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The Dowling Transform of Subspace Arrangements

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We define the Dowling transform of a real frame arrangement and show how the characteristic polynomial changes under this transformation. As a special case, the Dowling transform sends the braid arrangement A_n to the Dowling arrangement. Using Zaslavsky's characterization of supersolvability of signed graphs, we show supersolvability of an arrangement is preserved under the Dowling transform. We also give a direct proof of Zaslavsky's result on the number of chambers and bounded chambers in a real hyperplane arrangement. © 2000 Academic Press

1. INTRODUCTION

A number of techniques have been developed to compute the characteristic polynomial of subspace arrangements. These include Athanasiadis' modulo q method for real arrangements with integer coefficients [1], Blass and Sagan's lattice point counting for subarrangements of the braid arrangement B_n [2], Stanley's theory of supersolvability [13], and Terao's factorization theorem for free arrangements [15].

Inspired by Rota's treatment of valuations [9], the authors showed in [6] that there is a natural valuation v extending the Euler characteristic. It is defined on affine subspaces in $\mathbf{k}^n$ for $\mathbf{k}$ an infinite field. From this geometric viewpoint emerged a new combinatorial interpretation of the characteristic polynomial of a subspace arrangement as the valuation v applied to the complement of the arrangement. In particular, this applies to complex subspace arrangements. To demonstrate the power of this method,

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in Section 2 we give an enlightening proof of Zaslavsky's theorem on the number of chambers and bounded chambers in a real hyperplane arrangement.

The Dowling arrangement can be viewed as a natural generalization of the braid arrangements A_n and B_n; see [5; 7; 11, Example 6.29]. We continue to study this relationship by introducing a transformation which takes a real frame arrangement with coordinate hyperplanes into a complex one. We name this transformation the Dowling transform since it carries the braid arrangement A_n into the Dowling arrangement.

We show how the Dowling transform changes the characteristic polynomial. In fact, when the characteristic polynomial of a real arrangement factors, the resulting characteristic polynomial also factors. Moreover, the Dowling transform preserves supersolvability. At the end of this paper, we ask if there is a similar relationship between the homology groups of the corresponding intersection lattices.

2. VALUATIONS AND SUBSPACE ARRANGEMENTS

For a subspace arrangement $\mathscr{A} = \{A_1, ..., A_k\}$ let $L(\mathscr{A})$ denote its intersection lattice. The *characteristic polynomial* of $\mathscr{A}$ (see [10, 11, 14, 16]) is

$$\chi(\mathscr{A}) = \sum_{\substack{x \in L(\mathscr{A}) \\ x \neq \varnothing}} \mu(\hat{0}, x) \cdot t^{\dim(x)}.$$

Let V be a set and let G be a collection of subsets of V such that G is closed under finite intersections and the empty set $\varnothing$ belongs to G. Denote by $B(G)$ the *Boolean algebra* generated by G, that is, $B(G)$ is the smallest collection of subsets of V such that $B(G)$ contains G and is closed under finite intersections, finite unions, and complements. Observe that $B(G)$ forms a lattice where the meet and join are respectively intersection and union.

Let S be a set and let L be the collection of subsets of the underlying set S closed under intersection and union. A *valuation* v on L is a function $v : L \to R$, where R is a commutative ring, satisfying

$$v(A) + v(B) = v(A \cap B) + v(A \cup B),$$
$$v(\varnothing) = 0.$$

For a more detailed account of valuations, we refer the reader to [9, Chap. 2].

We now introduce a very useful valuation which was first defined in [6]. Let $\mathbf{k}$ be an infinite field and V an n-dimensional vector space over the field $\mathbf{k}$.

Let G be the set of all affine subspaces of V. Then $B(G)$ is the set of all sets that can be obtained by union, intersection and complement of affine subspaces. In [6] the following result was proved:

THEOREM 2.1. (a) *There exists a unique valuation* $v : B(G) \to \mathbb{Z}[t]$ *such that for* A *a nonempty affine subspace we have*

$$v(A) = t^{\dim(A)}.$$

(b) *Let* $\mathscr{A} = \{A_1, ..., A_k\}$ *be a subspace arrangement in* V. *Then the characteristic polynomial of* $\mathscr{A}$ *is given by*

$$\chi(\mathscr{A}) = v\left(V - \bigcup_{i=1}^{k} A_i\right).$$

Recall that a function f is *simple* if it can be written as $f = \sum_{i=1}^{k} \alpha_i \cdot I_{A_i}$ where $\alpha_i \in R$ and I_A denotes the indicator function on the set $A \in B(G)$. The *integral* of f is then defined to be $\int f \, dv = \sum_{i=1}^{k} \alpha_i \cdot v(A_i)$. The indicator functions of affine subspaces form a linear basis for all simple functions since we are working over the Boolean algebra $B(G)$.

