

# LEVEL EULERIAN POSETS

RICHARD EHRENBORG, GÁBOR HETYEI, AND MARGARET READDY

ABSTRACT. The notion of level posets is introduced. This class of infinite posets has the property that between every two adjacent ranks the same bipartite graph occurs. When the adjacency matrix is indecomposable, we determine the length of the longest interval one needs to check to verify Eulerianness. Furthermore, we show that every level Eulerian poset associated to an indecomposable matrix has even order. A condition for verifying shellability is introduced and is automated using the algebra of walks. Applying the Skolem–Mahler–Lech theorem, the  $\mathbf{ab}$ -series of a level poset is shown to be a rational generating function in the non-commutative variables  $\mathbf{a}$  and  $\mathbf{b}$ . In the case the poset is also Eulerian, the analogous result holds for the  $\mathbf{cd}$ -series. Using coalgebraic techniques a method is developed to recognize the  $\mathbf{cd}$ -series matrix of a level Eulerian poset.

## 1. INTRODUCTION

It is the instinct of every mathematician that whenever an infinite object is defined in terms of a finite object one should be able to describe various apparently infinite properties of the infinite object in finitely-many terms. For example, when one considers infinite random walks on a finite digraph it is satisfying to be able to describe the asymptotic properties of these random walks in terms of the finitely-many eigenvalues of the adjacency matrix.

In this paper we consider infinite partially ordered sets (posets) associated to finite directed graphs. The level poset of a graph that we introduce is an infinite voltage graph closely related to the finite voltage graphs studied by Gross and Tucker. It is a natural question to consider whether a level poset has Eulerian intervals, that is, every non-singleton interval satisfies the Euler–Poincaré relation. One method to form Eulerian posets is via the doubling operations. This corresponds to a standard trick widely used in the study of network flows. We extend these operations to level posets.

We look at questions that are often asked in the study of Eulerian posets: verifying Eulerianness, finding sufficient conditions which imply the order complex is shellable and describing the flag numbers. Usually these questions are aimed at a specific family of finite posets and explicit answers are given. Here we instead look at infinitely-many intervals defined by a single finite directed graph.

When the underlying graph of a level poset is strongly connected, we show that it is enough to verify the Eulerian condition for intervals up to a certain rank. This bound is linear in terms of the two parameters period and index of the level poset. Furthermore, for these Eulerian level posets we

---

*Date:* June 28, 2010.

*2000 Mathematics Subject Classification.* Primary 06A07; Secondary 05A15, 05C30, 52B22, 57M15.

also obtain that their order must be even. The order two Eulerian poset is the classical butterfly poset. See Example 4.7 for an order 4 example.

To show that a level poset has shellable intervals we introduce the vertex shelling order condition. This condition is an instance of Kozlov's  $CC$ -labelings and it implies shellability. Furthermore, we prove it is enough to verify this condition for intervals whose length is bounded by the sum of period and the index. This is still a large task. However, we automate it using the algebra of walks, reducing the problem of computing powers of a certain matrix modulo an ideal. See Example 6.10 for such a calculation. In this example we conclude that the order complexes of the intervals are not just homotopic equivalent to spheres, but homeomorphic to them.

The  $\mathbf{cd}$ -index is an invariant encoding the flag  $f$ -vector of an Eulerian poset which removes all linear relations among the flag  $f$ -vector entries. It is a non-commutative homogeneous polynomial in the two variables  $\mathbf{c}$  and  $\mathbf{d}$ . For level Eulerian posets there are infinitely-many intervals. We capture this information by summing all the  $\mathbf{cd}$ -indices. This gives a non-commutative formal power series which we call the  $\mathbf{cd}$ -series. We show that the  $\mathbf{cd}$ -series is a rational non-commutative generating function. See Theorem 7.4.

Recall that the infinite butterfly poset has the property that the  $\mathbf{cd}$ -index of any length  $m + 1$  interval equals  $\mathbf{c}^m$ . In our order 4 example of a level Eulerian poset, there are intervals of length  $m + 1$  whose  $\mathbf{cd}$ -index is the sum of every degree  $m$   $\mathbf{cd}$ -monomial. See Corollary 8.3.

In the concluding remarks we end with some open questions.

## 2. PRELIMINARIES

**2.1. Graded, Eulerian and half-Eulerian posets.** A partially ordered set  $P$  is *graded* if it has a unique minimum element  $\widehat{0}$ , a unique maximum element  $\widehat{1}$  and a rank function  $\rho : P \rightarrow \mathbb{N}$  such that  $\rho(\widehat{0}) = 0$  and for every cover relation  $x \prec y$  we have  $\rho(y) - \rho(x) = 1$ . The rank of  $\widehat{1}$  is called the *rank* of the poset. For two elements  $x \leq y$  in  $P$  define the rank difference  $\rho(x, y)$  by  $\rho(y) - \rho(x)$ . Given a graded poset  $P$  of rank  $n + 1$  and a subset  $S \subseteq \{1, \dots, n\}$ , define the  $S$ -rank selected subposet of  $P$  to be the poset  $P_S = \{x \in P : \rho(x) \in S\} \cup \{\widehat{0}, \widehat{1}\}$ . The *flag  $f$ -vector* ( $f_S(P) : S \subseteq \{1, \dots, n\}$ ) of  $P$  is the  $2^n$ -dimensional vector whose entry  $f_S(P)$  is the number of maximal chains in  $P_S$ . For further details about graded posets, see Stanley [29].

A graded partially ordered set  $P$  is *Eulerian* if every interval  $[x, y]$  in  $P$  of rank at least 1 satisfies  $\sum_{x \leq z \leq y} (-1)^{\rho(z)} = 0$ . Equivalently, the Möbius function  $\mu$  of the poset  $P$  satisfies  $\mu(x, y) = (-1)^{\rho(x, y)}$ . Classical examples of Eulerian posets include the face lattices of polytopes and the Bruhat order of a Coxeter group.

The *horizontal double*  $D_{\leftrightarrow}(P)$  of a graded poset  $P$  is obtained by replacing each element  $x \in P - \{\widehat{0}, \widehat{1}\}$  by two copies  $x_1$  and  $x_2$  and preserving the partial order of the original poset  $P$ , that is, we set  $x_i < y_j$  in  $D_{\leftrightarrow}(P)$  if and only if  $x < y$  holds in  $P$ . Following Bayer and Hetyei [4, 5], we call a

graded poset  $P$  *half-Eulerian* if its horizontal double is Eulerian. The following lemma appears in [4, Proposition 2.2].

**Lemma 2.1** (Bayer–Hetyei). *A graded partially ordered set  $P$  is half-Eulerian if and only if for every non-singleton interval  $[x, y]$  of  $P$*

$$\sum_{x < z < y} (-1)^{\rho(x, z)-1} = \begin{cases} 1 & \text{if } \rho(x, y) \text{ is even,} \\ 0 & \text{if } \rho(x, y) \text{ is odd.} \end{cases}$$

As noted in [5, Section 4], every graded poset  $P$  gives rise to a half-Eulerian poset via the “vertical doubling” operation.

**Definition 2.2.** *Given a graded poset  $P$ , the vertical double of  $P$  is the set  $D_{\uparrow}(P)$  obtained by replacing each  $x \in P - \{\widehat{0}, \widehat{1}\}$  by two copies  $x_1$  and  $x_2$ , with  $u <_{D_{\uparrow}(P)} v$  in  $Q$  exactly when one of the following conditions hold:*

- (i)  $u = \widehat{0}$ ,  $v \in P - \{\widehat{0}\}$ ;
- (ii)  $u \in P - \{\widehat{1}\}$ ,  $v = \widehat{1}$ ;
- (iii)  $u = x_1$  and  $v = x_2$  for some  $x \in P - \{\widehat{0}, \widehat{1}\}$ ; or
- (iv)  $u = x_i$  and  $v = y_j$  for some  $x, y \in P - \{\widehat{0}, \widehat{1}\}$ , with  $x <_P y$ .

**Lemma 2.3** (Bayer–Hetyei). *For a graded poset  $P$ , the vertical double  $D_{\uparrow}(P)$  is a half-Eulerian poset.*

**2.2. Shelling the order complex of a graded poset.** Recall a simplicial complex  $\Delta$  is a family of subsets (*faces*) of a finite vertex set  $V$  satisfying  $\{v\} \in \Delta$  for all  $v \in V$  and if  $\sigma \in \Delta$  and  $\tau \subseteq \sigma$  then  $\tau \in \Delta$ . Maximal faces are called *facets*. In this paper we will only consider order complexes of graded posets. The *order complex*  $\Delta(P)$  of a graded poset  $P$  is the simplicial complex with vertex set  $P - \{\widehat{0}, \widehat{1}\}$  whose faces are the chains contained in  $P - \{\widehat{0}, \widehat{1}\}$ , that is,

$$\Delta(P) = \{\{x_1, x_2, \dots, x_k\} : \widehat{0} < x_1 < x_2 < \dots < x_k < \widehat{1}\}.$$

A simplicial complex is *pure* if every facet has the same dimension. For a graded poset  $P$  of rank  $n + 1$ , the order complex  $\Delta(P)$  is pure of dimension  $n - 1$ . A pure simplicial complex  $\Delta$  is *shellable* if there is an ordering  $F_1, F_2, \dots, F_t$  of its facets such that for every  $k \in \{2, \dots, t\}$  the collection of faces of  $F_k$  contained in some earlier  $F_i$  is itself a pure simplicial complex of dimension  $\dim(\Delta) - 1$ . Equivalently, there exists a face  $R(F_k)$  of  $F_k$ , called the *facet restriction*, not contained in any earlier facet such that every face  $\sigma \subseteq F_k$  not contained in any earlier  $F_i$  contains  $R(F_k)$ . A complex being shellable implies it is homotopy equivalent to a wedge of spheres of the same dimension as the complex. For further details, we refer the reader to the articles of Björner and Wachs [8, 31].

A shelling of the order complex of a graded poset is usually found by labeling the cover relations in the maximal chains of  $P$ . The first such labelings were the *CL-labelings* introduced by Björner and Wachs [9, 10]. In this paper we will consider a special example of Kozlov’s *CC-labelings* [19], which were rediscovered independently by Hersh and Kleinberg (see the Introduction of [1]).

**2.3. Periodicity of nonnegative matrices.** We will need a few facts regarding sufficiently high powers of nonnegative square matrices. Unless noted otherwise, all statements cited in this subsection may be found in the monograph of Sachkov and Tarakanov [26, Chapter 6].

