for studying toric arrangements and the unbounded complex of non-central arrangements are similar. Hence, we present these results in the same paper. For example, the toric and non-central analogues of the Bayer–Sturmfels theorem only differ in which Zaslavsky invariant is used. The coalgebraic translations of the two analogues involve exactly the same argument, and the resulting underlying maps φ_t (in the toric case) and φ_{ub} (in the non-central case) only differ slightly in their definitions.

We end with many open questions about subdivisions of manifolds.

2 Preliminaries

All the posets we will work with are graded, that is, posets having a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$, and rank function ρ. For two elements x and z in a graded poset P such that $x \leq z$, let $[x, z]$ denote the interval $\{y \in P : x \leq y \leq z\}$. Observe that the interval $[x, z]$ is itself a graded poset. Given a graded poset P of rank $n + 1$ and $S \subseteq \{1, \ldots, n\}$, the *$S$-rank-selected poset* $P(S)$ is the poset consisting of the elements $P(S) = \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. The partial order of $[x, y]$ and $P(S)$ are each inherited from that of P. For standard poset terminology, we refer the reader to Stanley's work [39].

We now review important results about hyperplane arrangements, the **cd**-index and coalgebraic techniques that are essential for proving the main results of this paper.

2.1 Hyperplane arrangements

Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ be a hyperplane arrangement in $\mathbb{R}^n$, that is, a finite collection of affine hyperplanes in n-dimensional Euclidean space $\mathbb{R}^n$. For brevity, throughout this paper we will often refer to a hyperplane arrangement as an arrangement. We call the arrangement *essential* if there is no non-zero vector orthogonal to all the hyperplanes in $\mathcal{H}$. Otherwise we call the arrangement *inessential*. An inessential arrangement can be made essential by quotienting out by the subspace $V^{\perp}$ where V is the subspace orthogonal to the hyperplanes in $\mathcal{H}$. In this paper we are only interested in essential arrangements.

Observe that the intersection $\bigcap_{i=1}^{m} H_i$ of all of the hyperplanes in an essential arrangement is either the empty set $\emptyset$ or a singleton point. We call an arrangement *central* if the intersection of all the hyperplanes is one point. We may assume that this point is the origin $\mathbf{0}$ and hence all of the hyperplanes are codimension 1 subspaces. If the intersection is the empty set, we call the arrangement *non-central*.

The *intersection lattice* $\mathcal{L}$ is the lattice formed by ordering all the intersections of hyperplanes in $\mathcal{H}$ by reverse inclusion. If the intersection of all the hyperplanes in a given arrangement is empty, then we include the empty set $\emptyset$ as the the maximal element in the intersection lattice. If the arrangement is central the maximal element is $\{\mathbf{0}\}$. In all cases, the minimal element of $\mathcal{L}$ will be all of $\mathbb{R}^n$.

For a hyperplane arrangement $\mathcal{H}$ with intersection lattice $\mathcal{L}$ its *characteristic polynomial* is defined by

$$\chi(\mathcal{H}; t) = \sum_{\substack{x \in \mathcal{L} \\ x \neq \emptyset}} \mu(\hat{0}, x) \cdot t^{\dim(x)},$$

where μ denotes the Möbius function. The characteristic polynomial is a combinatorial invariant of the arrangement. The fundamental result of Zaslavsky [45] is that this invariant determines the number and type of regions.

Theorem 2.1 (Zaslavsky) *For a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ the number of regions is given by $(-1)^n \cdot \chi(\mathcal{H}; t = -1)$. Furthermore, the number of bounded regions is given by $(-1)^n \cdot \chi(\mathcal{H}; t = 1)$.*

For a graded poset P, define the two Zaslavsky invariants Z and Z_b by

$$\begin{aligned} Z(P) &= \sum_{\hat{0} \leq x \leq \hat{1}} (-1)^{\rho(x)} \cdot \mu(\hat{0}, x), \\ Z_b(P) &= (-1)^{\rho(P)} \cdot \mu(P). \end{aligned}$$

In order to work with Zaslavsky's result, we need the following reformulation of Theorem 2.1.

Theorem 2.2 *(i) For a central hyperplane arrangement $\mathcal{H}$ the number of regions is given by $Z(\mathcal{L})$, where $\mathcal{L}$ is the intersection lattice of the arrangement $\mathcal{H}$.*

(ii) For a non-central hyperplane arrangement $\mathcal{H}$ the number of regions is given by $Z(\mathcal{L}) - Z_b(\mathcal{L})$, where $\mathcal{L}$ is the intersection lattice of the arrangement $\mathcal{H}$. The number of bounded regions is given by $Z_b(\mathcal{L})$.

Given a central hyperplane arrangement $\mathcal{H}$ there are two associated lattices, namely the intersection lattice $\mathcal{L}$ and the lattice T of faces of the arrangement. The lattice of faces can be seen as the face poset of the CW-complex obtained by intersecting the arrangement $\mathcal{H}$ with a sphere of radius R centered at the origin. Each hyperplane corresponds to a great circle on the sphere. An alternative way to view the lattice of faces T is that the dual lattice T^* is the face lattice of the associated zonotope.

Let $\mathcal{L} \cup \{\hat{0}\}$ denote the intersection lattice with a new minimal element $\hat{0}$ adjoined. Define an order- and rank-preserving map z from the dual lattice T^* to the augmented lattice $\mathcal{L} \cup \{\hat{0}\}$ by sending a face of the arrangement to its affine hull. Note that under the map z the minimal element of T^* is mapped to the minimal element of $\mathcal{L} \cup \{\hat{0}\}$. Bayer and Sturmfels [6] proved the following result about the inverse image of a chain under the map z.

Theorem 2.3 (Bayer–Sturmfels) *Let $\mathcal{H}$ be a central hyperplane arrangement with intersection lattice $\mathcal{L}$. Let $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ be a chain in $\mathcal{L} \cup \{\hat{0}\}$. Then the cardinality of the inverse image of the chain c under the map $z : T^* \longrightarrow \mathcal{L} \cup \{\hat{0}\}$ is given by the product*

$$|z^{-1}(c)| = \prod_{i=2}^{k} Z([x_{i-1}, x_i]).$$

2.2 The cd-index and coalgebraic techniques

Let P be a graded poset of rank $n+1$ with rank function ρ and let $\mathbf{a}$ and $\mathbf{b}$ be two non-commutative variables. For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the poset P, define its *weight* to be

$$\mathrm{wt}(c) = (\mathbf{a}-\mathbf{b})^{\rho(x_0,x_1)-1} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x_1,x_2)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x_{k-1},x_k)-1}, \tag{2.1}$$

where $\rho(x, y)$ denotes the rank difference $\rho(y) - \rho(x)$. The **ab**-*index* of P is the noncommutative polynomial defined by

$$\Psi(P) = \sum_{c} \mathrm{wt}(c),$$

where the sum is over all chains c in the poset P. Equivalently, Stanley's recursion for the **ab**-index of a graded poset is [41, Equation (7)]

$$\Psi(P) = (\mathbf{a}-\mathbf{b})^{\rho(P)-1} + \sum_{\hat{0}<x<\hat{1}} (\mathbf{a}-\mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]). \tag{2.2}$$

The original way to describe the **ab**-index is in terms of the flag f- and h-vectors. For $S = \{s_1 < \cdots < s_{k-1}\}$ a subset of $\{1, \ldots, n\}$ define f_S to be the number of chains c that have elements with ranks in the set S, that is,

$$f_S = |\{c \ : \ \rho(x_1) = s_1, \ldots, \rho(x_{k-1}) = s_{k-1}\}|.$$

Observe that f_S is the number of maximal chains in the rank-selected poset $P(S)$. The flag h-vector is obtained by the relation (here we also present its inverse)

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T \quad \text{and} \quad f_S = \sum_{T \subseteq S} h_T.$$

Recall that by Philip Hall's theorem the Möbius function of the S-rank-selected poset $P(S)$ is given by $\mu(P(S)) = (-1)^{|S|-1} \cdot h_S$. For S a subset of $\{1, \ldots, n\}$ let u_S be the monomial $u_S = u_1 \cdots u_n$ where $u_i = \mathbf{b}$ if $i \in S$ and $u_i = \mathbf{a}$ if $i \notin S$. Then the **ab**-index is given by

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where the sum is over all subsets $S \subseteq \{1, \ldots, n\}$.

A poset P is *Eulerian* if every interval $[x, y]$, where $x < y$, satisfies the Euler-Poincaré relation, that is, there are the same number of elements of odd as even rank. Equivalently, the Möbius function of P is given by $\mu(x, y) = (-1)^{\rho(x,y)}$ for all $x \leq y$ in P. The quintessential result is that the **ab**-index of Eulerian posets has the following form.

Theorem 2.4 *The* **ab**-*index of an Eulerian poset P can be expressed in terms of the noncommutative variables* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$.

This theorem was originally proved for face lattices of convex polytopes by Bayer and Klapper [5]. Stanley provided a proof for all Eulerian posets [41]. There are proofs which have both used and revealed the underlying algebraic structure. See for instance [14, 21]. When the **ab**-index $\Psi(P)$ is written in terms of **c** and **d**, the resulting polynomial is called the **cd**-*index*. There are linear relations holding among the entries of the flag f-vector of an Eulerian poset, known as the generalized Dehn-Sommerville relations; see [3]. The importance of the **cd**-index is that it removes all of these linear redundancies among the flag f-vector entries.