Let V be the vector space $V_1 \times V_2$, where V_1 and V_2 are two vector spaces over the field $\mathbf{k}$. A simple function f defined on V can be viewed as a function of two variables $f(x, y)$, where $x \in V_1$ and $y \in V_2$. For a fixed element x in V_1 and f a function on V, let $f_x(y)$ denote the function f viewed as a function of the variable y. Moreover, let $\int f \, dv_i$ denote the integral on the space V_i.

THEOREM 2.2. *Let* f *be a simple function on* $V = V_1 \times V_2$. *Then* $f_x(y)$ *is a simple function on* V_2, *the function* $\int f_x(y) \, dv_2$ *is a simple function on* V_1 *and*

$$\int f \, dv = \iint f_x(y) \, dv_2 \, dv_1.$$

Theorem 2.2 is precisely the statement of Fubini's theorem in integration theory. It allows us to compute the valuation v componentwise. As a brief example, we can compute the characteristic polynomial of the two-dimensional line arrangement $x = 0$, $y = 0$, $y = x$, $y = \pi \cdot x$, by: we can choose x in $t - 1$ ways, namely $x \neq 0$. Next we can choose y in $t - 3$ ways, namely $y \neq 0, x, \pi \cdot x$. Hence the characteristic polynomial is $(t - 1) \cdot (t - 3)$.

Using the valuation v one can prove Zaslavsky's theorem enumerating the number of *chambers*, that is, maximal open regions, of a real hyperplane arrangement [16, Theorems A and C, Sect. 2].

THEOREM 2.3. (Zaslavsky). *Let $\mathcal{H} = \{H_1, ..., H_k\}$ be a hyperplane arrangement in $\mathbb{R}^n$. Then*

(i) *the number of chambers of the hyperplane arrangement $\mathcal{H}$ is equal to $(-1)^n \cdot \chi(\mathcal{H})|_{t=-1}$.*

(ii) *the number of bounded chambers of the hyperplane arrangement $\mathcal{H}$ is equal to $(-1)^n \cdot \chi(\mathcal{H})|_{t=1}$.*

Zaslavsky's original proof was based upon showing that the characteristic polynomial is a Tutte–Grothendieck invariant [16, Sect. 4]. Part (i) can be proved by recognizing that evaluating the valuation v at $t=-1$ for an affine subspace coincides to applying the Euler characteristic to the affine subspace; see [6] for details. For part (ii), let P denote the collection of all intersections of half-spaces of $\mathbb{R}^n$, where we allow the half-spaces to be open or closed. Observe that an element of P is a polyhedron and that P contains all affine subspaces. Define the valuation $\bar{\varepsilon}$ on the Boolean algebra $B(P)$ by

$$\bar{\varepsilon}(A) = \lim_{r \to \infty} \varepsilon(A \cap B_r),$$

where B_r denotes the n-dimensional closed ball of radius r and ε denotes the Euler characteristic. Recall that if a set A is homeomorphic to a k-dimensional open ball then $\varepsilon(A) = (-1)^k$, while if A is homeomorphic to a k-dimensional closed ball then $\varepsilon(A) = 1$. For A a bounded region we have, for large enough r, the set $A \cap B_r$ is the set A, which is homeomorphic to an open n-dimensional ball. In this case the valuation is $(-1)^n$. When A is an unbounded region, the set $A \cap B_r$, for large enough r, is homeomorphic to the n-dimensional closed ball minus the $(n-1)$-dimensional closed hemisphere. This has valuation $1-1$ equals zero. Hence $\bar{\varepsilon}(\mathbb{R}^n - \bigcup_{i=1}^k H_i)$ is equal to $(-1)^n$ times the number of bounded regions.

This proof of Zaslavsky's result on the number of bounded regions has been independently discovered by Chen [3]. In this paper Chen consolidates combinatorial interpretations of the characteristic polynomial appearing in [1, 2, 6] by applying different valuations to the underlying identity of indicator functions.