The *underlying digraph*  $\Gamma(A)$  of a square matrix  $A = (a_{i,j})_{1 \leq i,j \leq n}$  with nonnegative entries is the directed graph on the vertex set  $\{1, 2, \dots, n\}$  with  $(i, j)$  being an edge if and only if  $a_{i,j} > 0$ . Here and in the rest of the paper we use the notation  $A^k = (a_{i,j}^{(k)})_{1 \leq i,j \leq n}$ . Given a vertex  $i$  such that there is a directed walk of positive length from  $i$  to  $i$ , the *period*  $d(i)$  of the vertex  $i$  is the greatest common divisor of all positive integers  $k$  satisfying  $a_{i,i}^{(k)} > 0$ . The period is constant on strong components of  $\Gamma(A)$ .

**Lemma 2.4.** *If the vertices  $i \neq j$  of  $\Gamma(A)$  belong to the same strong component then  $d(i) = d(j)$ .*

The matrix  $A$  is *indecomposable* or *irreducible* if for any  $i, j \in \{1, 2, \dots, n\}$  there is a  $t > 0$  such that the  $(i, j)$  entry of  $A^t$  is positive. It is easy to see that  $A$  is indecomposable if and only if its underlying digraph is strongly connected, that is, for any pair of vertices  $i$  and  $j$  there is a directed walk from  $i$  to  $j$ . As a consequence of Lemma 2.4 all vertices of the underlying graph of an indecomposable matrix  $A$  have the same period. We call this number the *period of the indecomposable matrix*  $A$ . Given an  $n \times n$  indecomposable matrix of period  $d$ , for each  $i = 1, 2, \dots, n$  there exists an integer  $t_0(i)$  such that  $a_{i,i}^{(kd)} > 0$  holds for all  $k \geq t_0(i)$ . Using this observation is easy to show the following theorem. See [26, Theorem 6.2.2 and Lemma 6.2.3].

**Theorem 2.5.** *Let  $A$  be an  $n \times n$  indecomposable nonnegative matrix of period  $d$ . If  $i$  is a fixed vertex of the digraph  $\Gamma(A)$  then for any other vertex  $j$  there is a unique integer  $r_j$  such that  $0 \leq r_j \leq d - 1$  and the following two statements hold:*

- (1)  $a_{i,j}^{(s)} > 0$  implies  $s \equiv r_j \pmod{d}$ ,
- (2) there is a positive  $t(j)$  such that  $a_{i,j}^{(kd+r_j)} > 0$  for all  $k \geq t(j)$ .

Setting  $j \in C_r$  if and only if  $r_j = r$  provides a partitioning  $\{1, 2, \dots, n\} = \bigsqcup_{q=0}^{d-1} C_q$ . Replacing  $i$  with an arbitrary fixed vertex results in the same ordered list of subclasses, up to a cyclic rotation of the indices.

Ordering the elements of the set  $\{1, 2, \dots, n\}$  in such a way that the elements of each block  $C_q$  form a consecutive sublist results in a block matrix of the form

$$(2.1) \quad A = \begin{pmatrix} 0 & Q_{0,1} & 0 & \cdots & 0 \\ 0 & 0 & Q_{1,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{d-2,d-1} \\ Q_{d-1,0} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where  $Q_{q,q+1}$  occupies the rows indexed with  $C_q$  and the columns indexed by  $C_{q+1}$  (here we set  $C_d = C_0$ ). The block matrix in (2.1) is the *canonical form* of the indecomposable matrix  $A$  with

period  $d$ . By Theorem 2.5 there is a  $t > 0$  such that the canonical form of  $A^{td+1}$  is similar to the one given in (2.1) with the additional property that all entries in the blocks  $Q_{q,q+1}$  are strictly positive.

An  $n \times n$  matrix  $A$  with nonnegative entries is *primitive* if there is a  $\gamma > 0$  such that all entries of  $A^\gamma$  are positive. The smallest  $\gamma$  with the above property is the *exponent* of the primitive matrix  $A$ . It is straightforward to see that a nonnegative matrix is primitive if and only if it is indecomposable and *aperiodic*, i.e., its period  $d$  equals 1. There is a quadratic upper bound on the exponent of a primitive matrix due to Holladay and Varga [18]. See [26, Theorem 6.2.10].

**Theorem 2.6** (Holladay–Varga). *The exponent  $\gamma$  of an  $n \times n$  primitive matrix satisfies*

$$\gamma \leq n^2 - 2n + 2.$$

The matrix operator we are about to introduce will be frequently used in our paper and makes also stating the next few results easier.

**Definition 2.7.** *Given a matrix  $A$  with nonnegative entries, let  $\text{Bin}(A)$  denote the binary matrix formed by replacing each nonzero entry of  $A$  with 1. We call the resulting matrix the binary reduction of  $A$ .*

Thus a matrix  $A$  is primitive if there exists a power  $k$  such that  $\text{Bin}(A^k) = J$ , where  $J$  is the matrix consisting of all 1's.

A generalization of Theorem 2.5 may be found in the work of Heap and Lynn [17]. See also [23, 24, 25].

**Theorem 2.8.** *Given any non-negative square matrix  $A$  there exists integers  $d$  and  $\gamma$  such that  $\text{Bin}(A^{t+d}) = \text{Bin}(A^t)$  for all  $t \geq \gamma$ .*

The smallest integer  $\gamma$  such that for all  $t \geq \gamma$  we have  $\text{Bin}(A^{t+d}) = \text{Bin}(A^t)$  is known as the *index* of the matrix. For a primitive matrix this is the exponent. For estimates on the period  $d$  and the index  $\gamma$ , we refer the reader to [17]. Here we only wish to emphasize the following immediate generalization of Theorem 2.6. See [17, Equation (1.4)].

**Theorem 2.9.** *Let  $A$  be an  $n \times n$  indecomposable matrix with nonnegative entries having period  $d$ . An upper bound for the index  $\gamma$  of  $A$  is  $\gamma \leq (q^2 - 2q + 2)d + 2r$ , where  $n = qd + r$  with  $0 \leq r < d - 1$ .*

Let  $\Gamma$  be any digraph on a vertex set  $V$  such that its edge set  $E$  is a subset of  $V \times V$ , i.e.,  $\Gamma$  may have loops but no multiple edges. Recall the *adjacency matrix*  $A$  of  $\Gamma$  is a  $|V| \times |V|$  matrix whose rows and columns are indexed by the vertices. The entry in the row indexed by  $u \in V$  and in the column  $v \in V$  is 1 if  $(u, v) \in E$ , and zero otherwise. Clearly  $\Gamma$  is the underlying graph of its adjacency matrix. We may extend the above notions of period and aperiodicity from matrices to digraphs by defining the period of a digraph to be the period of its adjacency matrix. See for instance [22]. In particular, a directed graph is aperiodic if and only if it is strongly connected and there is no  $k > 1$  that divides the length of every directed cycle.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

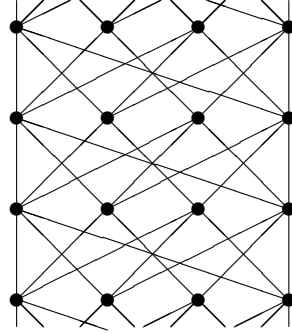


FIGURE 1. An adjacency matrix and its associated level poset. Note that this is an example of a level Eulerian poset.

### 3. LEVEL POSETS

**Definition 3.1.** A partially ordered set  $P$  is a level poset if the set of its elements is of the form  $V \times \mathbb{Z}$  for some finite nonempty set  $V$ , the projection onto the second coordinate is a rank function, and for any  $u, v \in V$  and  $i \in \mathbb{Z}$  we have  $(u, i) < (v, i + 1)$  if and only if  $(u, 0) < (v, 1)$  holds.

Informally speaking, the Hasse diagram of a level poset can be thought of as a graph containing a copy of the same vertex set  $V$  at each “level” such that the portion of the Hasse diagram containing the edges between elements of rank  $i$  and rank  $i + 1$  may be obtained by vertically shifting the edges in the Hasse diagram between the elements of rank 0 and 1. An example of a level poset is shown in Figure 1. (The meaning of the term Eulerian in this context will be explained in Section 4.)

Clearly it is sufficient to know the cover relations of the form  $(u, 0) \prec (v, 1)$  to obtain a complete description of a level poset. Introducing the digraph  $G$  with vertex set  $V$  and edge set  $E := \{(u, v) \in V \times V : (u, 0) < (v, 1)\}$ , we obtain a digraph representing a relation  $E \subseteq V \times V$  on the vertex set  $V$ , i.e., a digraph with no multiple edges but possibly containing loops. We call  $G$  the *underlying graph* of the level poset  $P$  and  $P$  the *level poset of  $G$* . The poset  $P$  and the digraph  $G$  determine each other uniquely.

**Lemma 3.2.** Let  $P$  be a level poset on  $V \times \mathbb{Z}$  and let  $G$  be its underlying digraph. Then for all  $i, j \in \mathbb{Z}$  and for all  $u, v \in V$  we have  $(u, i) < (v, j)$  in  $P$  if and only if  $i < j$  and there is a walk  $u = u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_{j-i} = v$  of length  $j - i$  from  $u$  to  $v$  in  $G$ .

The straightforward verification is left to the reader.

**Remark 3.3.** By directing all the edges upwards in the Hasse diagram of  $P$ , we obtain the (*right*) *derived graph* of the voltage graph obtained from  $G$  by assigning the voltage  $1 \in \mathbb{Z}$  to each directed edge. For a detailed discussion of the theory of voltage graphs, we refer the reader to the work of

Gross and Tucker [16]. The classical theory of voltage graphs focuses on the case where the voltages belong to a finite group. Here we have to consider  $\mathbb{Z}$ , that is, the simplest possible infinite group.

Since the underlying digraph of a level poset  $G$  is uniquely determined by its adjacency matrix, every level poset is uniquely determined by the adjacency matrix of its underlying digraph. For brevity, we will use the term *underlying matrix*  $M$  for “adjacency matrix of the underlying digraph” of a level poset  $P$ , and the term *level poset of  $M$*  for “level poset of the digraph whose adjacency matrix is  $M$ ”. The order of the rows and columns of the adjacency matrix corresponds to the order of vertices at the same level read from the left to the right in a Hasse diagram of the corresponding level poset. For any square matrix  $M$  whose rows and columns are indexed with elements of a set  $V$ , we will use the notation  $M_{u,v}$  for the entry in the row indexed by  $u \in V$  and in the column indexed by  $v \in V$ .