For a graded poset P define P^* to be the dual poset, that is, the poset having the same underlying set as P but with the order relation reversed: $x <_{P^*} y$ if and only if $y <_P x$. Define the reverse of an **ab**-monomial $u = u_1 u_2 \cdots u_n$ to be $u^* = u_n \cdots u_2 u_1$ and extend by linearity to an involution on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. Since $\mathbf{c}^* = \mathbf{c}$ and $\mathbf{d}^* = \mathbf{d}$, this involution applied to a **cd**-monomial reverses the **cd**-monomial. Finally, for a graded poset P we have $\Psi(P)^* = \Psi(P^*)$.

A coproduct Δ on a free $\mathbb{Z}$-module C is a linear map $\Delta : C \longrightarrow C \otimes C$. We say that the coproduct is coassociative if $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. In order to be explicit, we use the Sweedler notation [44] for writing the coproduct, that is, we write

$$\Delta(w) = \sum_w w_{(1)} \otimes w_{(2)}.$$

For instance, the condition for coassociativity can be written as

$$\sum_w \sum_{w_{(1)}} w_{(1,1)} \otimes w_{(1,2)} \otimes w_{(2)} = \sum_w \sum_{w_{(2)}} w_{(1)} \otimes w_{(2,1)} \otimes w_{(2,2)}.$$

In fact, coassociativity allows us to define the k-ary coproduct $\Delta^k : C \longrightarrow C^{\otimes k}$ by the recursion $\Delta^1 = \mathrm{id}$ and $\Delta^k = (\Delta^{k-1} \otimes \mathrm{id}) \circ \Delta$. The Sweedler notation for the k-ary coproduct is

$$\Delta^k(w) = \sum_w w_{(1)} \otimes w_{(2)} \otimes \cdots \otimes w_{(k)}.$$

Define a coproduct Δ on the algebra $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ by letting Δ satisfy the following identities: $\Delta(1) = 0$, $\Delta(\mathbf{a}) = \Delta(\mathbf{b}) = 1 \otimes 1$ and the Newtonian condition

$$\Delta(u \cdot v) = \sum_{u} u_{(1)} \otimes u_{(2)} \cdot v + \sum_{v} u \cdot v_{(1)} \otimes v_{(2)}. \tag{2.3}$$

For an **ab**-monomial $u = u_1 u_2 \cdots u_n$ we have that

$$\Delta(u) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n.$$

The fundamental result for this coproduct is that the **ab**-index is a coalgebra homomorphism [18]. We express this result as the following identity.

Theorem 2.5 (Ehrenborg–Readdy) *For a graded poset P with* **ab***-index w and k-multilinear map M on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$, the following coproduct identity holds:*

$$\sum_{c} M(\Psi([x_0, x_1]), \Psi([x_1, x_2]), \ldots, \Psi([x_{k-1}, x_k])) = \sum_{w} M(w_{(1)}, w_{(2)}, \ldots, w_{(k)}),$$

where the first sum is over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ of length k and the second sum is the Sweedler notation of the k-ary coproduct.

2.3 The cd-index of the face poset of a central arrangement

We recall the definition of the omega map [8].

Definition 2.6 *The linear map ω from $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ is formed by replacing every occurrence of* **ab** *in a given* **ab***-monomial by $2\mathbf{d}$ and replacing the remaining letters by $\mathbf{c}$.*

For a central hyperplane arrangement $\mathcal{H}$ the **cd**-index of the face poset is computed as follows [8]:

Theorem 2.7 (Billera–Ehrenborg–Readdy) *Let $\mathcal{H}$ be a central hyperplane arrangement with intersection lattice $\mathcal{L}$ and face lattice T. Then the* **cd***-index of the face lattice T is given by*

$$\Psi(T) = \omega(\mathbf{a} \cdot \Psi(\mathcal{L}))^*.$$

We review the basic ideas behind the proof of this theorem. We will refer back to them when we prove similar results for toric and affine arrangements in Sections 3 and 4.

Define three linear operators κ, β and η on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ by

$$\kappa(v) = \begin{cases} (\mathbf{a} - \mathbf{b})^m & \text{if } v = \mathbf{a}^m \text{ for some } m \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta(v) = \begin{cases} (\mathbf{a}-\mathbf{b})^m & \text{if } v = \mathbf{b}^m \text{ for some } m \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\eta(v) = \begin{cases} 2 \cdot (\mathbf{a}-\mathbf{b})^{m+k} & \text{if } v = \mathbf{b}^m\mathbf{a}^k \text{ for some } m, k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that κ and β are both algebra maps. The following relations hold for a poset P; see [8, Section 5]:

$$\begin{aligned} \kappa(\Psi(P)) &= (\mathbf{a}-\mathbf{b})^{\rho(P)-1}, && (2.4) \\ \beta(\Psi(P)) &= Z_b(P) \cdot (\mathbf{a}-\mathbf{b})^{\rho(P)-1}, && (2.5) \\ \eta(\Psi(P)) &= Z(P) \cdot (\mathbf{a}-\mathbf{b})^{\rho(P)-1}. && (2.6) \end{aligned}$$

For $k \geq 1$ the operator φ_k is defined by the coalgebra expression

$$\varphi_k(v) = \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(k)}),$$

where the coproduct splits v into k parts. Finally φ is defined as the sum

$$\varphi(v) = \sum_{k \geq 1} \varphi_k(v).$$

Note that in this expression only a finite number of terms are non-zero. The connection with hyperplane arrangements is given by the following proposition.

Proposition 2.8 *The* **ab***-index of the lattice of faces of a central hyperplane arrangement is given by*

$$\Psi(T) = \varphi(\Psi(\mathcal{L} \cup \{\hat{0}\}))^*.$$

The function φ satisfies the initial conditions $\varphi(1) = 1$ and $\varphi(\mathbf{b}) = 2 \cdot \mathbf{b}$ and the recursions:

$$\begin{aligned} \varphi(v \cdot \mathbf{a}) &= \varphi(v) \cdot \mathbf{c}, && (2.7) \\ \varphi(v \cdot \mathbf{bb}) &= \varphi(v \cdot \mathbf{b}) \cdot \mathbf{c}, && (2.8) \\ \varphi(v \cdot \mathbf{ab}) &= \varphi(v) \cdot \mathbf{d}, && (2.9) \end{aligned}$$

for an **ab**-monomial v; see [8, Section 5]. These recursions culminate in the following result.

Proposition 2.9 *For an* **ab***-monomial w that begins with* **a***, the two maps φ and ω coincide, that is, $\varphi(w) = \omega(w)$.*

Finally, Theorem 2.7 follows by Proposition 2.9 and from the fact that $\Psi(\mathcal{L} \cup \{\hat{0}\}) = \mathbf{a} \cdot \Psi(\mathcal{L})$.

2.4 Regular subdivisions of manifolds

A regular subdivision of the sphere has an Eulerian face poset and hence a **cd**-index. For regular subdivisions of compact manifolds, a similar result holds. This was independently observed by Ed Swartz [43].

Theorem 2.10 *Let Ω be a regular CW-complex whose geometric realization is a compact n-dimensional manifold M. Let $\chi(M)$ denote the Euler characteristic of M. Then the* **ab***-index of the face poset P of Ω has the following form.*

(i) If n is odd then P is an Eulerian poset and hence $\Psi(P)$ can written in terms of **c** *and* **d***.*

(ii) If n is even then $\Psi(P)$ has the form

$$\Psi(P) = \left(1 - \frac{\chi(M)}{2}\right) \cdot (\mathbf{a} - \mathbf{b})^{n+1} + \frac{\chi(M)}{2} \cdot \mathbf{c}^{n+1} + \Phi,$$

where Φ is a homogeneous **cd***-polynomial of degree $n+1$ and where the term $\mathbf{c}^{n+1}$ does not occur.*

Proof: Observe that the poset P has rank $n+2$. By [39, Theorem 3.8.9] we know that every interval $[x, y]$ strictly contained in P is Eulerian. When the rank of P is odd this implies that P is also Eulerian and hence its **ab**-index can be expressed as a **cd**-index. When n is even, we use [14, Theorem 4.2] to conclude that the **ab**-index of P belongs to $\mathbb{R}\langle \mathbf{c}, \mathbf{d}, (\mathbf{a} - \mathbf{b})^{n+1} \rangle$. Since $\Psi(P)$ has degree $n+1$, the **ab**-index $\Psi(P)$ can be written in the form

$$\Psi(P) = c_1 \cdot (\mathbf{a} - \mathbf{b})^{n+1} + c_2 \cdot \mathbf{c}^{n+1} + \Phi,$$

where Φ is a homogeneous **cd**-polynomial of degree $n+1$ not contain any $\mathbf{c}^{n+1}$ terms. By looking at the coefficients of $\mathbf{a}^{n+1}$ and $\mathbf{b}^{n+1}$, we have $c_1 + c_2 = 1$ and $c_2 - c_1 = \mu(P) = \chi(M) - 1$, where the last identity is again [39, Theorem 3.8.9]. Solving for c_1 and c_2 proves the result. $\square$

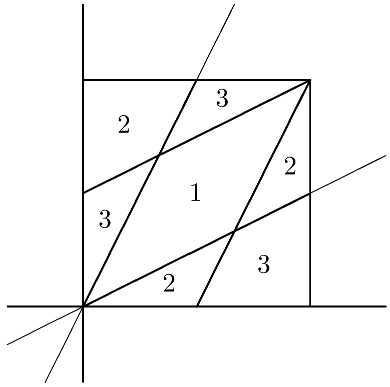
For the n-dimensional torus Theorem 2.10 can be expressed as follows.