Note that from the valuation v the *deletion–restriction* formula [11, Theorem 2.56] follows immediately: for a subspace arrangement $\mathcal{A}$ one has $\chi(\mathcal{A}) = \chi(\mathcal{A}') - \chi(\mathcal{A}'')$, where $\mathcal{A}'$ is obtained by deleting a hyperplane

H from the arrangement $\mathscr{A}$ and $\mathscr{A}''$ is the arrangement resulting from restricting the arrangement $\mathscr{A}$ to the hyperplane H.

3. THE DOWLING TRANSFORM

We now study a transformation which takes a real hyperplane arrangement into a complex one. We call this transformation the *Dowling transform* since as a special case it sends the braid arrangement A_n to the Dowling arrangement [4, 5]. Following the language of Zaslavsky [17], we call a subspace arrangement a *frame arrangement with coordinate hyperplanes* if it contains the coordinate hyperplanes and every subspace not contained in the coordinate hyperplanes is constructed by intersecting hyperplanes of the form $x_i = a \cdot x_j$, where a is a nonzero constant.

Let W be a subspace of dimension $n-p$ not contained in any coordinate hyperplane and determined by the intersection of the p hyperplanes

$$\begin{cases} x_{i_1} = a_1 \cdot x_{j_1}, \\ \quad\vdots \\ x_{i_p} = a_p \cdot x_{j_p}. \end{cases}$$

Such a subspace W is called a *frame subspace* and the constants a_i the *defining constants*. One may view a frame subspace W as a directed graph on the vertices $1, ..., n$. A directed edge from vertex i to vertex j is labeled a if W is contained in the hyperplane $x_i = a \cdot x_j$. Observe that the graph is the disjoint union of cliques and the number of cliques is the dimension of W. Also, the product of the labels along a directed cycle is equal to 1.

Let ζ be a primitive mth root of unity. The *Dowling transform* of the frame subspace W, denoted $D_m(W)$, is the collection of subspaces described by

$$\begin{cases} x_{i_1} = \zeta^{h_1} \cdot a_1 \cdot x_{j_1}, \\ \quad\vdots \\ x_{i_p} = \zeta^{h_p} \cdot a_p \cdot x_{j_p}, \end{cases}$$

and indexed by $(h_1, ..., h_p) \in \{0, ..., m-1\}^p$. Note these m^p subspaces are distinct. We let the Dowling transform of the coordinate hyperplane $x_i = 0$ be itself. The Dowling transform of the frame arrangement is simply the union of the Dowling transform of each subspace.

EXAMPLE 3.1. A frame hyperplane arrangement with coordinate hyperplanes is described by

$$\begin{cases} x_i = a_{i,j,l} \cdot x_j, & 1 \leqslant i < j \leqslant n \quad \text{and} \quad 1 \leqslant l \leqslant k(i, j), \\ x_i = 0, & 1 \leqslant i \leqslant n, \end{cases}$$

where the $k(i, j)$ are constants depending on i and j. This hyperplane arrangement is best encoded as a gain graph on the vertices $1, ..., n$, where the hyperplane $x_i = a_{i,j,l} \cdot x_j$ is encoded by a directed edge from vertex i to vertex j labeled $a_{i,j,l}$. The Dowlingization of this hyperplane arrangement is then

$$\begin{cases} x_i = \zeta^h \cdot a_{i,j,l} \cdot x_j, & 1 \leqslant i < j \leqslant n, 1 \leqslant l \leqslant k(i, j) \quad \text{and} \quad 0 \leqslant h \leqslant m-1, \\ x_i = 0, & 1 \leqslant i \leqslant n. \end{cases}$$

The Dowlingization corresponds to replacing each edge in the gain graph with m copies and having the label of each copy multiplied by a distinct power of the unit root ζ.

We are now ready to state our main theorem.

THEOREM 3.2. *Let $\mathscr{A}$ be a frame arrangement with coordinate hyperplanes where the defining constants are positive real numbers. Then the characteristic polynomial of the Dowlingization $D_m(\mathscr{A})$ is given by*

$$\chi(D_m(\mathscr{A})) = m^n \cdot f((t-1)/m),$$

where $f(t-1)$ is the characteristic polynomial of $\mathscr{A}$.