Using Lemma 3.2 we may describe the partial order of  $P$  in terms of its underlying matrix  $M$  as follows.

**Corollary 3.4.** *Given a level poset  $P$  with underlying matrix  $M$ , we have  $(u, i) < (v, j)$  in  $P$  if and only if  $i < j$  and  $M_{u,v}^{j-i} > 0$  hold.*

Using the operation  $\text{Bin}$  we may rephrase Corollary 3.4 as follows.

**Corollary 3.5.** *Given a level poset  $P$  with underlying matrix  $M$ , we have  $(u, i) < (v, j)$  in  $P$  if and only if  $i < j$  and  $\text{Bin}(M^{j-i})_{u,v} = 1$  hold.*

Clearly the underlying digraph of a level poset is strongly connected if and only if the underlying matrix  $M$  is indecomposable. Equivalently, for any pair of vertices  $u, v \in V$  such that  $u \neq v$  there is a  $p > 0$  such that the adjacency matrix  $M$  satisfies  $\text{Bin}(M^p)_{u,v} = 1$ . If  $|V| > 1$  then the adjacency matrix  $M$  of a strongly connected digraph must also satisfy  $\text{Bin}(M^p)_{u,u} = 1$  for some  $p$ , given any  $u \in V$ .

Having a strongly connected underlying digraph is neither a necessary nor sufficient condition for the Hasse diagram of a level poset (considered as an undirected graph) to be a connected graph. An example of a connected level poset whose underlying digraph is not strongly connected is the level poset with the underlying adjacency matrix  $M = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ . For level posets with strongly connected underlying digraphs, a necessary and sufficient condition for the connectivity of their Hasse diagram may be stated using the notion of aperiodic graphs. A digraph on  $n$  vertices is *aperiodic* if and only if the underlying adjacency matrix  $M$  is primitive.

**Theorem 3.6.** *Assume that  $P$  is the level poset of a strongly connected digraph  $G$ . Then the Hasse diagram of  $P$  is connected if and only if  $G$  is aperiodic.*

*Proof.* Assume first  $G$  is aperiodic and that its adjacency matrix  $M$  satisfies  $\text{Bin}(M^p) = J$ . Clearly  $\text{Bin}(M^n) = J$  for all  $n \geq p$ . Given any  $(u, i)$  and  $(v, j)$  in  $P$  there is a directed walk of length  $p + \max(i, j) - i$  from  $u$  to  $u$  and a directed walk of length  $p + \max(i, j) - j$  from  $v$  to  $u$  in  $G$ . The first walk lifts to a walk from  $(u, i)$  to  $(u, \max(i, j) + p)$  in the Hasse diagram of  $P$ , whereas the second

walk lifts to a walk from  $(v, j)$  to  $(u, \max(i, j) + p)$  in the Hasse diagram. Thus we may walk from  $(u, i)$  to  $(v, j)$  by first walking along the edges of the walk from  $(u, i)$  to  $(u, \max(i, j) + p)$  and then following the edges of the walk from  $(v, j)$  to  $(u, \max(i, j) + p)$  backwards.

Assume next that  $G$  is not aperiodic. Let  $k > 1$  be an integer dividing the length of every cycle. It is easy to see that we may color the vertex set  $V$  of  $G$  using  $k$  colors in such a way that  $(u, v)$  is an edge only if  $u$  and  $v$  have different colors. This coloring may be lifted to the Hasse diagram of  $P$  by setting the color of  $(u, i)$  to be the color of  $u$  for each  $(u, i) \in P$ . There is no walk between elements of the same color in the Hasse diagram of  $P$ .  $\square$

Let  $\alpha = (\alpha_1, \dots, \alpha_r)$  be a composition of  $m$ , that is,  $\alpha_1, \dots, \alpha_r$  are positive integers whose sum is  $m$ . Let  $S$  be the associated subset of  $\{1, \dots, m-1\}$ , that is,

$$S = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{r-1}\}.$$

The flag  $f$ -vector entry  $S$  of any interval  $[(u, i), (v, i+m)] \cong [(u, 0), (v, m)]$  in a level poset may be computed using its underlying adjacency matrix as follows.

**Lemma 3.7.** *Let  $P$  be a level poset whose underlying digraph has vertex set  $V$  of cardinality  $n$ . Let  $F_S$  be the  $n \times n$  matrix whose  $(u, v)$  entry is  $f_S([(u, 0), (v, m)])$  if  $(u, 0) \leq (v, m)$  in  $P$  and 0 otherwise. Then the matrix  $F_S$  is given by*

$$F_S = \text{Bin}(M^{\alpha_1}) \cdot \text{Bin}(M^{\alpha_2}) \cdots \text{Bin}(M^{\alpha_r}),$$

where  $(\alpha_1, \dots, \alpha_r)$  is the composition associated with the subset  $S \subseteq \{1, \dots, m-1\}$ .

Note that every interval  $[(u, 0), (v, m)]$  in  $P$  is isomorphic to all intervals of the form  $[(u, i), (v, i+m)]$  where  $i \in \mathbb{Z}$  is an arbitrary integer.

#### 4. LEVEL EULERIAN POSETS

**Definition 4.1.** *We call a level poset  $P$  a level Eulerian poset if every interval is Eulerian.*

As a consequence of Lemma 3.7 we have the following condition for Eulerianness.

**Lemma 4.2.** *A level poset is Eulerian if and only if its adjacency matrix  $M$  satisfies*

$$(4.1) \quad \sum_{i=0}^p (-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p-i}) = 0$$

holds for all  $p \geq 1$ .

As it was noted in [11, Lemma 4.4] and [14, Lemma 2.6], a graded poset of odd rank is Eulerian if all of its proper intervals are Eulerian. Thus it suffices to verify the condition in Lemma 4.2 for even integers  $p$ . As a consequence of Theorem 2.8, equation (4.1) only needs to be verified for finitely-many values of  $p$ .



**Theorem 4.3.** *Let  $P$  be the level poset of an  $n \times n$  indecomposable matrix  $M$  with period  $d$  and index  $\gamma$ . Then  $P$  is level Eulerian if and only if  $M$  satisfies the Eulerian condition (4.1) for  $p < 2\gamma + 4d$ . For odd  $d$ , the bound for  $p$  may be improved to  $p < 2\gamma + 2d$ .*

*Proof.* We introduce  $\Sigma(p)$  as a shorthand for  $\sum_{i=0}^p (-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p-i})$ . We wish to calculate  $\Sigma(p+2d) - \Sigma(p)$  for an arbitrary  $p \geq 2\gamma$ .

For  $i = 0, 1, \dots, \gamma - 1$ , the term  $(-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p-i})$  in  $\Sigma(p)$  cancels with the term  $(-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p+2d-i})$  in  $\Sigma(p+2d)$  since  $i < \gamma(d)$  and  $p \geq 2\gamma$  imply  $p-i \geq \gamma$ . For  $i = \gamma, \gamma+1, \dots, p$ , the term  $(-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p-i})$  in  $\Sigma(p)$  cancels with the term  $(-1)^i \cdot \text{Bin}(M^{i+2d}) \cdot \text{Bin}(M^{p-i})$  in  $\Sigma(p+2d)$  since  $\text{Bin}(M^i) = \text{Bin}(M^{i+2d})$  for  $i \geq \gamma$ . After these cancellations, we obtain

$$(4.2) \quad \Sigma(p+2d) - \Sigma(p) = \sum_{i=\gamma}^{\gamma+2d-1} (-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p+2d-i}) \quad \text{for } p \geq 2\gamma.$$

If  $d$  is odd, then the right-hand side of (4.2) is zero. Indeed, for each  $i$  satisfying  $\gamma \leq i \leq \gamma+d-1$ , the term  $(-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p+2d-i})$  cancels with  $(-1)^{d+i} \cdot \text{Bin}(M^{d+i}) \cdot \text{Bin}(M^{p+d-i})$  since  $(-1)^d = -1$ ,  $\text{Bin}(M^i) = \text{Bin}(M^{d+i})$  and  $\text{Bin}(M^{p+2d-i}) = \text{Bin}(M^{p+d-i})$ . This concludes the proof of the theorem in the case when  $d$  is odd.

Assume from now on that  $d$  is even. Substituting any  $p \geq 2\gamma + 2d$  in (4.2) yields

$$\begin{aligned} \Sigma(p+2d) - \Sigma(p) &= \sum_{i=\gamma}^{\gamma+2d-1} (-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p+2d-i}) \\ &= \sum_{i=\gamma}^{\gamma+2d-1} (-1)^i \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p-i}) = \Sigma(p) - \Sigma(p-2d). \end{aligned}$$

We obtain that

$$(4.3) \quad \Sigma(p+2d) - \Sigma(p) = \Sigma(p) - \Sigma(p-2d) \quad \text{holds for } p \geq 2\gamma + 2d.$$

Therefore, if we verify that  $\Sigma(p) = 0$  holds for  $p \leq 2\gamma + 4d$ , the equality  $\Sigma(p) = 0$  for  $p > 2(\gamma + d)$  may be shown by induction on  $p$  using (4.3).  $\square$

As a consequence of Theorems 2.6 and 4.3 we obtain the following upper bound.

**Corollary 4.4.** *To determine whether the level poset  $P$  of an  $n \times n$  primitive binary matrix  $M$  is a level Eulerian poset one must only verify the Eulerian condition (4.1) for  $p \leq 2n^2 - 4n + 6$ .*

**Remark 4.5.** For a general  $n \times n$  adjacency matrix it seems hard to give a better than exponential estimate as a function of  $n$  for the bounds given in Theorem 4.3. However, for indecomposable matrices we may still obtain a polynomial estimate using Theorem 2.9.

**Example 4.6.** The simplest level Eulerian poset is the butterfly poset whose underlying adjacency matrix is

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has exponent  $\gamma = 1$  and hence by Theorem 4.3 it is enough to verify the Eulerian condition (4.1) for  $p = 2$ .

**Example 4.7.** Consider the level poset shown in Figure 1. Its underlying adjacency matrix  $M$  satisfies

$$\text{Bin}(M^2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and  $\text{Bin}(M^3) = J$ . Thus  $M$  is primitive and the exponent is given by  $\gamma = 3$ . Theorem 4.3 gives the bound  $p < 8$ . Hence to show that this matrix produces a level Eulerian poset, we need verify the Eulerian condition (4.1) for the three values  $p = 2, 4, 6$ , which is a straightforward task.