Corollary 2.11 *Let Ω be a regular CW-complex whose geometric realization is the n-dimensional torus T^n. Then the* **ab***-index of the face poset P of Ω has the following form:*

$$\Psi(P) = (\mathbf{a} - \mathbf{b})^{n+1} + \Phi,$$

where Φ is a homogeneous **cd***-polynomial of degree $n+1$ and where the term $\mathbf{c}^{n+1}$ does not occur.*

Proof: When n is even this is Theorem 2.10. When n is odd this is Theorem 2.10 together with the two facts that $\chi(T^n) = 0$ and $(\mathbf{a} - \mathbf{b})^{n+1} = (\mathbf{c}^2 - 2\mathbf{d})^{(n+1)/2}$. $\square$



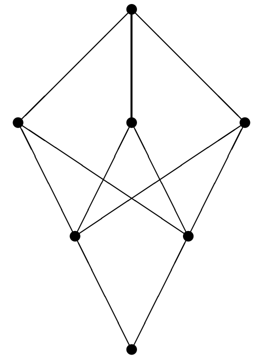


Figure 1: A toric line arrangement which subdivides the torus T^2 into a non-regular CW-complex and its intersection poset.

3 Toric arrangements

3.1 Toric subspaces and arrangements

The n-dimensional torus T^n is defined as the quotient $\mathbb{R}^n/\mathbb{Z}^n$. Let V be a k-dimensional affine subspace in $\mathbb{R}^n$ with rational coefficients. That is, V has the form

$$V = \{\vec{v} \in \mathbb{R}^n \ : \ A\vec{v} = \vec{b}\},$$

where the matrix A has rational entries and the vector $\vec{b}$ is allowed to have real entries. Let $\overline{V}$ denote the image of V under the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. We call the image $\overline{V}$ a *toric subspace* of the torus T^n. When we remove the condition that the matrix A is rational, the image is no longer homeomorphic to a torus.

The intersection of two toric subspaces is in general not a toric subspace, but instead is the disjoint union of a finite number of toric subspaces. For two affine subspaces V and W with rational coefficients, we have that $\overline{V \cap W} \subseteq \overline{V} \cap \overline{W}$. In general, this containment is strict.

A *toric hyperplane arrangement* $\mathcal{H} = \{H_1, \ldots, H_m\}$ is a finite collection of toric hyperplanes. Define the *intersection poset* $\mathcal{P}$ of a toric arrangement to be the set of all connected components in all possible intersections of the toric hyperplanes, that is, all connected components of $\bigcap_{i \in S} H_i$ where $S \subseteq \{1, \ldots, m\}$, together with the empty set. We order the elements of the intersection poset $\mathcal{P}$ by reverse inclusion, that is, the torus T^n is the minimal element of $\mathcal{P}$ corresponding to the empty intersection, and the empty set is the maximal element. A toric subspace V is contained in the intersection poset $\mathcal{P}$ if there are toric hyperplanes $H_{i_1}, \ldots, H_{i_k}$ in the arrangement such that $V \subseteq H_{i_1} \cap \cdots \cap H_{i_k}$ and there is no toric subspace W satisfying $V \subset W \subseteq H_{i_1} \cap \cdots \cap H_{i_k}$. In other words, V has to be a maximal toric subspace in some intersection of toric hyperplanes from the arrangement.

The notion of using the intersection poset can be found in work of Zaslavsky, where he considers topological dissections [46]. In this setting there there is not an intersection lattice, but rather an

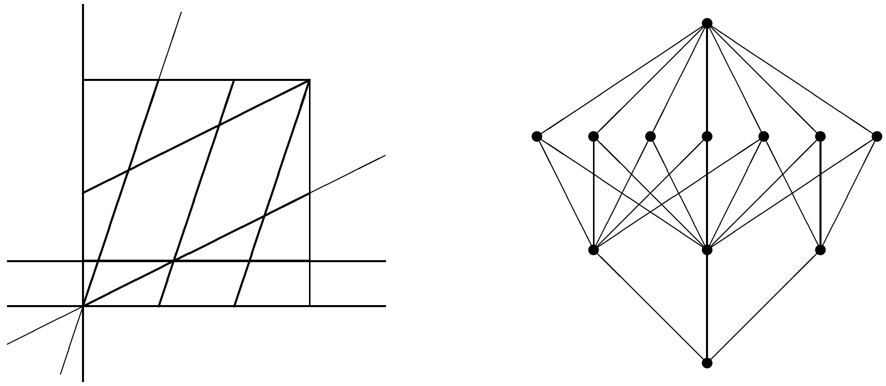


Figure 2: A toric line arrangement and its intersection poset.

intersection poset.

To every toric hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_m\}$ there is an associated periodic hyperplane arrangement $\widetilde{\mathcal{H}}$ in the Euclidean space $\mathbb{R}^n$. Namely, the inverse image of the toric hyperplane H_i under the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is the union of parallel integer translates of a real hyperplane. Let $\widetilde{\mathcal{H}}$ be the collection of all these integer translates. Observe that every face of the toric arrangement $\mathcal{H}$ can be lifted to a parallel class of faces in the periodic real arrangement $\widetilde{\mathcal{H}}$.

For a toric hyperplane arrangement $\mathcal{H}$ define the *toric characteristic polynomial* to be

$$\chi(\mathcal{H}; t) = \sum_{\substack{x \in \mathcal{P} \\ x \neq \emptyset}} \mu(\hat{0}, x) \cdot t^{\dim(x)}.$$

Example 3.1 Consider the line arrangement consisting of the two lines $y = 2 \cdot x$ and $x = 2 \cdot y$ in the plane $\mathbb{R}^2$. In $\mathbb{R}^2$ they intersect in one point, namely the origin, whereas on the torus T^2 they intersect in three points, namely $(0, 0)$, $(2/3, 1/3)$ and $(1/3, 2/3)$. The characteristic polynomial is given by $\chi(\mathcal{H}; t) = t^2 - 2 \cdot t + 3$. However, this arrangement is not regular, since the induced subdivision is not regular. The boundary of each region is a wedge of two circles. See Figure 1.

Example 3.2 Consider the line arrangement consisting of the three lines $y = 3 \cdot x$, $x = 2 \cdot y$ and $y = 1/5$. It subdivides the torus into a regular CW-complex. The subdivision and the associated intersection poset are shown in Figure 2. The characteristic polynomial is given by $\chi(\mathcal{H}; t) = t^2 - 3 \cdot t + 8$. Furthermore, the **ab**-index of the subdivision of the torus is given by $\Psi(T_t) = (\mathbf{a} - \mathbf{b})^3 + 7 \cdot \mathbf{dc} + 8 \cdot \mathbf{cd}$, as the following calculation shows.

S	f_S	h_S	u_S	$(\mathbf{a}-\mathbf{b})^3$	$7 \cdot \mathbf{dc}$	$8 \cdot \mathbf{cd}$
$\emptyset$	1	1	**aaa**	1	0	0
$\{1\}$	7	6	**baa**	-1	7	0
$\{2\}$	15	14	**aba**	-1	7	8
$\{3\}$	8	7	**aab**	-1	0	8
$\{1,2\}$	30	9	**bba**	1	0	8
$\{1,3\}$	30	16	**bab**	1	7	8
$\{2,3\}$	30	8	**abb**	1	7	0
$\{1,2,3\}$	60	-1	**bbb**	-1	0	0

Observe that the sum of the three last columns is equal to the flag h-vector.

We now give a natural interpretation of the toric characteristic polynomial. Let G be the collection of all toric subspaces of the n-dimensional torus T^n together with the empty set. Observe that T^n also belongs to G and that G is closed under finite intersections. Let L be the distributive lattice consisting of all subsets of the torus T^n that are obtained from the collection G by finite intersections, finite unions and complements. The set G is the generating set for the lattice L. A *valuation* v is a function on the lattice L such that $v(\emptyset) = 0$ and $v(A) + v(B) = v(A \cap B) + v(A \cup B)$ for all sets $A, B \in L$.

Similar to Theorem 2.1 in [19] we have:

Theorem 3.3 *There is a valuation v on the distributive lattice L such that the valuation v applied to a k-dimensional toric subspace V is t^k, that is, $v(V) = t^k$.*

Proof: Groemer's integral theorem [26] (see also [33, Theorem 2.2.1]) states that a function v defined on a generating set G extends to a valuation on the distributive lattice if for all $V_1, \ldots, V_m$ in G such that $V_1 \cup \cdots \cup V_m \in G$ the inclusion-exclusion formula holds:

$$v(V_1 \cup \cdots \cup V_m) = \sum_i v(V_i) - \sum_{i<j} v(V_i \cap V_j) + \cdots . \qquad (3.1)$$

For toric subspaces the condition that $V_1 \cup \cdots \cup V_m$ belongs to the generating set G implies that $V_1 \cup \cdots \cup V_m = V_i$ for some index i. It follows that the inclusion-exclusion formula (3.1) holds trivially. $\square$

By Möbius inversion we directly have the following theorem. The proof is standard. See the references [1, 10, 19, 29].