COROLLARY 3.3. *Let $\mathscr{A}$ be a frame arrangement with coordinate hyperplanes where the defining constants are positive real numbers. If the characteristic polynomial of $\mathscr{A}$ has the roots*

$$r_1 + 1, r_2 + 1, ..., r_n + 1,$$

then the characteristic polynomial of the Dowlingization $D_m(\mathscr{A})$ has the roots

$$m \cdot r_1 + 1, m \cdot r_2 + 1, ..., m \cdot r_n + 1.$$

We first prove a result about the characteristic polynomial of a frame subspace arrangement which will be used in the proof of Theorem 3.2.

PROPOSITION 3.4. *Let η be a valuation defined by $\eta(A) = v(A \cap \mathbf{k}^{*n})$. Then for W a frame subspace we have*

$$\eta(W) = (t-1)^{\dim W}.$$

Moreover, let $\{A_1, ..., A_k\}$ be a collection of frame subspaces and let L be the intersection lattice of the subspace arrangement

$$\mathscr{A} = \{A_1, ..., A_k\} \cup \{x_i = 0\}_{i=1}^{n}.$$

Let K be the sublattice of the lattice L consisting of all frame subspaces. Then the characteristic polynomial of the subspace arrangement $\mathscr{A}$ is given by

$$\chi(\mathscr{A}) = \sum_{z \in K} \mu(\hat{0}, z) \cdot \eta(z).$$

Proof. To compute $\eta(W) = v(W \cap \mathbf{k}^{*n})$ for W a frame subspace, we need to consider the corresponding directed graph of W. We choose the values for the variables x_i clique by clique. In a given clique, once we choose the value of one of the variables, the values of the other variables in the same clique are determined. For each clique, this can be done in $t-1$ ways. Hence, the result is $(t-1)^k$, where k is the number of cliques, that is, the dimension of W.

To obtain the second statement, begin to observe that

$$\mathbf{k}^n - \bigcup_{A \in \mathscr{A}} A = \mathbf{k}^{*n} - \bigcup_{A \in \mathscr{A}} A.$$

Apply now the valuation v. The characteristic polynomial of the subspace arrangement $\mathscr{A}$ is equal to

$$\begin{aligned} \chi(\mathscr{A}) &= v\left(\mathbf{k}^{*n} - \bigcup_{A \in \mathscr{A}} A\right) \\ &= \eta\left(\mathbf{k}^{*n} - \bigcup_{A \in \mathscr{A}} A\right) \\ &= \sum_{z \in L} \mu(\hat{0}, z) \cdot \eta(z). \end{aligned}$$

Since for elements z not belonging to K we have $\eta(z) = 0$, the result follows. ∎

A *balanced cycle* in the gain graph is a directed cycle such that the products of the labels are equal to 1. Recall that we view the directed edge from i to j labeled with a to be equivalent to the directed edge from j to i labeled with a^{-1}.

EXAMPLE 3.5. The *generic arrangement* $\mathscr{G}$ with parameters $(k(i, j))_{1 \leqslant i < j \leqslant n}$ is the frame hyperplane arrangement

$$\begin{cases} x_i = a_{i,j,l} \cdot x_j, & 1 \leqslant i < j \leqslant n \quad \text{and} \quad 1 \leqslant l \leqslant k(i, j), \\ x_i = 0, & 1 \leqslant i \leqslant n, \end{cases}$$

such that there are no balanced cycles in the associated gain graph. For instance, one can obtain such an arrangement by choosing the defining constants $a_{i,j,l}$ so that no algebraic relations hold between them. By applying Proposition 3.4 we know that the characteristic polynomial is given by

$$\chi(\mathscr{G}) = \sum_F (-1)^{n-|F|} \cdot k^F \cdot (t-1)^{n-|F|}, \tag{3.1}$$

where the sum is over all forests F on the vertices 1 through n, $|F|$ denotes the number of edges in the forest, and $k^F = \prod_{(i,j) \in F} k_{i,j}$. See, for instance, Postnikov and Stanley [12].

To prove Theorem 3.2, we need to introduce a new valuation on a new Boolean algebra. Let S be an infinite multiplicative subgroup of an infinite field $\mathbf{k}$ and let V be the vector space $\mathbf{k}^n$. A frame subspace W of V is called an S-*subspace* if W is the intersection of hyperplanes $x_i = a \cdot x_j$, where a belongs to S. Let $\mathscr{B}$ be the smallest Boolean algebra containing all S-subspaces and the coordinate hyperplanes. Observe that the valuation η is defined on the Boolean algebra $\mathscr{B}$.