Starting from the butterfly poset, for each  $n \geq 2$  we may construct a level Eulerian poset whose underlying digraph has  $n$  vertices by repeatedly using the following lemma. The drawback to this construction is that it does not add any more strongly connected components to the underlying graph.

**Lemma 4.8.** *Let  $M$  be a  $n \times n$  matrix whose poset is level Eulerian and let  $\vec{v}$  be a column vector of  $M$ . Then*

- (i) *the level poset of the transpose matrix  $M^T$  is also level Eulerian.*
- (ii) *the  $(n + 1) \times (n + 1)$  matrix*

$$\begin{pmatrix} M & \vec{v} \\ 0 & 0 \end{pmatrix}$$

*is also level Eulerian.*

If we restrict our attention to level posets with strongly connected underlying digraphs, we obtain the following restriction on the order.

**Theorem 4.9.** *Let  $P$  be a level Eulerian poset whose underlying matrix  $M$  is indecomposable. Then the order of the matrix  $M$  is even.*

*Proof.* Let  $d$  and  $\gamma$  be respectively the period and index of the matrix  $M$ . Let  $\delta$  be the least multiple of  $d$  which is greater than or equal to  $\gamma$ . (An upper bound for  $\delta$  is  $\gamma + d - 1$ .) By reordering the vertices of the graph  $G$ , we may assume that the matrix  $M$  has the block form given in equation (2.1). Hence  $M^\delta$  is also a block matrix. Since  $\delta \geq \gamma$ , each block in  $\text{Bin}(M^\delta)$  is either the zero matrix or the matrix  $J$  of all ones. Since  $\delta$  is a multiple of  $d$ , the matrix  $\text{Bin}(M^\delta)$  has the form

$$(4.4) \quad \text{Bin}(M^\delta) = \begin{pmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{pmatrix},$$

where the  $q$ th block is a  $c_q \times c_q$  square matrix whose entries are all 1's. Apply the trace to the Eulerian condition (4.1) for  $p = 2\delta$  and consider this equation modulo 2. Recall  $\text{trace}(AB) = \text{trace}(BA)$  holds for any pair of square matrices, and in particular, it holds for  $A = \text{Bin}(M^i)$  and  $B = \text{Bin}(M^{2\delta-i})$ . Hence the Eulerian condition (4.1) collapses to

$$\text{trace} \left( \text{Bin}(M^\delta)^2 \right) \equiv 0 \pmod{2}.$$

Note that the square of the matrix  $\text{Bin}(M^\delta)$  is given by

$$\text{Bin}(M^\delta)^2 = \begin{pmatrix} c_0 \cdot J & 0 & \cdots & 0 \\ 0 & c_1 \cdot J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{q-1} \cdot J \end{pmatrix}.$$

Hence the trace of the above matrix is  $\sum_{q=0}^{d-1} c_q^2 \equiv \sum_{q=0}^{d-1} c_q \pmod{2}$ . Hence we conclude that the order of  $M$  is an even number.  $\square$

## 5. LEVEL HALF-EULERIAN POSETS

In analogy to level Eulerian posets (Definition 4.1) we define level half-Eulerian posets as follows.

**Definition 5.1.** *A level poset  $P$  is said to be a level half-Eulerian poset if every interval is half-Eulerian.*

In analogy to the horizontal doubling operation introduced in [4, 5], we define the *horizontal double*  $D_{\leftrightarrow}(P)$  of a level poset  $P$  as the poset obtained by replacing each  $(u, i) \in P$  by two copies  $(u_1, i)$ ,  $(u_2, i)$  and preserving the partial order of  $P$ , i.e., setting  $(u_k, i) < (v_l, j)$  in  $D_{\leftrightarrow}(P)$  if and only if  $(u, i) < (v, j)$  holds in  $P$ .

**Proposition 5.2.** *The horizontal double  $D_{\leftrightarrow}(P)$  of a level poset  $P$  is a level poset. In particular, if the underlying adjacency matrix of  $P$  is  $M$  then the underlying adjacency matrix  $D_{\leftrightarrow}(M)$  of  $D_{\leftrightarrow}(P)$  is*

$$(5.1) \quad D_{\leftrightarrow}(M) = \begin{pmatrix} M & M \\ M & M \end{pmatrix}.$$

The straightforward verification is left to the reader. As an immediate consequence of Lemma 2.1, we obtain the following corollary.

**Corollary 5.3.** *A level poset  $P$  is half-Eulerian if and only if its horizontal double  $D_{\leftrightarrow}(P)$  is level Eulerian.*

Lemma 4.2 has the following half-Eulerian analogue.

**Lemma 5.4.** *A level poset is half-Eulerian if and only if its adjacency matrix  $M$  satisfies*

$$(5.2) \quad \sum_{i=1}^{p-1} (-1)^{i-1} \cdot \text{Bin}(M^i) \cdot \text{Bin}(M^{p-i}) = \begin{cases} J & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd,} \end{cases}$$

for all  $p \geq 1$ .

Directly from Lemmas 2.1 and 3.7, a level poset  $P$  is half-Eulerian if and only if its underlying adjacency matrix  $M$  satisfies (5.2) for all  $p > 0$ . It is straightforward to show directly that the adjacency matrix  $M$  of a level poset  $P$  satisfies (5.2) for a given  $p > 0$  if and only if the matrix  $D_{\leftrightarrow}(M)$  given in (5.1) satisfies (4.1) for the same  $p$ . As a consequence, we only need to verify (5.2) for values of  $p$  up to the bound stated in Theorem 4.3.

Unfortunately the natural generalization of the vertical doubling operation to a level poset by replacing each element  $(u, i)$  of a level poset  $P$  with two copies  $(u_1, 2i)$  and  $(u_2, 2i + 1)$ , setting  $(u_1, 2i) < (u_2, 2i + 1)$  for each  $u$ , and setting  $(u_2, 2i + 1) < (v_1, 2j)$  whenever  $(u, i) < (v, j)$  does not work. This operation does not result in a level poset because the cover relations between levels  $2i$  and  $2i + 1$  would be different from the cover relations between levels  $2i - 1$  and  $2i$ . However, we may perform this operation, take two copies, shift the Hasse diagram of one of the copies one step up, and finally intertwine the two copies. In short, consider the level poset with the adjacency matrix

$$(5.3) \quad D_{\uparrow}(M) = \begin{pmatrix} 0 & I \\ M & 0 \end{pmatrix},$$

where  $M$  is the  $n \times n$  underlying adjacency matrix of the level poset  $P$  and  $I$  is the  $n \times n$  identity matrix.

**Definition 5.5.** *Let  $P$  be a level poset with adjacency matrix  $M$ . Define the vertical double  $D_{\uparrow}(P)$  of  $P$  to be the level poset whose adjacency matrix  $D_{\uparrow}(M)$  is given by (5.3).*

As a direct consequence of Lemma 2.3, we have the following corollary.

**Corollary 5.6.** *The vertical double  $D_{\uparrow}(P)$  of an arbitrary level poset  $P$  is level half-Eulerian.*

Combining Corollaries 5.3 and 5.6, we obtain the following statement.

**Corollary 5.7.** *For any square binary matrix  $M$ , the matrix*

$$D_{\leftrightarrow}(D_{\uparrow}(M)) = \begin{pmatrix} 0 & I & 0 & I \\ M & 0 & M & 0 \\ 0 & I & 0 & I \\ M & 0 & M & 0 \end{pmatrix}$$

is the adjacency matrix of a level Eulerian poset.

**Remark 5.8.** The vertical doubling operation induces a widely used operation on the underlying digraph. If  $G$  is a digraph with adjacency matrix  $M$  then  $D_{\uparrow}(M)$  is the adjacency matrix of the digraph  $D_{\uparrow}(G)$  obtained from  $G$  as follows.

- (1) Replace each vertex  $u$  of  $G$  with two copies  $u_1$  and  $u_2$ .
- (2) The edge set of  $D_{\uparrow}(G)$  consists of all edges of the form  $u_1 \rightarrow u_2$  and of all edges of the form  $u_2 \rightarrow v_1$  where  $u \rightarrow v$  is an edge in  $G$ .

Introducing a graph identical to or very similar to  $D_{\uparrow}(G)$  is often used in the study of network flows. These type of constructions also appear in proofs of the vertex-disjoint path variant of Menger's theorem as a way to reduce the study of vertex capacities to that of edge capacities.

Every half-Eulerian poset arising as a vertical double of a level poset has an even number of elements at each level, and each canonical block of its underlying adjacency matrix has even period. This observation may be complemented by the following analogue of Theorem 4.9 for half-Eulerian posets.

**Theorem 5.9.** *Let  $P$  be a level half-Eulerian poset whose underlying matrix  $M$  is primitive. Then the order of the matrix  $M$  is odd.*

*Proof.* Since  $M$  is primitive, let  $\gamma$  be the exponent of the matrix  $M$ . Recall that  $\text{Bin}(M^\gamma) = \text{Bin}(M^{\gamma+1}) = \dots = \text{Bin}(M^{2\gamma}) = J$ . Hence we may rewrite the half-Eulerian condition (5.2) for  $p = 2\gamma$  as

$$(5.4) \quad XJ + (-1)^{\gamma-1}J^2 + JX = J,$$

where  $X = \sum_{i=1}^{\gamma-1} (-1)^{i-1} \cdot \text{Bin}(M^i)$ . Similarly, the half-Eulerian condition (5.2) for  $p = 2\gamma + 1$  yields

$$(5.5) \quad XJ - JX = 0.$$

Combining equations (5.4) and (5.5) modulo 2 yields that  $J^2 \equiv J \pmod{2}$ . Since  $J^2 = nJ$ , the order  $n$  must be odd.  $\square$

It is interesting to note that unlike the proof of Theorem 4.9, the trace operation does not appear in the argument for Theorem 5.9.

## 6. SHELLABLE LEVEL POSETS

Labelings that induce a shelling of the order complex of a graded poset have a vast literature. In the case of level posets it is natural to seek a labeling that may be defined in a uniform fashion for the order complex of every interval. The next definition is an example of such a uniform labeling.

**Definition 6.1.** *Let  $G$  be a directed graph on the vertex set  $V$ . A linear order on  $V$  is a vertex shelling order if for any  $u, v \in V$  and every pair of walks  $u = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_k = v$  and  $u = v'_0 \rightarrow v'_1 \rightarrow \dots \rightarrow v'_k = v$  of the same length such that  $v'_1 < v_1$  holds, there is a  $j \in [1, k-1]$  and a vertex  $w \in V$  such that  $w < v_j$  holds and  $v_{j-1} \rightarrow w$  and  $w \rightarrow v_{j+1}$  are edges of  $G$ .*

The term ‘‘vertex shelling order’’ is justified by the following result.