Theorem 3.4 *The characteristic polynomial of a toric arrangement is given by*

$$\chi(\mathcal{H}) = v\left(T^n - \bigcup_{i=1}^{m} H_i\right).$$

Observe that the Euler valuation of a k-dimensional torus is given by the Kronecker delta $\delta_{k,0}$. This corresponds to setting $t = 0$ in the valuation. Using that the Euler valuation of a n-dimensional region is $(-1)^n$, we have the next result. The proof is analogous to the proofs in [19, 20].

Theorem 3.5 *Let $\mathcal{H}$ be a toric hyperplane arrangement on the n-dimensional torus T^n that subdivides the torus into regions that are open n-dimensional balls. Then the number of regions of the arrangement is given by $(-1)^n \cdot \chi(\mathcal{H}; t = 0)$.*

Continuation of Example 3.1 Setting $t = 0$ in the characteristic polynomial in Example 3.1 we obtain 3, which is indeed is the number of regions of this arrangement.

We call a toric hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_m\}$ *rational* if each hyperplane H_i is of the form $\vec{a}_i \cdot \vec{x} = b_i$ where the vector $\vec{a}_i$ has integer entries and b_i is an integer. This is equivalent to assuming every constant b_i is rational since every vector $\vec{a}_i$ was already assumed to be rational. In what follows it will be convenient to assume every coefficient is integral in a given rational arrangement.

Define $M(\mathcal{H})$ to be the least common multiple of all the $n \times n$ minors of the $n \times m$ matrix $(\vec{a}_1, \ldots, \vec{a}_m)$. We can now give different interpretation of the toric chromatic polynomial by counting lattice points.

Theorem 3.6 *For a rational hyperplane arrangement $\mathcal{H}$ there exists a constant k such that for every $q > k$ and q a multiple of $M(\mathcal{H})$, the toric characteristic polynomial evaluated at q is given by the number of lattice points in $\left(\frac{1}{q}\mathbb{Z}\right)^n / \mathbb{Z}^n$ that do not lie on any of the toric hyperplanes H_i, that is,*

$$\chi(\mathcal{H}; t = q) = \left| \left(\frac{1}{q}\mathbb{Z}\right)^n / \mathbb{Z}^n - \bigcup_{i=1}^{m} H_i \right| .$$

The condition that q is a multiple of $M(\mathcal{H})$ implies that every subspace x in the intersection poset $\mathcal{P}$ intersects the toric lattice $\left(\frac{1}{q}\mathbb{Z}\right)^n / \mathbb{Z}^n$ in exactly $q^{\dim(x)}$ points. Theorem 3.6 now follows by Möbius inversion. This theorem is the toric analogue of Athanasiadis' finite field method. See especially [2, Theorem 2.1].

In the case when $M(\mathcal{H}) = 1$, the toric arrangement $\mathcal{H}$ is called *unimodular.* Novik, Postnikov and Sturmfels [37] state Theorem 3.5 in the special case of unimodular arrangements. Their first proof is based upon Zaslavsky's result on the number of bounded regions in an affine arrangement. The second proof, due to Vic Reiner, is equivalent to our proof for arbitrary toric arrangements.

We end this subsection by discussing an application to graphical arrangements. For a graph G on the vertex set $\{1, \ldots, n\}$ define the graphical arrangement $\mathcal{H}_G$ to be the collection of hyperplanes of the form $x_i = x_j$ for each edge ij in the graph G.

Corollary 3.7 *For a connected graph G on n vertices the regions of the complement of the graphical arrangement $\mathcal{H}_G$ on the torus T^n are each homotopy equivalent to the 1-dimensional torus T^1 and the number of regions is given by $(-1)^{n-1}$ times the linear coefficient of the chromatic polynomial of G.*

Proof: The chromatic polynomial of the graph G is equal to the characteristic polynomial of the graphical arrangement $\mathcal{H}_G$. Furthermore, the intersection lattice of the real arrangement $\mathcal{H}_G$ is the same as the intersection poset of the toric arrangement $\mathcal{H}_G$. Translating the graphic arrangement in the direction $(1, \ldots, 1)$ leaves the arrangement on the torus invariant. Since G is connected this is the only direction that leaves the arrangement invariant. Hence each region is homotopy equivalent to T^1. By adding the hyperplane $x_1 = 0$ to the arrangement we obtain a new arrangement $\mathcal{H}'$ with the same number of regions, but with each region homeomorphic to a ball. Since the intersection lattice of $\mathcal{H}'$ is just the Cartesian product of the two-element poset with the intersection lattice of $\mathcal{H}_G$, we have

$$\chi(\mathcal{H}', t) = (t-1) \cdot \chi(\mathcal{H}_G, t)/t.$$

The number of regions is obtained by setting $t = 0$ in this equality. $\square$

A similar statement holds for graphs that are not connected. The result follows from the fact that the complement of the graphical arrangement is the product of the complements of each connected component.

Corollary 3.8 *For a graph G on n vertices consisting of k components, the regions of the complement of the graphical arrangement $\mathcal{H}_G$ on the torus T^n are each homotopy equivalent to the k-dimensional torus T^k and the number of regions is given by $(-1)^{n-k}$ times the coefficient of t^k in the chromatic polynomial of G.*

Stanley [38] proved the celebrated result that the chromatic polynomial of a graph evaluated at $t = -1$ is $(-1)^n$ times the number of acyclic orientations of the graph. A similar interpretation for the linear coefficient of the chromatic polynomial is due to Greene and Zaslavsky [25]:

Theorem 3.9 (Greene–Zaslavsky) *Let G be a connected graph and v a given vertex of the graph. Then the linear coefficient of the chromatic polynomial is $(-1)^{n-1}$ times the number of acyclic orientations of the graph such that the only sink is the vertex v.*

Proof: It is enough to give a bijection between regions of the complement of the graphical arrangement on the torus T^n and acyclic orientations with the vertex v as the unique sink. For a region R of the arrangement intersect it with the hyperplane $x_v = 0$ to obtain the face S. Let $\mathcal{H}'$ be the arrangement $\mathcal{H}_G$ together with the hyperplane $x_v = 0$. Lift S to a face $\widetilde{S}$ in the periodic arrangement $\widetilde{\mathcal{H}'}$ in $\mathbb{R}^n$. Observe that $\widetilde{S}$ is the interior of a polytope. When minimizing the linear functional $L(x) = x_1 + \cdots + x_n$ on the closure of the face $\widetilde{S}$, the optimum is a lattice point $k = (k_1, \ldots, k_n)$. Pick a point $x = (x_1, \ldots, x_n)$ in $\widetilde{S}$ close to the optimum, that is, each coordinate x_i lies in the interval $[k_i, k_i + \epsilon)$ for some small $\epsilon > 0$.

Let $y = (y_1, \ldots, y_n)$ be the image of the point x on the torus T^n, that is, $y_i = x_i \bmod 1$. Note that each entry y_i lies in the half open interval $[0, 1)$ and that $y_v = 0$. Construct an orientation of the graph G by letting the edge ij be oriented $i \to j$ if $y_i > y_j$. Note that this orientation is acyclic and has the vertex v as a sink.

To show that the vertex v is the unique sink, assume that the vertex i is also a sink, where $i \neq v$. In other words, for all neighbors j of the vertex i we have that $y_i < y_j$. We can continuously move the point x in $\widetilde{S}$ by decreasing the value of the ith coordinate x_i. Observe that there is no hyperplane in the periodic arrangement blocking the coordinate x_i from passing through the integer value k_i and continuing down to $k_i - 1 + \epsilon$. This contradicts the fact that we chose the original point x close to the optimum of the linear functional L. Hence the vertex i is not a sink.

It is straightforward to verify that this map from regions to the set of acyclic orientations with the unique sink at v is a bijection. □

The technique of assigning a point to every region of a toric arrangement using a linear functional was used by Novik, Postnikov and Sturmfels in their paper [37]. See their first proof of the number of regions of a toric arrangement.

3.2 The toric Bayer–Sturmfels result

Define the *toric Zaslavsky invariant* of a graded poset P by

$$Z_t(P) = \sum_{x \text{ coatom of } P} (-1)^{\rho(\hat{0},x)} \cdot \mu(\hat{0},x) = (-1)^{\rho(P)-1} \cdot \sum_{x \text{ coatom of } P} \mu(\hat{0},x).$$

We reformulate Theorem 3.5 as follows.

Theorem 3.10 *For a toric hyperplane arrangement $\mathcal{H}$ on the torus T^n that subdivides the torus into open n-dimensional balls, the number of regions is given by $Z_t(\mathcal{P})$, where $\mathcal{P}$ is the intersection poset of the arrangement $\mathcal{H}$.*

As a corollary of the Theorem 3.10 we can describe the f-vector of the subdivision T_t of the torus.

Corollary 3.11 *The number of i-dimensional regions in the subdivision T_t of the n-dimensional torus is given by the sum*

$$f_{i+1}(T_t) = (-1)^i \cdot \sum_{\substack{x \leq y \\ \dim(x)=i \\ \dim(y)=0}} \mu(x,y),$$

where μ denotes the Möbius function in the intersection poset $\mathcal{P}$.

Proof: Each i-dimensional region is contained in a unique i-dimensional subspace x. By restricting the arrangement to the subspace x and applying Theorem 3.5, we have that the number of i-dimensional regions in x is given by $(-1)^i \cdot \sum_{x \leq y, \dim(y)=0} \mu(x,y)$. Summing over all x, the result follows. □

For the remainder of this section we will assume that the induced subdivision of the torus is a regular CW-complex. Let T_t be the face poset of the subdivision of the torus induced by the toric arrangement. Define the map $z_t : T_t^* \longrightarrow \mathcal{P} \cup \{\hat{0}\}$ by sending each face to the smallest toric subspace in the arrangement that contains the face and sending the minimal element in T_t^* to $\hat{0}$. Observe that the map z_t is order- and rank-preserving, as well as being surjective.