Proof of Theorem 3.2. Set $S_{\mathbb{R}}$ to be the positive reals viewed as an open ray in the complex plane. Similarly, set $S_{\mathbb{C}}$ to be $S = \bigcup_{i=0}^{m-1} \zeta^i \cdot S_{\mathbb{R}}$. For the two corresponding Boolean algebras $\mathscr{B}_{\mathbb{R}}$ and $\mathscr{B}_{\mathbb{C}}$ there is a natural map $\varphi : \mathscr{B}_{\mathbb{C}} \to \mathscr{B}_{\mathbb{R}}$ sending the hyperplane $x_i = a \cdot x_j$ in $\mathscr{B}_{\mathbb{C}}$ to the hyperplane $x_i = |a| \cdot x_j$ in $\mathscr{B}_{\mathbb{R}}$. Moreover, the coordinate hyperplane $x_i = 0$ is mapped to itself. Now extend φ by the Boolean operations.

Let $\eta_{\mathbb{R}}$ and $\eta_{\mathbb{C}}$ be the valuation η on the Boolean algebras $\mathscr{B}_{\mathbb{R}}$, respectively $\mathscr{B}_{\mathbb{C}}$. Let $K_{\mathbb{R}}$ denote the $S_{\mathbb{R}}$-subspaces in the intersection lattice of $\mathscr{A}$ and let $\mu_{\mathbb{R}}$ denote the Möbius function of the intersection lattice. Similarly, let $K_{\mathbb{C}}$ and $\mu_{\mathbb{C}}$ denote the corresponding notions in the intersection lattice of the Dowlingization $D_m(\mathscr{A})$.

Let w be an element in $K_{\mathbb{R}}$ of rank k. That is, w is an $S_{\mathbb{R}}$-subspace of codimension k. Since w is determined by the k equations describing the k hyperplanes intersecting to form w, under the map φ^{-1} there are m^k ways to sign these k equations. Observe also that for all of these complex subspaces z we have $\mu_{\mathbb{C}}(\hat{0}, z) = \mu_{\mathbb{R}}(\hat{0}, w)$. Define the linear map Φ on polynomials of degree at most n by $\Phi(f(t-1)) = m^n \cdot f((t-1)/m)$. We have that $\Phi((t-1)^{n-k}) = m^k \cdot (t-1)^{n-k}$ and hence $\sum_z \eta_{\mathbb{C}}(z) = \Phi(\eta_{\mathbb{R}}(w))$ where the sum ranges over all subspaces z such that $\varphi(z) = w$. Now we have

$$\begin{aligned}
\chi(D_m(\mathscr{A})) &= \sum_{z \in K_{\mathbb{C}}} \mu_{\mathbb{C}}(\hat{0}, z) \cdot \eta_{\mathbb{C}}(z) \\
&= \sum_{w \in K_{\mathbb{R}}} \sum_{\varphi(z) = w} \mu_{\mathbb{C}}(\hat{0}, z) \cdot \eta_{\mathbb{C}}(z) \\
&= \sum_{w \in K_{\mathbb{R}}} m^{\mathrm{codim}(w)} \cdot \mu_{\mathbb{R}}(\hat{0}, w) \cdot \eta_{\mathbb{R}}(w) \\
&= \sum_{w \in K_{\mathbb{R}}} \mu_{\mathbb{R}}(\hat{0}, w) \cdot \Phi(\eta_{\mathbb{R}}(w)) \\
&= \Phi(\chi(\mathscr{A})),
\end{aligned}$$

which completes the proof. ∎

EXAMPLE 3.6. The braid arrangement A_n is given by

$$\begin{aligned}
x_i &= x_j, & 1 \leqslant i < j \leqslant n, \\
x_i &= 0, & 1 \leqslant i \leqslant n.
\end{aligned}$$

It is well-known that the characteristic polynomial of A_n has the roots $1, \ldots, n$. The Dowlingization of A_n is the Dowling arrangement:

$$\begin{aligned}
x_i &= \zeta^h \cdot x_j, & 1 \leqslant i < j \leqslant n \quad \text{and} \quad 0 \leqslant h \leqslant m-1, \\
x_i &= 0, & 1 \leqslant i \leqslant n.
\end{aligned}$$

Thus by Corollary 3.3 we know that the characteristic polynomial of the Dowling arrangement has the roots $1, m+1, \ldots, (n-1) \cdot m + 1$. There are many computations of this fact. See, for instance, [4–7].