**Theorem 6.2.** *Let  $P$  be a level poset and  $<$  be a vertex shelling order on the vertex set of the underlying digraph of  $P$ . Associate to each maximal chain  $(u, i) = (v_i, i) \prec (v_{i+1}, i+1) \prec \cdots \prec (v_{j-1}, j-1) \prec (v_j, j) = (v, j)$  in the interval  $[(u, i), (v, j)]$  the word  $v_i \cdots v_j$ . Then ordering the maximal chains of the interval  $[(u, i), (v, j)]$  in  $P$  by increasing lexicographic order of the associated words is a shelling of the order complex  $\Delta([(u, i), (v, j)])$ .*

*Proof.* The maximal chains of an interval  $[(u, i), (v, j)]$  in a level poset  $P$  are in a one-to-one correspondence with walks  $u = v_i \rightarrow v_{i+1} \rightarrow \cdots \rightarrow v_{j-1} \rightarrow v_j = v$  of length  $j - i$  in the underlying digraph. Assume that the maximal chain encoded by the word  $v_i \cdots v_j$  is preceded by the chain encoded by the word  $v'_i \cdots v'_j$ . Let  $k \in [i+1, j-1]$  be the least index such that  $v'_k \neq v_k$  and let  $l \in [k+1, j]$  be the least index such that  $v_l = v'_l$ . Since  $v_i \cdots v_j$  is preceded by  $v'_i \cdots v'_j$  in the lexicographic order, we must have  $v'_k < v_k$ . As a consequence of Definition 6.1 applied to the walks  $v_{k-1} \rightarrow \cdots \rightarrow v_l$  and  $v'_{k-1} \rightarrow \cdots \rightarrow v'_l$ , there is an  $m \in [k, l-1]$  and a vertex  $w$  such that  $w < v_m$  holds and  $v_{m-1} \rightarrow w$  and  $w \rightarrow v_{m+1}$  are edges of the underlying digraph. The maximal chain associated to the word  $v_i v_{i+1} \cdots v_{m-1} w v_{m+1} \cdots v_j$  precedes the maximal chain associated to  $v_i \cdots v_j$ , the intersection of the two chains has codimension one, and contains the intersection of the chain associated to  $v_i \cdots v_j$  with the chain associated to  $v'_i \cdots v'_j$ .  $\square$

**Remark 6.3.** The shelling order used in Theorem 6.2 is induced by labeling each cover relation  $(u, i) \prec (v, i+1)$  by the vertex  $v$ . This is an example of Kozlov's *CC-labelings* [19], discovered independently by Hersh and Kleinberg. See the Introduction of [1]. Even if the linear order  $<$  is not a vertex shelling order, labeling each cover relation  $(u, i) \prec (v, i+1)$  by the vertex  $v$  and using the linear order  $<$  to lexicographically order the maximal chains in each interval induces an *FA-labeling*, as defined by Billera and Hetyei [7]. See also [6]. Essentially the same labelings were used by Babson and Hersh [1] to construct a discrete Morse matching, a technique which helps determine the homotopy type of the order complex of an arbitrary graded poset. We refer the reader to the above cited sources for further information.

In analogy to Theorem 4.3 the vertex shelling order condition needs to be verified only for finitely-many values of  $k$ . We prove this for strongly connected digraphs.

**Theorem 6.4.** *Let  $P$  be the level poset of an indecomposable  $n \times n$  matrix  $M$  with period  $d$  and index  $\gamma$ . Then the underlying digraph  $G$  of  $P$  is vertex shellable if and only if it satisfies the vertex shelling order condition stated in Definition 6.1 for  $k \leq \gamma + d$ .*

*Proof.* Assume  $G$  satisfies the condition stated in Definition 6.1 for  $k \leq \gamma + d$  and let  $k$  be the least integer for which the condition is violated. We must have  $k \geq \gamma + d + 1$ . Consider any pair of walks  $u = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = v$  and  $u = v'_0 \rightarrow v'_1 \rightarrow \cdots \rightarrow v'_k = v$  such that  $v'_1 < v_1$ . As in the proof of Theorem 4.9, let  $\delta \leq \gamma + d - 1$  be the least multiple of  $d$  which is greater than or equal to  $\gamma$ . After rearranging the rows and columns if necessary,  $\text{Bin}(M)$  takes the form given in (2.1) and  $\text{Bin}(M^\delta)$  takes the form given in (4.4). As in (2.1), we may assume that the block  $Q_{q, q+1}$  occupies the rows indexed by  $C_q$  and the columns indexed by  $C_{q+1}$ . The vertex set of  $G$  is the disjoint union of the sets  $C_0, \dots, C_{d-1}$  and every edge starting in  $C_q$  ends in  $C_{q+1}$ . Since there is a walk of length  $k - 1$  from  $v'_1$  to  $v_k = v$  and there is a walk of length  $k - \delta - 1$  from  $v_{\delta+1}$  to  $v_k$ , it follows that  $v'_1$  and  $v_{\delta+1}$  belong to the same set  $C_q$ . As a consequence of equation (4.4), there is a walk  $v'_1 \rightarrow v''_2 \rightarrow \cdots \rightarrow v''_\delta \rightarrow v_{\delta+1}$

of length  $\delta$  from  $v'_1$  to  $v_{\delta+1}$ . Note that  $\delta + 1 \leq \gamma + d < k$ . By the minimality of  $k$ , the walks  $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_\delta \rightarrow v_{\delta+1}$  and  $v_0 \rightarrow v'_1 \rightarrow v''_2 \rightarrow \cdots \rightarrow v''_\delta \rightarrow v_{\delta+1}$  still satisfying  $v'_1 < v_1$  cannot violate the vertex shelling order condition stated in Definition 6.1. Hence there is a  $j \in [1, \delta]$  and a vertex  $w \in V$  such that  $w < v_j$  holds and  $v_{j-1} \rightarrow w$  and  $w \rightarrow v_{j+1}$  are edges of  $G$ , and the pair of walks  $u = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = v$  and  $u = v'_0 \rightarrow v'_1 \rightarrow \cdots \rightarrow v'_k = v$  do not violate the vertex shelling order condition, in contradiction with our assumption.  $\square$

The verification whether a linear order on the vertices of a digraph is a vertex shelling order may be automated by introducing the algebra of walks.

**Definition 6.5.** Let  $G$  be a digraph with edge set  $E$  on the vertex set  $V$ . Assume  $G$  has no multiple edges. The algebra of walks  $\mathbb{Q}\langle G \rangle$  is the quotient of the free non-commutative algebra over  $\mathbb{Q}$  generated by the set of variables  $\{x_{u,v} : (u,v) \in E\}$  by the ideal generated by the set of monomials  $\{x_{u_1,v_1}x_{u_2,v_2} : v_1 \neq u_2\}$ .

A vector space basis for  $\mathbb{Q}\langle G \rangle$  may be given by 1, which labels the trivial walk, and all monomials  $x_{v_0,v_1}x_{v_1,v_2} \cdots x_{v_{k-1},v_k}$  such that  $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k$  is a walk in  $G$ .

**Notation 6.6.** We introduce  $x_{v_0,v_1,\dots,v_k}$  as a shorthand for  $x_{v_0,v_1}x_{v_1,v_2} \cdots x_{v_{k-1},v_k}$ .

**Theorem 6.7.** Let  $G$  be a digraph on the vertex set  $V$  of cardinality  $n$  having no multiple edges and let  $<$  be a linear order on  $V$ . Let  $I_<$  be the ideal in  $\mathbb{Q}\langle G \rangle$  generated by all monomials  $x_{v_0,v_1}x_{v_1,v_2}$  such that there is a vertex  $v'_1 < v_1$  such that  $v_0 \rightarrow v'_1 \rightarrow v_2$  is a walk in  $G$ . Let  $Z = (z_{u,v})_{u,v \in V}$  be the  $n \times n$  matrix whose rows and columns are indexed by the vertices of  $G$  such that  $z_{u,v} = x_{u,v}$  if  $(u,v)$  is an edge and it is zero otherwise. If over the ring  $\mathbb{Q}\langle G \rangle/I_<$  every entry in every power of the matrix  $Z$  is a single monomial or zero, then  $<$  is a vertex shelling order.

*Proof.* We first calculate the powers of the matrix  $Z$  over the ring  $\mathbb{Q}\langle G \rangle$ . It is straightforward to see by induction on  $k$  that the entry  $z_{u,v}^{(k)}$  in  $Z^k$  is the sum of all monomials of the form  $x_{v_0,v_1,\dots,v_k}$  where  $u = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = v$  is a walk of length  $k$  from  $u$  to  $v$ .

The effect of factoring by the ideal  $I_<$  may be easily described by introducing the following *flip operators*  $\sigma_i$  for  $i \geq 1$ . Given a monomial  $x_{v_0,v_1,\dots,v_k}$ , set

$$\sigma_i(x_{v_0,v_1,\dots,v_k}) = x_{v_0,v_1,\dots,v_{i-1},v'_i,v_{i+1},\dots,v_k},$$

if  $v'_i$  is the least vertex in the linear order such that  $v_{i-1} \rightarrow v'_i \rightarrow v_{i+1}$  is a walk in the digraph. Clearly a monomial belongs to  $I_<$  if and only if it is not fixed by some  $\sigma_i$ . The order  $<$  induces a lexicographic order on all walks of length  $k$  from  $u$  to  $v$ . Applying a flip  $\sigma_i$  to a monomial  $x_{v_0,v_1,\dots,v_k}$  either leaves the monomial unchanged or replaces it with a monomial that represents a lexicographically smaller walk of length  $k$  from  $u$  to  $v$ . In particular, the monomial representing the lexicographically least walk of length  $k$  from  $u$  to  $v$  does not belong to  $I_<$  and so  $z_{u,v}^{(k)} \neq 0$  in  $\mathbb{Q}\langle G \rangle/I_<$  if there is a walk of length  $k$  from  $u$  to  $v$ .