The toric analogue of Theorem 2.3 is as follows.

Theorem 3.12 *Let P be the intersection poset of a toric hyperplane arrangement. Let $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ be a chain in $\mathcal{P} \cup \{\hat{0}\}$ with $k \geq 2$. Then the cardinality of the inverse image of the chain c is given by the product*

$$|z_t^{-1}(c)| = \prod_{i=2}^{k-1} Z([x_{i-1}, x_i]) \cdot Z_t([x_{k-1}, x_k]).$$

Proof: We need to count the number of ways we can select a chain $d = \{\hat{0} = y_0 < y_1 < \cdots < y_k = \hat{1}\}$ in T_t^* such that $z_t(y_i) = x_i$. The number of ways to select the element y_{k-1} in T_t^* is the number of regions in the arrangement restricted to the toric subspace x_{k-1}. By Theorem 3.10 this can be done in $Z_t([x_{k-1}, x_k])$ number of ways. Observe now that all other elements in the chain d contain the face y_{k-1}.

To count the number of ways to select the element y_{k-2}, we follow the original argument of Bayer–Sturmfels. Namely, this equals the number of regions in the arrangement having the intersection poset $[x_{k-2}, x_{k-1}]$, which is $Z([x_{k-2}, x_{k-1}])$. By iterating this procedure until we reach the element y_1, the result follows. □

Corollary 3.13 *The flag f-vector entry $f_S(T_t)$ of the face poset T_t of a toric arrangement is divisible by $2^{|S|-1}$ for S any nonempty index set.*

Proof: The proof follows from the fact that the Zaslavsky invariant Z is an even integer and that a given flag f-vector entry is the appropriate sum of products appearing in Theorem 3.12. □

3.3 The connection between posets and coalgebras

For an **ab**-monomial v define the linear map λ_t by letting

$$\lambda_t(v) = \begin{cases} (\mathbf{a} - \mathbf{b})^m & \text{if } v = \mathbf{b}^m \text{ for some } m \geq 0, \\ (\mathbf{a} - \mathbf{b})^{m+1} & \text{if } v = \mathbf{b}^m\mathbf{a} \text{ for some } m \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define the linear operator H' on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to be the one which removes the last letter in each monomial, that is, $H'(w \cdot \mathbf{a}) = H'(w \cdot \mathbf{b}) = w$ and $H'(1) = 0$. We use the prime in the notation to distinguish it from the H map defined in [8, Section 8] which instead removes the first letter in each **ab**-monomial. From [8] we have the following lemma.

Lemma 3.14 *For a poset P, the following identity holds:*

$$H'(\Psi(P)) = \sum_{x \text{ coatom of } P} \Psi([\hat{0}, x]).$$

The next lemma gives the relation between the toric Zaslavsky invariant Z_t and the map λ_t.

Lemma 3.15 *For a poset P, the following identity holds:*

$$\lambda_t(\Psi(P)) = Z_t(P) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1}.$$

Proof: When P has rank 1, both sides are equal to 1. For an $\mathbf{ab}$-monomial v different from 1, we have that $\lambda_t(v) = \beta(H'(v)) \cdot (\mathbf{a} - \mathbf{b})$. Hence

$$\begin{aligned} \lambda_t(\Psi(P)) &= \beta(H'(\Psi(P))) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \sum_{x \text{ coatom of } P} \beta(\Psi([\hat{0}, x])) \cdot (\mathbf{a} - \mathbf{b}) \\ &= (-1)^{\rho(P)} \cdot \sum_{x \text{ coatom of } P} \mu(\hat{0}, x) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1}, \end{aligned}$$

which concludes the proof. $\square$

Define a sequence of functions $\varphi_{t,k} \colon \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by $\varphi_{t,1} = \kappa$, and for $k \geq 2$,

$$\varphi_{t,k}(v) = \sum_{v} \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \cdot \mathbf{b} \cdot \eta(v_{(3)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(k-1)}) \cdot \mathbf{b} \cdot \lambda_t(v_{(k)}).$$

Finally, let $\varphi_t(v)$ be the sum $\varphi_t(v) = \sum_{k \geq 1} \varphi_{t,k}(v)$.

Theorem 3.16 *The* $\mathbf{ab}$*-index of the face poset T_t of a toric arrangement is given by*

$$\Psi(T_t)^* = \varphi_t(\Psi(\mathcal{P} \cup \{\hat{0}\})).$$

Proof: The $\mathbf{ab}$-index of the poset T_t is given by the sum $\Psi(T_t) = \sum_c |z_t^{-1}(c)| \cdot \mathrm{wt}(c)$. Fix $k \geq 2$ and sum over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ of length k. We then have

$$\begin{aligned} & \sum_c |z_t^{-1}(c)| \cdot \mathrm{wt}(c) \\ = \; & \sum_c \prod_{i=2}^{k-1} Z([x_{i-1}, x_i]) \cdot Z_t([x_{k-1}, x_k]) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_0, x_1)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_{k-1}, x_k)-1} \\ = \; & \sum_c \kappa(\Psi([x_0, x_1])) \cdot \prod_{i=2}^{k-1} (\mathbf{b} \cdot \eta(\Psi([x_{i-1}, x_i]))) \cdot \mathbf{b} \cdot \lambda_t(\Psi([x_{k-1}, x_k])) \\ = \; & \sum_w \kappa(w_{(1)}) \cdot \prod_{i=2}^{k-1} \big(\mathbf{b} \cdot \eta(w_{(i)})\big) \cdot \mathbf{b} \cdot \lambda_t(w_{(k)}) \\ = \; & \varphi_{t,k}(w), \end{aligned}$$

where we let w denote the **ab**-index of the augmented intersection poset $\mathcal{P} \cup \{\hat{0}\}$. For $k = 1$ we have that $(\mathbf{a} - \mathbf{b})^{\rho(T_t)-1} = \varphi_{t,1}(\Psi(\mathcal{P} \cup \{\hat{0}\}))$. Summing over all $k \geq 1$ we obtain the result. $\square$

3.4 Evaluating the function φ_t

Proposition 3.17 *For an* **ab**-*monomial* v, *the following identity holds:*

$$\varphi_t(v) = \kappa(v) + \sum_v \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \lambda_t(v_{(2)}).$$

Proof: Observe that for $k \geq 2$ we have that

$$\begin{aligned}
\varphi_{t,k}(v) &= \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(k-1)}) \cdot \mathbf{b} \cdot \lambda_t(v_{(k)}) \\
&= \sum_v \sum_{v_{(1)}} \kappa(v_{(1,1)}) \cdot \mathbf{b} \cdot \eta(v_{(1,2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(1,k-1)}) \cdot \mathbf{b} \cdot \lambda_t(v_{(2)}) \\
&= \sum_v \varphi_{k-1}(v_{(1)}) \cdot \mathbf{b} \cdot \lambda_t(v_{(2)}).
\end{aligned}$$

Here we have used the coassociativity of the coproduct. By summing over all $k \geq 1$, the result follows. $\square$

Lemma 3.18 *Let* v *be an* **ab**-*monomial that begins with* **a** *and let* x *be either* **a** *or* **b**. *Then*

$$\varphi_t(v \cdot \mathbf{a} \cdot x) = \kappa(v \cdot \mathbf{a} \cdot x) + 1/2 \cdot \omega(v \cdot \mathbf{ab}).$$

Proof: Using Proposition 3.17 we have

$$\begin{aligned}
\varphi_t(v \cdot \mathbf{a} \cdot x) &= \kappa(v \cdot \mathbf{a} \cdot x) + \varphi(v \cdot \mathbf{a}) \cdot \mathbf{b} \cdot \lambda_t(1) + \varphi(v) \cdot \mathbf{b} \cdot \lambda_t(x) \\
&\quad + \sum_v \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \lambda_t(v_{(2)} \cdot \mathbf{b} \cdot x) \\
&= \kappa(v \cdot \mathbf{a} \cdot x) + \varphi(v) \cdot \mathbf{c} \cdot \mathbf{b} + \varphi(v) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \\
&= \kappa(v \cdot \mathbf{a} \cdot x) + \omega(v) \cdot \mathbf{d} \\
&= \kappa(v \cdot \mathbf{a} \cdot x) + 1/2 \cdot \omega(v \cdot \mathbf{ab}),
\end{aligned}$$

since $\lambda_t(v_{(2)} \cdot \mathbf{b} \cdot x) = 0$. $\square$

Lemma 3.19 *Let v be an* **ab**-*monomial that begins with* **a**, *let k be a positive integer and let x be either* **a** *or* **b**. *Then the following evaluation holds:*

$$\varphi_t(v \cdot \mathbf{ab}^k \cdot x) = \kappa(v \cdot \mathbf{ab}^k \cdot x) + 1/2 \cdot \omega(v \cdot \mathbf{ab}^{k+1}).$$