EXAMPLE 3.5 REVISITED. Let $\mathscr{G}$ be a real generic arrangement, that is, the defining constants are real. Observe that the products of the labels along a directed cycle in the gain graph associated with the Dowlingization $D_m(\mathscr{G})$ is a power of ζ times the product along the underlying cycle in the original gain graph. Hence the Dowlingization of a real generic arrangement is again a generic arrangement. The characteristic polynomial of

$D_m(\mathscr{G})$ is given by Eq. (3.1) with the term k^F replaced with $m^{|F|} \cdot k^F$. This agrees with Theorem 3.2.

4. SUPERSOLVABILITY AND A CONCLUDING REMARK

In this section we show that the Dowling transform preserves supersolvability. For readers not familiar with this concept, see [13]. We begin by discussing further properties the valuation v satisfies.

For T a subset of $\mathbf{k}^n$ recall that the valuation of v of T can be calculated by

$$v(T) = \int I_T \, dv = \int \cdots \int I_T \, dx_1 \cdots dx_n,$$

where I_T is the indicator function of the set T. Let T_i be the projection of the set T onto the linear subspace spanned by the variables $x_i, ..., x_n$, for $i = 1, ..., n$. That is, T_i is given by

$$T_i = \{(x_i, ..., x_n) : \text{there exist elements } x_1, ..., x_{i-1} \in \mathbf{k} \text{ such that } (x_1, ..., x_n) \in T\}.$$

We say that the valuation v of the set T can be *evaluated coordinatewise in the order* $x_1, ..., x_n$ if

$$\int I_{T_i} \, dx_i = b_i \cdot I_{T_{i+1}} \tag{4.2}$$

for a suitable sequence $b_1, ..., b_n$ of polynomials in t of degree at most one. Observe that both sides of this identity are functions in the variables $x_{i+1}, ..., x_n$. In this case we have $v(T_i) = b_i \cdots b_n$ and especially $v(T) = b_1 \cdots b_n$. Finally, we say that $v(T)$ can be *evaluated coordinatewise* if there is an ordering of the coordinates such that $v(T)$ can be evaluated in that order.

THEOREM 4.1. *If the frame hyperplane arrangement $\mathscr{H}$ is supersolvable then so is the Dowlingization $D_m(\mathscr{H})$.*

Proof. By [17] a hyperplane arrangement $\mathscr{H}$ is supersolvable if and only if the characteristic polynomial $\chi(\mathscr{H}) = v(\mathbf{k}^n - \bigcup_{H \in \mathscr{H}} H)$ can be computed coordinatewise. We tacitly assume that the order is $x_1, ..., x_n$. In this case we have $b_i = t - 1 - r_i$ and

$$\chi(\mathscr{H}) = \prod_{i=1}^{n} (t - 1 - r_i). \tag{4.3}$$

Here the r_i count the number of hyperplanes of the form $x_i = a \cdot x_p$ where p is less than i. Moreover, the 1 occurring in each term corresponds to the hyperplane $x_i = 0$.

Observe that Eq. (4.2) is equivalent to the following condition: If the hyperplanes $x_i = a \cdot x_p$ and $x_i = b \cdot x_q$, with $p, q < i$, belong to the arrangement then so does the hyperplane $x_p = b/a \cdot x_q$. But if $\mathscr{H}$ satisfies this condition so does the Dowlingization $D_m(\mathscr{H})$ and the result follows. We also obtain that the characteristic polynomial of $D_m(\mathscr{H})$ is

$$\chi(D_m(\mathscr{H})) = \prod_{i=1}^{n} (t - 1 - m \cdot r_i). \quad \blacksquare \tag{4.4}$$

Gottlieb and Wachs [8] computed the homology of the Dowling lattice and gave a basis for the non-trivial top homology. This is an extension of what is known for the partition lattice. We ask if their computations can be extended to supersolvable lattices coming from frame arrangements $\mathscr{H}$. Moreover, could a basis for the homology of the intersection lattice of $D_m(\mathscr{H})$ be determined from knowing a basis of the homology of $\mathscr{H}$?

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