Assume first that each  $z_{u,v}^{(k)}$  contains at most one monomial that does not belong to  $I_<$ . As noted above, in this case  $z_{u,v}^{(k)}$  contains exactly one monomial not belonging to  $I_<$  and this monomial represents

the lexicographically least walk of length  $k$  from  $u$  to  $v$ . Consider a pair of walks  $u = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = v$  and  $u = v'_0 \rightarrow v'_1 \rightarrow \cdots \rightarrow v'_k = v$  satisfying  $v'_1 < v_1$ . Since  $u = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = v$  is not the lexicographically least walk of length  $k$  from  $u$  to  $v$ , the monomial  $x_{v_0, v_1, \dots, v_k}$  must belong to  $I_{<}$ . Thus there is a  $j \in [1, k-1]$  and a vertex  $w < v_j$  such that

$$\sigma_j(x_{v_0, v_1, \dots, v_k}) = x_{v_0, v_1, \dots, v_{j-1}, w, v_{j+1}, \dots, v_k}.$$

This  $j$  and  $w$  show that the vertex shelling order condition stated in Definition 6.1 is satisfied.

Assume that  $<$  is a vertex shelling order and, by way of contradiction, assume that  $z_{u,v}^{(k)}$  contains at least two monomials  $x_{v_0, v_1, \dots, v_k}$  and  $x_{v_0, v'_1, \dots, v'_{k-1}, v_k}$  not belonging to  $I_{<}$ . Without loss of generality we may assume  $v_0 = v'_0, v_1 = v'_1, \dots, v_{i-1} = v'_{i-1}$  and  $v'_i < v_i$ . Applying the vertex shelling order condition given to the pair of walks  $v_{i-1} \rightarrow v_i \rightarrow \cdots \rightarrow v_k$  and  $v_{i-1} \rightarrow v'_i \rightarrow \cdots \rightarrow v_k$ , we obtain a  $j \in [i, k-1]$  and a vertex  $w$  such that  $w < v_j$  and  $v_{j-1} \rightarrow w$  and  $w \rightarrow v_{j+1}$  are edges. But then  $\sigma_j(x_{v_0, v_1, \dots, v_k}) \neq x_{v_0, v_1, \dots, v_k}$  and  $x_{v_0, v_1, \dots, v_k}$  does not belong to  $I_{<}$ .  $\square$

Using Theorem 6.4, the proof of Theorem 6.7 may be modified to show the following.

**Proposition 6.8.** *Let  $M$  be an indecomposable matrix having period  $d$  and index  $\gamma$ . When applying Theorem 6.7 to decide whether an order of the vertices is a vertex shelling order, one needs to verify the condition on the matrices  $Z^k$  only for  $k \leq \gamma + d$ .*

For a shellable Eulerian posets we can conclude more.

**Theorem 6.9.** *If  $P$  is an Eulerian poset of rank  $n+1$  whose order complex  $\Delta(P - \{\widehat{0}, \widehat{1}\})$  is shellable, then the order complex is homeomorphic to an  $n$ -dimensional sphere.*

*Proof.* Since every interval of rank 2 in an Eulerian poset is a diamond, every subfacet of the order complex is contained in exactly two facets. Hence the order complex  $\Delta(P - \{\widehat{0}, \widehat{1}\})$  is a pseudo-manifold without boundary. Let  $F_1, \dots, F_t$  be a shelling of the order complex. Since the reduced Euler characteristic equals the Möbius function  $\mu(P)$ , which in turn equals  $(-1)^{n+1}$  as  $P$  is Eulerian, the order complex is homotopy equivalent to one sphere. Hence there is one facet that changes the topology during shelling. We can move this facet to be last facet of the shelling, that is,  $F_t$ . Hence the previous facets form a contractible complex. The shelling implies that the complex  $F_1 \cup \cdots \cup F_{t-1}$  is collapsible to a point. Lastly, the complex  $F_1 \cup \cdots \cup F_{t-1}$  is a pseudo-manifold with boundary. By a result of J.H.C. Whitehead [15, Theorem 1.6], such a pseudo-manifold is homeomorphic to an  $n$ -dimensional ball. The result follows by gluing back the last facet  $F_t$  along the common boundary.  $\square$

**Example 6.10.** Consider again the level poset shown in Figure 1. See also Example 4.7. Consider the linear order  $1 < 2 < 3 < 4$  where  $i$  is the vertex associated to row (and column)  $i$ . We claim this is a vertex shelling order. By Proposition 6.8 we need to check that all entries of  $Z^k$  are zero or monomials for  $k \leq 4$ . Direct calculation shows

$$Z = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,1} & 0 & x_{2,3} & x_{2,4} \\ 0 & x_{3,2} & 0 & x_{3,4} \\ x_{4,1} & 0 & x_{4,3} & x_{4,4} \end{pmatrix}, \quad Z^2 = \begin{pmatrix} x_{1,1,1} & x_{1,1,2} & x_{1,1,3} & x_{1,2,4} \\ x_{2,1,1} & x_{2,1,2} & x_{2,1,3} & x_{2,3,4} \\ x_{3,2,1} & 0 & x_{3,2,3} & x_{3,2,4} \\ x_{4,1,1} & x_{4,1,2} & x_{4,1,3} & x_{4,3,4} \end{pmatrix}$$



$$Z^3 = \begin{pmatrix} x_{1,1,1,1} & x_{1,1,1,2} & x_{1,1,1,3} & x_{1,1,2,4} \\ x_{2,1,1,1} & x_{2,1,1,2} & x_{2,1,1,3} & x_{2,1,2,4} \\ x_{3,2,1,1} & x_{3,2,1,2} & x_{3,2,1,3} & x_{3,2,3,4} \\ x_{4,1,1,1} & x_{4,1,1,2} & x_{4,1,1,3} & x_{4,1,2,4} \end{pmatrix} \quad \text{and}$$

$$Z^4 = \begin{pmatrix} x_{1,1,1,1,1} & x_{1,1,1,1,2} & x_{1,1,1,1,3} & x_{1,1,1,2,4} \\ x_{2,1,1,1,1} & x_{2,1,1,1,2} & x_{2,1,1,1,3} & x_{2,1,1,2,4} \\ x_{3,2,1,1,1} & x_{3,2,1,1,2} & x_{3,2,1,1,3} & x_{3,2,1,2,4} \\ x_{4,1,1,1,1} & x_{4,1,1,1,2} & x_{4,1,1,1,3} & x_{4,1,1,2,4} \end{pmatrix}.$$

Thus we conclude that each interval is shellable and hence by Theorem 6.9 that the order complex of each interval is homeomorphic to a sphere.

We conclude this section with an example of a level Eulerian poset that has a strongly connected underlying digraph and non-shellable intervals.

**Example 6.11.** Consider the level poset  $P$  whose underlying graph has the adjacency matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We then have

$$\text{Bin}(M^2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \text{Bin}(M^3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and  $\text{Bin}(M^4) = J$ . Thus  $M$  is primitive and the exponent is  $\gamma = 4$ . By the half-Eulerian analogue of Theorem 4.3, to check whether  $P$  is half-Eulerian we only need to verify the half-Eulerian condition (5.2) for  $p < 10$ . Furthermore, just as in the Eulerian case, we only need to check (5.2) holds for even values of  $p$ . We leave this to the reader as an exercise. The level poset  $P$  is half-Eulerian, so its horizontal double  $D_{\leftrightarrow}(P)$  is Eulerian. Let 1 denote the vertex corresponding to the first row in  $M$  and consider the interval  $[(1, 0), (1, 3)]$  in  $D_{\leftrightarrow}(P)$ . The order complex of this interval has two connected components and is thus not shellable.

## 7. THE **ab**- AND **cd**-SERIES OF LEVEL AND LEVEL EULERIAN POSETS

Level Eulerian posets are infinite in nature, so one must encode their face incidence data using a non-commutative series. For a reference on non-commutative formal power series, see [30, Section 6.5]. In this section we review the notions of the flag  $h$ -vector, the **ab**-index for finite posets and, in the case the poset is Eulerian, the **cd**-index. We extend these notions to the **ab**-series and **cd**-series of a level Eulerian poset. The main result of this section is that the **cd**-series of any level Eulerian poset is a rational generating function.

For a finite graded poset  $P$  of rank  $m + 1$ , the flag  $f$ -vector has  $2^m$  entries. When the poset  $P$  is Eulerian, there are linear relations among these entries known as the generalized Dehn–Sommerville relations [2]. They describe a subspace whose dimension is given by the  $m$ th Fibonacci number. The

**cd**-index offers an explicit bases for this subspace. In order to describe it, we begin by defining the flag  $h$ -vector and the **ab**-index. The *flag  $h$ -vector* of the poset  $P$  is defined by the invertible relation

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

Hence the flag  $h$ -vector encodes the same information as the flag  $f$ -vector. Let  $\mathbf{a}$  and  $\mathbf{b}$  be two non-commutative variables each of degree one. For  $S$  a subset of  $\{1, \dots, m\}$  define the **ab**-monomial  $u_S = u_1 u_2 \cdots u_m$  by letting  $u_i = \mathbf{b}$  if  $i \in S$  and  $u_i = \mathbf{a}$  otherwise. The **ab**-index of the poset  $P$  is defined by

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where the sum is over all subsets  $S \subseteq \{1, \dots, m\}$ .

Bayer and Klapper [3] proved that for an Eulerian poset  $P$  the **ab**-index can be written in terms of the non-commutative variables  $\mathbf{c} = \mathbf{a} + \mathbf{b}$  and  $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$  of degree one and two, respectively. There are several proofs of this fact in the literature [11, 13, 28]. When  $\Psi(P)$  is written in terms of  $\mathbf{c}$  and  $\mathbf{d}$ , it is called the **cd**-index of the poset  $P$ .

Another way to approach the **ab**-index is by chain enumeration. For a chain in the poset  $P$   $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_{k+1} = \widehat{1}\}$  define its weight to be

$$\text{wt}(c) = (\mathbf{a} - \mathbf{b})^{\rho(x_0, x_1)-1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_1, x_2)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{\rho(x_k, x_{k+1})-1}.$$

The **ab**-index is then given by

$$(7.1) \quad \Psi(P) = \sum_c \text{wt}(c),$$

where the sum ranges over all chains  $c$  in the poset  $P$ .

For a level poset  $P$  and two vertices  $x$  and  $y$  in the underlying digraph, set  $\Psi([(x, i), (y, j)])$  to be zero if  $(x, i) \not\leq (y, j)$ . Define the **ab**-series  $\Psi_{x,y}$  of the level poset  $P$  to be the non-commutative formal power series

$$\Psi_{x,y} = \sum_{m \geq 0} \Psi([(x, 0), (y, m+1)]).$$

Since the  $m$ th term in this sum is homogeneous of degree  $m$ , the sum is well-defined.