Proof: Using Proposition 3.17 we have

$$\begin{aligned}
\varphi_t(v \cdot \mathbf{ab}^k \cdot x) - \kappa(v \cdot \mathbf{ab}^k \cdot x) &= \varphi(v \cdot \mathbf{ab}^k) \cdot \mathbf{b} \cdot \lambda_t(1) + \varphi(v \cdot \mathbf{a}) \cdot \mathbf{b} \cdot \lambda_t(\mathbf{b}^{k-1} \cdot x) \\
&\quad + \varphi(v) \cdot \mathbf{b} \cdot \lambda_t(\mathbf{b}^k \cdot x) + \sum_{i+j=k-2} \varphi(v \cdot \mathbf{ab}^{i+1}) \cdot \mathbf{b} \cdot \lambda_t(\mathbf{b}^j \cdot x) \\
&= \varphi(v) \cdot \Bigg(2\mathbf{dc}^{k-1} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^k + \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{k+1} \\
&\quad + \sum_{i+j=k-2} 2\mathbf{dc}^i \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{j+1} \Bigg). \qquad (3.2)
\end{aligned}$$

In order to simplify this expression, consider the butterfly poset of rank k. Recall this is the poset consisting of two elements of rank i, for $i = 1, \ldots, k-1$ adjoined with a minimal and maximal element. Each of the rank i elements of cover the rank $i-1$ element(s) for $i = 1, \ldots, k-1$. The butterfly poset is the unique poset with the **cd**-index $\mathbf{c}^{k-1}$ and it is Eulerian. Applying (2.2) to the butterfly poset, we have

$$\mathbf{c}^{k-1} = (\mathbf{a}-\mathbf{b})^{k-1} + 2 \cdot \sum_{i+j=k-2} \mathbf{c}^i \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^j.$$

Using this relation to simplify equation (3.2), we obtain

$$\begin{aligned}
\varphi_t(v \cdot \mathbf{ab}^k \cdot x) - \kappa(v \cdot \mathbf{ab}^k \cdot x) &= \varphi(v) \cdot \mathbf{d} \cdot \mathbf{c}^k \\
&= 1/2 \cdot \omega(v \cdot \mathbf{ab}^{k+1}).
\end{aligned}$$

□

By combining Lemmas 3.18 and 3.19, we have the following proposition.

Proposition 3.20 *For an* **ab**-*monomial v that begins with the letter* **a**, *the following holds:*

$$\varphi_t(v) = \kappa(v) + 1/2 \cdot \omega(H'(v) \cdot \mathbf{b}).$$

We now obtain the main result for computing the **ab**-index of the face poset of a toric arrangement.

Theorem 3.21 *Let $\mathcal{H}$ be a toric hyperplane arrangement on the n-dimensional torus T^n that subdivides the torus into a regular CW-complex. Then the* **ab**-*index of the face poset T_t can be computed from the* **ab**-*index of the intersection poset $\mathcal{P}$ as follows:*

$$\Psi(T_t) = (\mathbf{a}-\mathbf{b})^{n+1} + \frac{1}{2} \cdot \omega(\mathbf{a} \cdot H'(\Psi(\mathcal{P})) \cdot \mathbf{b})^*.$$

Observe that in Lemmas 3.18 and 3.19, Proposition 3.20 and Theorem 3.21 no rational coefficients were introduced. Only the **ab**-monomial $\mathbf{a}^n$ is mapped to a **cd**-polynomial with an odd coefficient, hence $1/2 \cdot \omega(v \cdot \mathbf{b})$ has all integer coefficients.

Continuation of Example 3.2 The flag f-vector of the intersection poset $\mathcal{P}$ in Example 3.2 is given by $(f_\emptyset, f_1, f_2, f_{12}) = (1, 3, 7, 15)$, the flag h-vector by $(h_\emptyset, h_1, h_2, h_{12}) = (1, 2, 6, 6)$, and so the **ab**-index is $\Psi(P) = \mathbf{a}^2 + 2 \cdot \mathbf{ba} + 6 \cdot \mathbf{ab} + 6 \cdot \mathbf{b}^2$. Thus

$$\begin{aligned}
\Psi(T_t) &= (\mathbf{a}-\mathbf{b})^3 + 1/2 \cdot \omega(\mathbf{a} \cdot H'(\mathbf{a}^2 + 2 \cdot \mathbf{ba} + 6 \cdot \mathbf{ab} + 6 \cdot \mathbf{b}^2) \cdot \mathbf{b})^* \\
&= (\mathbf{a}-\mathbf{b})^3 + 1/2 \cdot \omega(\mathbf{a} \cdot (7 \cdot \mathbf{a} + 8 \cdot \mathbf{b}) \cdot \mathbf{b})^* \\
&= (\mathbf{a}-\mathbf{b})^3 + 1/2 \cdot \omega(7 \cdot \mathbf{a}^2\mathbf{b} + 8 \cdot \mathbf{ab}^2)^* \\
&= (\mathbf{a}-\mathbf{b})^3 + 7 \cdot \mathbf{dc} + 8 \cdot \mathbf{cd},
\end{aligned}$$

which agrees with the calculation in Example 3.2.

Theorem 3.21 gives a different approach than Corollary 3.11 for determining the f-vector of T_t. For notational ease, for positive integers i and j, let $[i, j] = \{i, i+1, \ldots, j\}$ and $[j] = \{1, \ldots, j\}$.

Corollary 3.22 *The number of i-dimensional regions in the subdivision T_t of the n-dimensional torus is given by the following sum of flag h-vector entries from the intersection poset $\mathcal{P}$:*

$$f_{i+1}(T_t) = h_{[n-i,n]}(\mathcal{P}) + h_{[n-i,n-1]}(\mathcal{P}) + h_{[n-i+1,n]}(\mathcal{P}) + h_{[n-i+1,n-1]}(\mathcal{P}),$$

for $1 \leq i \leq n-1$. The number of vertices is given by $f_1(T_t) = 1 + h_n(\mathcal{P})$ and the number of maximal regions by $f_{n+1}(T_t) = h_{[n-1]}(\mathcal{P}) + h_{[n]}(\mathcal{P})$.

Proof: Let $\langle \cdot | \cdot \rangle$ denote the inner product on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ defined by $\langle u | v \rangle = \delta_{u,v}$ for two **ab**-monomials u and v. For $1 \leq i \leq n-1$ we have

$$\begin{aligned}
f_{i+1}(T_t) &= 1 + h_{i+1}(T_t) \\
&= 1 + \left\langle \mathbf{a}^i\mathbf{b}\mathbf{a}^{n-i} \middle| \Psi(T_t) \right\rangle \\
&= \frac{1}{2} \cdot \left\langle \mathbf{a}^i\mathbf{b}\mathbf{a}^{n-i} \middle| \omega(\mathbf{a} \cdot H'(\Psi(\mathcal{P})) \cdot \mathbf{b})^* \right\rangle \\
&= \frac{1}{2} \cdot [\mathbf{c}^{i-1}\mathbf{d}\mathbf{c}^{n-i}]\omega(\mathbf{a} \cdot H'(\Psi(\mathcal{P})) \cdot \mathbf{b})^* + \frac{1}{2} \cdot [\mathbf{c}^{i}\mathbf{d}\mathbf{c}^{n-i-1}]\omega(\mathbf{a} \cdot H'(\Psi(\mathcal{P})) \cdot \mathbf{b})^* \\
&= \left\langle \mathbf{a}^{n-i} \cdot \mathbf{ab} \cdot \mathbf{b}^{i-1} + \mathbf{a}^{n-i-1} \cdot \mathbf{ab} \cdot \mathbf{b}^{i} \middle| \mathbf{a} \cdot H'(\Psi(\mathcal{P})) \cdot \mathbf{b} \right\rangle \\
&= \left\langle \mathbf{a}^{n-i-1} \cdot (\mathbf{a}+\mathbf{b}) \cdot \mathbf{b}^{i-1} \middle| H'(\Psi(\mathcal{P})) \right\rangle \\
&= \left\langle \mathbf{a}^{n-i-1} \cdot (\mathbf{a}+\mathbf{b}) \cdot \mathbf{b}^{i-1} \cdot (\mathbf{a}+\mathbf{b}) \middle| \Psi(\mathcal{P}) \right\rangle .
\end{aligned}$$

Expanding in terms of the flag h-vector the result follows. The expressions for f_1 and f_{n+1} are obtained by similar calculations. $\square$

The fact that Corollaries 3.11 and 3.22 are equivalent follows from the coalgebra techniques in Theorem 2.5.

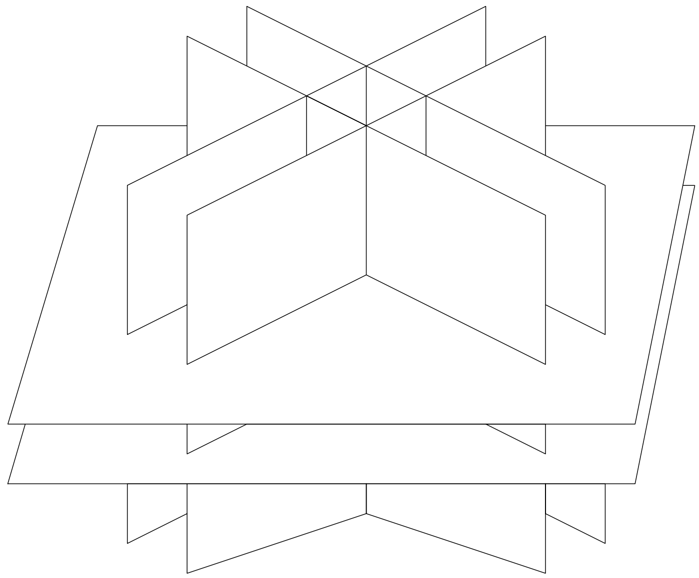


Figure 3: The non-central arrangement $x, y, z = 0, 1$.

4 The complex of unbounded regions

4.1 Zaslavsky and Bayer–Sturmfels

The *unbounded Zaslavsky invariant* is defined by

$$Z_{ub}(P) = Z(P) - 2 \cdot Z_b(P).$$

As the name suggests, the number of unbounded regions in a non-central arrangement is given by this invariant. By taking the difference of the two statements in Theorem 2.2 part (ii), we have:

Lemma 4.1 *For a non-central hyperplane arrangement $\mathcal{H}$ the number of unbounded regions is given by $Z_{ub}(\mathcal{L})$, where $\mathcal{L}$ is the intersection lattice of the arrangement $\mathcal{H}$.*

Let $\mathcal{H}$ be a non-central hyperplane arrangement in $\mathbb{R}^n$ with intersection lattice $\mathcal{L}$. Let $\mathcal{L}_{ub}$ denote the *unbounded intersection lattice*, that is, the subposet of the intersection lattice consisting of all affine subspaces with the points (dimension zero affine subspaces) omitted. Equivalently, the poset $\mathcal{L}_{ub}$ is the rank-selected poset $\mathcal{L}([1, n-1])$, that is, the poset $\mathcal{L}$ with the coatoms removed. Similarly, let T_{ub} denote all of the faces in the hyperplane arrangement $\mathcal{H}$ which are unbounded. We observe that T_{ub} is the face poset of an $(n-1)$-dimensional sphere. Pick R large enough so that all of the bounded faces are strictly inside a ball of radius R. Intersect the arrangement $\mathcal{H}$ with a sphere of radius R. The resulting CW-complex has face poset T_{ub}. Our goal is to compute the **cd**-index of T_{ub} in terms of the **ab**-index of $\mathcal{L}_{ub}$.

The collection of unbounded regions of the arrangement $\mathcal{H}$ forms an ideal in the poset T^*. Let Q be the subposet of T^* consisting of this ideal with a maximal element $\hat{1}$ adjoined and let Q inherit the rank function ρ from the poset T^*. Note that Q contains none of the coatoms from the poset T^*. Thus as posets we have that T^*_{ub} and Q are isomorphic. However, since their rank functions differ, their **ab**-indexes satisfy $\Psi(T_{ub})^* \cdot (\mathbf{a} - \mathbf{b}) = \Psi(Q)$.

We now restrict the zero map $z : T^* \longrightarrow \mathcal{L} \cup \{\hat{0}\}$ to form the map $z_{ub} : Q \longrightarrow \mathcal{L} \cup \{\hat{0}\}$. Observe that z_{ub} is order- and rank-preserving. Also note that z_{ub} is not necessarily surjective. Analogous to the Bayer–Sturmfels result, Theorem 2.3, we have the following theorem:

Theorem 4.2 *Let $\mathcal{H}$ be a non-central hyperplane arrangement with intersection lattice $\mathcal{L}$. Let $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ be a chain in $\mathcal{L} \cup \{\hat{0}\}$ with $k \geq 2$. Then the cardinality of its inverse image of the chain c under z_{ub} is given by*

$$|z_{ub}^{-1}(c)| = \prod_{i=2}^{k-1} Z([x_{i-1}, x_i]) \cdot Z_{ub}([x_{k-1}, x_k]).$$

Proof: We need to count the number of ways we can select a chain $d = \{\hat{0} = y_0 < y_1 < \cdots < y_k = \hat{1}\}$ in the poset of unbounded regions Q such that $z_{ub}(y_i) = x_i$. The number of ways to select the element y_{k-1} in Q is the number of unbounded regions in the arrangement restricted to the subspace x_{k-1}. By Lemma 4.1 this can be done in $Z_{ub}([x_{k-1}, x_k])$ number of ways. Since y_{k-1} is an unbounded face of the arrangement and all other elements in the chain d contain the face y_{k-1}, the other elements must be unbounded.

The remainder of the proof is the same as that of Theorem 3.12. □

Corollary 4.3 *The flag f-vector entry $f_S(T_{ub})$ is divisible by $2^{|S|}$ for any index set S.*

Proof: The proof is the same as Corollary 3.13 with the extra observation that the Zaslavsky invariant Z_{ub} is even. □

4.2 The connection between posets and coalgebras

Define λ_{ub} by $\lambda_{ub} = \eta - 2 \cdot \beta$. By equations (2.5) and (2.6) we have for a poset P

$$\lambda_{ub}(\Psi(P)) = Z_{ub}(P) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1}.$$

Define a sequence of functions $\varphi_{ub,k} \colon \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by $\varphi_{ub,1} = \kappa$ and for $k > 1$,

$$\varphi_{ub,k}(v) = \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \cdot \mathbf{b} \cdot \eta(v_{(3)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(k-1)}) \cdot \mathbf{b} \cdot \lambda_{ub}(v_{(k)}).$$

Finally, let $\varphi_{ub}(v)$ be the sum $\varphi_{ub}(v) = \sum_{k \geq 1} \varphi_{ub,k}(v)$.

Similar to Theorem 3.16 we have the next result. The proof only differs in replacing the map $z_t : T_t^* \longrightarrow \mathcal{P} \cup \{\hat{0}\}$ with $z_{ub} : Q \longrightarrow \mathcal{L} \cup \{\hat{0}\}$ and the invariant Z_t by Z_{ub}.

Theorem 4.4 *The* **ab**-*index of the poset* Q *of unbounded regions of a non-central arrangement is given by*

$$\Psi(Q) = \varphi_{ub}(\Psi(\mathcal{L} \cup \{\hat{0}\})).$$

4.3 Evaluating the function φ_{ub}

In this subsection we analyze the behavior of φ_{ub}.

Lemma 4.5 *For any* **ab**-*monomial* v,

$$\varphi_{ub}(v) = \varphi(v) - 2 \cdot \sum_{v} \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \beta(v_{(2)}).$$

Proof: For $k \geq 2$ we have that

$$\begin{aligned}
\varphi_{ub,k}(v) &= \varphi_k(v) - 2 \cdot \sum_{v} \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \eta(v_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(k-1)}) \cdot \mathbf{b} \cdot \beta(v_{(k)}) \\
&= \varphi_k(v) - 2 \cdot \sum_{v} \sum_{v_{(1)}} \kappa(v_{(1,1)}) \cdot \mathbf{b} \cdot \eta(v_{(1,2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(v_{(1,k-1)}) \cdot \mathbf{b} \cdot \beta(v_{(2)}) \\
&= \varphi_k(v) - 2 \cdot \sum_{v} \varphi_{k-1}(v_{(1)}) \cdot \mathbf{b} \cdot \beta(v_{(2)}),
\end{aligned}$$

using coassociativity. The result then follows by summing over all $k \geq 2$ and adding $\varphi_{ub,1}(v) = \kappa(v) = \varphi_1(v)$. $\square$

Lemma 4.6 *Let* v *be an* **ab**-*monomial. Then*

$$\varphi_{ub}(v \cdot \mathbf{a}) = \varphi(v) \cdot (\mathbf{a} - \mathbf{b}).$$

Proof: By Lemma 4.5 and the Newtonian relation (2.3) we have

$$\varphi_{ub}(v \cdot \mathbf{a}) = \varphi(v \cdot \mathbf{a}) - 2 \cdot \varphi(v) \cdot \mathbf{b} \cdot \beta(1) - 2 \cdot \sum_{v} \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \beta(v_{(2)} \cdot \mathbf{a}).$$

By equation (2.7) $\varphi(v \cdot \mathbf{a}) = \varphi(v) \cdot \mathbf{c}$. The summation above is zero because $\beta(v_{(2)} \cdot \mathbf{a})$ is always zero. Hence $\varphi_{ub}(v \cdot \mathbf{a}) = \varphi(v) \cdot (\mathbf{c} - 2\mathbf{b}) = \varphi(v) \cdot (\mathbf{a} - \mathbf{b})$. $\square$

Lemma 4.7 *Let v be an* **ab***-monomial. Then*

$$\varphi_{ub}(v \cdot \mathbf{bb}) = \varphi_{ub}(v \cdot \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}).$$

Proof: Let $u = v \cdot \mathbf{b}$. Applying Lemma 4.5 and the Newtonian relation (2.3) to u gives:

$$\begin{aligned}
\varphi_{ub}(u \cdot \mathbf{b}) &= \varphi(u \cdot \mathbf{b}) - 2 \cdot \varphi(u) \cdot \mathbf{b} \cdot \beta(1) - 2 \cdot \sum_u \varphi(u_{(1)}) \cdot \mathbf{b} \cdot \beta(u_{(2)} \cdot \mathbf{b}) \\
&= \varphi(u) \cdot (\mathbf{c} - 2\mathbf{b}) - 2 \cdot \sum_u \varphi(u_{(1)}) \cdot \mathbf{b} \cdot \beta(u_{(2)}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \left(\varphi(u) - 2 \cdot \sum_u \varphi(u_{(1)}) \cdot \mathbf{b} \cdot \beta(u_{(2)}) \right) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \varphi_{ub}(u) \cdot (\mathbf{a} - \mathbf{b}).
\end{aligned}$$

Here we have used the two facts $\varphi(u \cdot \mathbf{b}) = \varphi(u) \cdot \mathbf{c}$ and $\beta(u_{(2)} \cdot \mathbf{b}) = \beta(u_{(2)}) \cdot (\mathbf{a} - \mathbf{b})$. $\square$