Finally, for a level poset  $P$  let  $\Psi$  be the matrix whose  $(x, y)$  entry is the **ab**-series  $\Psi_{x,y}$ . Our goal is to show that the **ab**-series  $\Psi_{x,y}$  is a rational non-commutative formal power series.

Let  $K(t)$  denote the matrix

$$K(t) = M + \text{Bin}(M^2) \cdot t + \text{Bin}(M^3) \cdot t^2 + \cdots,$$

and let  $K_{x,y}(t)$  denote the  $(x, y)$  entry of the matrix  $K(t)$ . Observe  $K_{x,y}(t)$  is the generating function having the coefficient of  $t^{m-1}$  to be 1 if there is a walk of length  $m$  from the vertex  $x$  to the vertex  $y$  and zero otherwise.

To prove the main result of this section we need the following classical result due to Skolem [27], Mahler [21] and Lech [20]. The formulation here is the same as that given in [29, Chapter 4, Exercise 3]). In fact, since we are only dealing with integer coefficients, it is sufficient to use Skolem's original result [27].

**Theorem 7.1** (Skolem–Mahler–Lech). *Let  $\sum_{n \geq 0} a_n \cdot t^n$  be a rational generating function and let*

$$b_n = \begin{cases} 1 & \text{if } a_n \neq 0, \\ 0 & \text{if } a_n = 0. \end{cases}$$

*Then the generating function  $\sum_{n \geq 0} b_n \cdot t^n$  is rational.*

**Lemma 7.2.** *The generating function  $K_{x,y}(t)$  is rational.*

*Proof.* Let  $G(t)$  denote the matrix

$$G(t) = M + M^2 \cdot t + M^3 \cdot t^2 + \cdots = M \cdot (I - M \cdot t)^{-1}$$

and let  $G_{x,y}(t)$  denote the  $(x, y)$  entry of this matrix. Clearly  $G_{x,y}(t)$  is the generating function for the number of walks from the vertex  $x$  to the vertex  $y$  where the coefficient of  $t^{m-1}$  is the number of walks of length  $m$ . Furthermore, it is clear that  $G_{x,y}(t)$  is a rational function. Hence by Theorem 7.1 the result follows.  $\square$

When the underlying digraph is strongly connected and has period  $d$ , it is easy to observe that  $K_{x,y}(t)$  is the rational function  $t^r/(1-t^d)$  minus a finite number of terms, where the lengths of the walks from  $x$  to  $y$  are congruent to  $r$  modulo  $d$ .

**Theorem 7.3.** *The **ab**-series  $\Psi_{x,y}$  is a rational generating function in the non-commutative variables **a** and **b**.*

*Proof.* We first restrict ourselves to summing weights of chains which has length  $k + 1$  in the level poset, that is, after excluding the minimal and maximal element, those chains consisting of  $k$  elements. The matrix enumerating such chains is given by the product

$$K(\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdot K(\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot K(\mathbf{a} - \mathbf{b}) = K(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} \cdot K(\mathbf{a} - \mathbf{b}))^k.$$

Summing over all  $k \geq 0$ , we obtain

$$(7.2) \quad \Psi = K(\mathbf{a} - \mathbf{b}) \cdot (I - \mathbf{b} \cdot K(\mathbf{a} - \mathbf{b}))^{-1}.$$

Hence each entry of the matrix  $\Psi$  is a rational generating function in **a** and **b**.  $\square$

We turn our attention to the **cd**-index of level Eulerian posets.

**Theorem 7.4.** *For a level Eulerian poset the **ab**-series  $\Psi_{x,y}$  is a rational generating function in the non-commutative variables **c** and **d**.*

We call the resulting generating function guaranteed in Theorem 7.4 the **cd**-series.

*Proof of Theorem 7.4.* Observe equation (7.2) is equivalent to

$$(7.3) \quad \Psi = K(\mathbf{a} - \mathbf{b}) + K(\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdot \Psi.$$

Consider the involution that exchanges the variables  $\mathbf{a}$  and  $\mathbf{b}$ . Note that this involution leaves series expressed in  $\mathbf{c}$  and  $\mathbf{d}$  invariant. Apply this involution to equation (7.3) gives

$$(7.4) \quad \Psi = K(\mathbf{b} - \mathbf{a}) + K(\mathbf{b} - \mathbf{a}) \cdot \mathbf{a} \cdot \Psi.$$

Add the two equations (7.3) and (7.4) and divide by 2.

$$(7.5) \quad \Psi = (K(\mathbf{a} - \mathbf{b}) + K(\mathbf{b} - \mathbf{a}))/2 + (K(\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} + K(\mathbf{b} - \mathbf{a}) \cdot \mathbf{a})/2 \cdot \Psi.$$

Divide the generating function  $K(t)$  into its even, respectively odd, generating function, that is, let

$$K_0(t) = \frac{K(\sqrt{t}) + K(-\sqrt{t})}{2} \quad \text{and} \quad K_1(t) = \frac{K(\sqrt{t}) - K(-\sqrt{t})}{2 \cdot \sqrt{t}}.$$

We have  $K(t) = K_0(t^2) + K_1(t^2) \cdot t$  and  $K(-t) = K_0(t^2) - K_1(t^2) \cdot t$ . Note that

$$\begin{aligned} (K(\mathbf{a} - \mathbf{b}) + K(\mathbf{b} - \mathbf{a}))/2 &= K_0(\mathbf{c}^2 - 2 \cdot \mathbf{d}), \\ (K(\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} + K(\mathbf{b} - \mathbf{a}) \cdot \mathbf{a})/2 &= K_0(\mathbf{c}^2 - 2 \cdot \mathbf{d}) \cdot \mathbf{c} + K_1(\mathbf{c}^2 - 2 \cdot \mathbf{d}) \cdot (2 \cdot \mathbf{d} - \mathbf{c}^2). \end{aligned}$$

The result now follows since  $K_0$  and  $K_1$  are rational generating functions and we can solve for  $\Psi$  in equation (7.5).  $\square$

Bayer and Hetyei [5] proved that the  $\mathbf{ab}$ -index of a half-Eulerian poset is a polynomial in the two variables  $\mathbf{a}$  and  $(\mathbf{a} - \mathbf{b})^2$ . We now show this also holds for the rational series of a level half-Eulerian poset. Define the algebra morphism  $f_{\leftrightarrow}$  on  $\mathbb{R}\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle$  by  $f_{\leftrightarrow}(\mathbf{a} - \mathbf{b}) = \mathbf{a} - \mathbf{b}$  and  $f_{\leftrightarrow}(\mathbf{b}) = 2\mathbf{b}$ . It is then easy to observe from the chain definition (7.1) of the  $\mathbf{ab}$ -index that for any poset  $P$  the  $\mathbf{ab}$ -index of the poset  $P$  and its horizontal double are related by  $\Psi(D_{\leftrightarrow}(P)) = f_{\leftrightarrow}(\Psi(P))$ .

**Corollary 7.5.** *The  $\mathbf{ab}$ -series  $\Psi_{x,y}$  of a level half-Eulerian poset is a rational generating function in the non-commutative variables  $\mathbf{a}$  and  $(\mathbf{a} - \mathbf{b})^2$ .*

*Proof.* Consider the horizontal double of the level half-Eulerian poset. Its  $\mathbf{ab}$ -series is a rational generating function in terms of  $\mathbf{a} + \mathbf{b} = \mathbf{c}$  and  $(\mathbf{a} - \mathbf{b})^2 = \mathbf{c}^2 - 2\mathbf{d}$ . The result follows by applying the inverse morphism  $f_{\leftrightarrow}^{-1}$  to this rational series.  $\square$

We similarly define the algebra morphism  $f_{\uparrow}$  by  $f_{\uparrow}(\mathbf{a} - \mathbf{b}) = (\mathbf{a} - \mathbf{b})^2$  and  $f_{\uparrow}(\mathbf{b}) = \mathbf{b}(\mathbf{a} - \mathbf{b}) + (\mathbf{a} - \mathbf{b})\mathbf{b} + \mathbf{b}^2 = \mathbf{ab} + \mathbf{ba} + \mathbf{b}^2$ . By the chain definition (7.1) of the  $\mathbf{ab}$ -index we can conclude that  $\Psi(D_{\uparrow}(P)) = f_{\uparrow}(\Psi(P))$ . We end this section by presenting the corresponding results for horizontal- and vertical-doubling of a level poset. The proof is straightforward and hence omitted.

**Proposition 7.6.** *Let  $P$  a level poset with underlying matrix  $M$ . Then we have*

$$\Psi(D_{\leftrightarrow}(P)) = \begin{pmatrix} f_{\leftrightarrow}(\Psi(P)) & f_{\leftrightarrow}(\Psi(P)) \\ f_{\leftrightarrow}(\Psi(P)) & f_{\leftrightarrow}(\Psi(P)) \end{pmatrix}$$

and

$$\Psi(D_{\uparrow}(P)) = \begin{pmatrix} \mathbf{a} \cdot f_{\uparrow}(\Psi(P)) & I + \mathbf{a} \cdot f_{\uparrow}(\Psi(P)) \cdot \mathbf{a} \\ f_{\uparrow}(\Psi(P)) & f_{\uparrow}(\Psi(P)) \cdot \mathbf{a} \end{pmatrix},$$

where the two morphisms  $f_{\leftrightarrow}$  and  $f_{\downarrow}$  are applied entrywise to the matrices.

## 8. COMPUTING THE **cd**-SERIES

The recursions (7.3), (7.4) and (7.5) are not very practical for explicitly computing the **cd**-series of a level Eulerian poset. In this section we offer a different method to show the **cd**-series has a given expression based upon the coalgebraic techniques developed in [12].

Define a derivation  $\Delta : \mathbb{R}\langle\langle \mathbf{a}, \mathbf{b} \rangle\rangle \longrightarrow \mathbb{R}\langle\langle \mathbf{a}, \mathbf{b}, \mathbf{t} \rangle\rangle$  by  $\Delta(\mathbf{a}) = \Delta(\mathbf{b}) = \mathbf{t}$ ,  $\Delta(1) = 0$  and require that it satisfy the product rule  $\Delta(u \cdot v) = \Delta(u) \cdot v + u \cdot \Delta(v)$ . It is straightforward to verify that this derivation is well-defined. Observe that the coefficient of a monomial  $u$  in  $\Delta(v)$  is zero unless  $u$  contains exactly one  $\mathbf{t}$ .