Lemma 4.8 *Let v be an* **ab***-monomial. Then $\varphi_{ub}(v \cdot \mathbf{ab}) = 0$.*

Proof: Directly we have

$$\begin{aligned}
\varphi_{ub}(v \cdot \mathbf{ab}) &= \varphi(v \cdot \mathbf{ab}) - 2 \cdot \varphi(v) \cdot \mathbf{b} \cdot \beta(\mathbf{b}) - 2 \cdot \varphi(v \cdot \mathbf{a}) \cdot \mathbf{b} \cdot \beta(1) - 2 \cdot \sum_v \varphi(v_{(1)}) \cdot \mathbf{b} \cdot \beta(v_{(2)} \cdot \mathbf{ab}) \\
&= \varphi(v) \cdot 2\mathbf{d} - 2 \cdot \varphi(v) \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) - 2 \cdot \varphi(v) \cdot \mathbf{cb} \\
&= 2 \cdot \varphi(v) \cdot (\mathbf{d} - \mathbf{b}(\mathbf{a} - \mathbf{b}) - \mathbf{cb}) \\
&= 0,
\end{aligned}$$

where we have used the facts $\varphi(v \cdot \mathbf{ab}) = \varphi(v) \cdot 2\mathbf{d}$ and $\beta(v_{(2)} \cdot \mathbf{ab}) = 0$. $\square$

The previous three lemmas enable us to determine φ_{ub}. In order to obtain more compact notation, define a map $r \colon \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \to \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by $r(1) = 0$, $r(v \cdot \mathbf{a}) = v$, and $r(v \cdot \mathbf{b}) = 0$. The map r divides an **ab**-polynomial on the right by **a**. By using the chain definition of the **ab**-index, it is straightforward to observe that $\Psi(\mathcal{L}_{ub}) = r(\Psi(\mathcal{L}))$.

Proposition 4.9 *Let w be an* **ab***-polynomial without constant term. Then*

$$\varphi_{ub}(\mathbf{a} \cdot w) = \omega(\mathbf{a} \cdot r(w)) \cdot (\mathbf{a} - \mathbf{b}).$$

Proof: The case $w = v \cdot \mathbf{a}$ follows from Lemma 4.6. The remaining case is $w = v \cdot \mathbf{b}$. Note that $\mathbf{a} \cdot v \cdot \mathbf{b}$ can be factored as $u \cdot \mathbf{ab} \cdot \mathbf{b}^k$ for a monomial u. Hence $\varphi_{ub}(u \cdot \mathbf{ab} \cdot \mathbf{b}^k) = \varphi_{ub}(u \cdot \mathbf{ab}) \cdot (\mathbf{a} - \mathbf{b})^k = 0$ by Lemmas 4.7 and 4.8. $\square$

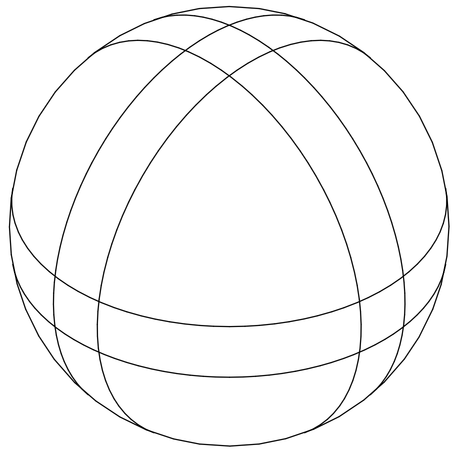


Figure 4: The spherical subdivision obtained from the non-central arrangement $x, y, z = 0, 1$.

We combine all of these results to conclude that the **cd**-index of the poset of unbounded regions T_{ub} can be computed in terms of the **ab**-index of the unbounded intersection lattice $\mathcal{L}_{ub}$.

Theorem 4.10 *Let $\mathcal{H}$ be a non-central hyperplane arrangement with the unbounded intersection lattice $\mathcal{L}_{ub}$ and poset of unbounded regions T_{ub}. Then*

$$\Psi(T_{ub}) = \omega(\mathbf{a} \cdot \Psi(\mathcal{L}_{ub}))^*.$$

Proof: We have that

$$\begin{aligned} \Psi(T_{ub})^* \cdot (\mathbf{a} - \mathbf{b}) &= \Psi(Q) \\ &= \varphi_{ub}(\mathbf{a} \cdot \Psi(\mathcal{L})) \\ &= \omega(\mathbf{a} \cdot r(\Psi(\mathcal{L}))) \cdot (\mathbf{a} - \mathbf{b}) \\ &= \omega(\mathbf{a} \cdot \Psi(\mathcal{L}_{ub})) \cdot (\mathbf{a} - \mathbf{b}). \end{aligned}$$

By cancelling $\mathbf{a} - \mathbf{b}$ on both sides, the result follows. $\square$

Example 4.11 Consider the non-central hyperplane arrangement consisting of the six hyperplanes $x = 0, 1$, $y = 0, 1$ and $z = 0, 1$. See Figure 3. Intersecting this with a sphere of large enough radius we have the CW-complex in Figure 4. The polytopal realization of this complex is known as the rhombicuboctahedron. The dual of the face lattice of the spherical complex is not realized by a zonotope. However, the dual lattice can be viewed as the face lattice of a $2 \times 2 \times 2$ pile of cubes.

The intersection lattice $\mathcal{L}$ is the face lattice of the three-dimensional crosspolytope, in other words, the octahedron. Hence the lattice of unbounded intersection $\mathcal{L}_{ub}$ has the flag f-vector $(f_\emptyset, f_1, f_2, f_{12}) =$

$(1, 6, 12, 24)$ and the flag h-vector $(h_\emptyset, h_1, h_2, h_{12}) = (1, 5, 11, 7)$. The **ab**-index is given by $\Psi(\mathcal{L}_{ub}) = \mathbf{a}^2 + 5 \cdot \mathbf{ba} + 11 \cdot \mathbf{ab} + 7 \cdot \mathbf{b}^2$. Hence the **cd**-index of T_{ub} is given by

$$\begin{aligned}\Psi(T_{ub}) &= \omega(\mathbf{a}^3 + 5 \cdot \mathbf{aba} + 11 \cdot \mathbf{a}^2\mathbf{b} + 7 \cdot \mathbf{ab}^2)^* \\ &= \mathbf{c}^3 + 22 \cdot \mathbf{dc} + 24 \cdot \mathbf{cd}.\end{aligned}$$

5 Concluding remarks

For regular subdivisions of manifolds there is now a plethora of questions to ask.

(i) What is the right analogue of a regular subdivision in order that it be polytopal? Can flag f-vectors be classified for polytopal subdivisions?

(ii) Is there a Kalai convolution for manifolds that will generate more inequalities for flag f-vectors? [31]

(iii) Is there a lifting technique that will yield more inequalities for higher dimensional manifolds? [16]

(iv) Are there minimization inequalities for the **cd**-coefficients in the polynomial Ψ? As a first step, can one prove the non-negativity of Ψ? [7, 17]

(v) Is there an extension of the toric g-inequalities to manifolds? [4, 30, 32, 40]

(vi) Can the coefficients for Ψ be minimized for regular toric arrangements as was done in the case of central hyperplane arrangements? [8]

The most straightforward manifold to study is n-dimensional projective space P^n. We offer the following result in obtaining the **ab**-index of subdivisions of P^n.

Theorem 5.1 *Let Ω be a centrally symmetric regular subdivision of the n-dimensional sphere S^n. Assume that when antipodal points of the sphere are identified, a regular subdivision Ω' of the projective space P^n is obtained. Then the* **ab***-index of Ω' is given by*

$$\Psi(\Omega') = \frac{\mathbf{c}^{n+1} + (\mathbf{a} - \mathbf{b})^{n+1}}{2} + \frac{\Phi}{2},$$

where the **cd***-index of Ω is $\Psi(\Omega) = \mathbf{c}^{n+1} + \Phi$.*

The results in this paper have been stated for hyperplane arrangements. In true generality one could work with the underlying oriented matroid, especially since there are nonrealizable ones such as the non-Pappus oriented matroid. All of these can be represented as pseudo-hyperplane arrangements. However, we have chosen to work with hyperplane arrangements in order not to lose the geometric intuition.

Other poset transformations that have been considered appear in [15, 22, 28]. Each uses a map related to the ω map. Are there toric or affine analogues of these posets transforms?

Another way to encode the flag f-vector data of a poset is to use the quasisymmetric function of a poset [13]. In this language the ω map is translated to Stembridge's ϑ map; see [9, 42]. Would the results of Theorems 3.21 and 4.10 be appealing in the quasisymmetric function viewpoint?

Richard Stanley has asked if the coefficients of the toric characteristic polynomial are alternating. If so, is there any combinatorial interpretation of the absolute values of the coefficients.

A far reaching generalization of Zaslavsky's results for hyperplane arrangements is by Goresky and MacPherson [24]. Their results determine the cohomology groups of the complement of a complex hyperplane arrangement. For a toric analogue of the Goresky–MacPherson results, see work of De Concini and Procesi [11]. For algebraic considerations of toric arrangements, see [12, 34, 35, 36].

In Section 3 we restricted ourselves to studying arrangements that cut the torus into regular CW-complexes. In a future paper [23], two of the authors are developing the notion of a **cd**-index for non-regular CW-complexes.

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