Note that for a formal power series  $u$  without constant term we have that

$$\Delta\left(\frac{1}{1-u}\right) = \frac{1}{1-u} \cdot \Delta(u) \cdot \frac{1}{1-u},$$

since  $\Delta(u^m) = \sum_{i=0}^{m-1} u^i \cdot \Delta(u) \cdot u^{m-1-i}$  and then by summing over all  $m$ .

When restricting the derivation  $\Delta$  to non-commutative polynomials  $\mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$ , it becomes equivalent to the coproduct on **ab**-polynomials introduced by Ehrenborg and Readdy in [12]. To see this fact, observe that the subspace of  $\mathbb{R}\langle \mathbf{a}, \mathbf{b}, \mathbf{t} \rangle$  spanned by monomials containing exactly one  $\mathbf{t}$  is isomorphic to  $\mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle \otimes \mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$  by mapping the variable  $\mathbf{t}$  to the tensor sign, that is,  $u \cdot \mathbf{t} \cdot v \longmapsto u \otimes v$ .

We need two properties of the derivation  $\Delta$ . The first is that the **ab**-index is a coalgebra homomorphism, that is, for a poset  $P$  we have

$$(8.1) \quad \Delta(\Psi(P)) = \sum_{\widehat{0} < x < \widehat{1}} \Psi([\widehat{0}, x]) \cdot \mathbf{t} \cdot \Psi([x, \widehat{1}]).$$

See [12, Proposition 3.1]. Applying (8.1) to all the rank  $m+1$  intervals of a level poset, we have that

$$\Delta(\Psi_m) = \sum_{i=0}^{m-1} \Psi_i \cdot \mathbf{t} \cdot \Psi_{m-1-i},$$

where  $\Psi_m$  denotes the degree  $m$  terms of the **ab**-series  $\Psi$ . The second property of the derivation is that when restricting the derivation to **ab**-polynomials of degree  $n$ , the kernel of the map is spanned by  $(\mathbf{a} - \mathbf{b})^n$ . See [12, Lemma 2.2].

We now prove the main result of this section. It is a method to recognize the **ab**-series matrix of a level poset.

**Theorem 8.1.** *The  $n \times n$  matrix  $\Psi$  of the **ab**-series of a level poset is the unique solution to the equation system*

$$(8.2) \quad \Psi|_{\mathbf{a}=\mathbf{t}, \mathbf{b}=0} = K(\mathbf{t}),$$

$$(8.3) \quad \Delta(\Psi) = \Psi \cdot \mathbf{t} \cdot \Psi.$$

*Proof.* Let  $\Gamma$  be a solution to the two equations (8.2) and (8.3). Write  $\Gamma$  as the sum  $\sum_{m \geq 0} \Gamma_m$  where the entries of the matrix  $\Gamma_m$  are homogeneous of degree  $m$ . By induction on  $m$  we will prove that  $\Gamma_m$  is equal to  $\Psi_m$ , the  $m$ th homogeneous component of the matrix  $\Psi$ . The base case  $m = 0$  is as follows.

$$\Gamma_0 = \Gamma|_{\mathbf{a}=\mathbf{b}=0} = K(t)|_{t=0} = M = \Psi_0.$$

Now assume the statement is true for all values less than  $m$ . Observe that the  $m$ th component of equation (8.3) is

$$\Delta(\Gamma_m) = \sum_{i=0}^{m-1} \Gamma_i \cdot \mathbf{t} \cdot \Gamma_{m-1-i} = \sum_{i=0}^{m-1} \Psi_i \cdot \mathbf{t} \cdot \Psi_{m-1-i} = \Delta(\Psi_m).$$

Hence the difference  $\Gamma_m - \Psi_m$  is a constant matrix  $N$  times the  $\mathbf{ab}$ -polynomial  $(\mathbf{a} - \mathbf{b})^m$ . However, the matrix  $N$  is zero since  $\Gamma_m|_{\mathbf{b}=0} = \Psi_m|_{\mathbf{b}=0}$ , proving that  $\Gamma_m$  is equal to  $\Psi_m$ , completing the induction.  $\square$

**Example 8.2.** Consider the level Eulerian poset in Figure 1. We claim that its  $\mathbf{cd}$ -series matrix  $\Psi$  is given by

$$\Psi = \begin{pmatrix} \frac{1}{1-\mathbf{c}-\mathbf{d}} & \frac{1}{1-\mathbf{c}-\mathbf{d}} \cdot \mathbf{c} + 1 & \frac{1}{1-\mathbf{c}-\mathbf{d}} & \frac{1}{1-\mathbf{c}-\mathbf{d}} - 1 \\ \frac{1}{1-\mathbf{c}-\mathbf{d}} & \frac{1}{1-\mathbf{c}-\mathbf{d}} \cdot \mathbf{c} & \frac{1}{1-\mathbf{c}-\mathbf{d}} & \frac{1}{1-\mathbf{c}-\mathbf{d}} \\ \mathbf{c} \cdot \frac{1}{1-\mathbf{c}-\mathbf{d}} & \mathbf{c} \cdot \frac{1}{1-\mathbf{c}-\mathbf{d}} \cdot \mathbf{c} + 1 & \mathbf{c} \cdot \frac{1}{1-\mathbf{c}-\mathbf{d}} & \mathbf{c} \cdot \frac{1}{1-\mathbf{c}-\mathbf{d}} + 1 \\ \frac{1}{1-\mathbf{c}-\mathbf{d}} & \frac{1}{1-\mathbf{c}-\mathbf{d}} \cdot \mathbf{c} & \frac{1}{1-\mathbf{c}-\mathbf{d}} & \frac{1}{1-\mathbf{c}-\mathbf{d}} \end{pmatrix}.$$

It is straightforward to check the first condition in Theorem 8.1:

$$\Psi|_{\mathbf{a}=t, \mathbf{b}=0} = \Psi|_{\mathbf{c}=t, \mathbf{d}=0} = \begin{pmatrix} \frac{1}{1-t} & \frac{1}{1-t} & \frac{1}{1-t} & \frac{1}{1-t} - 1 \\ \frac{1}{1-t} & \frac{1}{1-t} - 1 & \frac{1}{1-t} & \frac{1}{1-t} \\ \frac{1}{1-t} - 1 & \frac{1}{1-t} - t & \frac{1}{1-t} - 1 & \frac{1}{1-t} \\ \frac{1}{1-t} & \frac{1}{1-t} - 1 & \frac{1}{1-t} & \frac{1}{1-t} \end{pmatrix} = K(t).$$

To verify the second condition, define the four vectors

$$x = \begin{pmatrix} 1 \\ 1 \\ \mathbf{c} \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad z = (1 \quad \mathbf{c} \quad 1 \quad 1) \quad \text{and} \quad w = (0 \quad 1 \quad 0 \quad 0),$$

and the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have the following relations between this matrix and these vectors:

$$z \cdot \mathbf{t} \cdot x = 2 \cdot \mathbf{t} + \mathbf{c}\mathbf{t} + \mathbf{t}\mathbf{c} = \Delta(\mathbf{c} + \mathbf{d}), \quad A \cdot x = 2 \cdot y, \quad z \cdot A = 2 \cdot w \quad \text{and} \quad A \cdot \mathbf{t} \cdot A = 0.$$

Furthermore, the derivative  $\Delta$  acts as follows

$$\Delta(x) = 2 \cdot \mathbf{t} \cdot y, \quad \Delta(z) = 2 \cdot \mathbf{t} \cdot w \quad \text{and} \quad \Delta(A) = 0.$$

Observe now that

$$\Psi = x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z + A.$$

Hence we have the following calculation

$$\begin{aligned} \Psi \cdot \mathbf{t} \cdot \Psi &= \left( x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z + A \right) \cdot \mathbf{t} \cdot \left( x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z + A \right) \\ &= x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z \cdot \mathbf{t} \cdot x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z \\ &\quad + x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z \cdot A \cdot \mathbf{t} + \mathbf{t} \cdot A \cdot x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z + A \cdot \mathbf{t} \cdot A \\ &= x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot \Delta(\mathbf{c} + \mathbf{d}) \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z \\ &\quad + x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot 2 \cdot w \cdot \mathbf{t} + \mathbf{t} \cdot 2 \cdot y \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z \\ &= x \cdot \Delta \left( \frac{1}{1 - \mathbf{c} - \mathbf{d}} \right) \cdot z + x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot \Delta(z) + \Delta(x) \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z \\ &= \Delta \left( x \cdot \frac{1}{1 - \mathbf{c} - \mathbf{d}} \cdot z \right) \\ &= \Delta(\Psi), \end{aligned}$$

proving our claim.

As a corollary to this example we obtain an interesting Eulerian poset whose  $\mathbf{cd}$ -index has all of its coefficients to be 1.

**Corollary 8.3.** *The  $\mathbf{cd}$ -index of the interval  $[(1, 0), (1, m + 1)]$  in the level poset in Figure 1 is the sum of all  $\mathbf{cd}$ -monomials of degree  $m$ .*

## 9. CONCLUDING REMARKS

Given a non-commutative rational formal power series in the variables  $\mathbf{a}$  and  $\mathbf{b}$  which can be expressed in terms of  $\mathbf{c}$  and  $\mathbf{d}$ , is it necessarily a non-commutative rational formal power series in the variables  $\mathbf{c}$  and  $\mathbf{d}$ ? In other words, is the following equality true

$$F_{\text{rat}} \langle \langle \mathbf{a}, \mathbf{b} \rangle \rangle \cap F \langle \langle \mathbf{c}, \mathbf{d} \rangle \rangle = F_{\text{rat}} \langle \langle \mathbf{c}, \mathbf{d} \rangle \rangle,$$

where  $F$  is a field? It is clear that right-hand side of the above is contained in the left-hand side.

Corollary 8.3 suggests a question about the existence of Eulerian posets. For which subsets  $M$  of  $\mathbf{cd}$ -monomials of degree  $m$  is there an Eulerian poset whose  $\mathbf{cd}$ -index is the sum of the monomials in  $M$ ? The two extreme cases  $\mathbf{c}^m$  and  $\sum_{\deg(w)=m} w$  both arrive from level Eulerian posets.

An open question is if the eigenvalues or other classical matrix invariants carry information about the corresponding level poset, such as if the level poset is Eulerian or shellable.

#### ACKNOWLEDGMENTS

The first author was partially funded by National Science Foundation grant DMS-0902063. The authors thank the Department of Mathematics at the University of Kentucky for funding a research visit for the second author to the University of Kentucky, where part of this research was carried out.

#### REFERENCES

- [1] E. BABSON AND P. HERSH, Discrete Morse functions from lexicographic orders, *Trans. Amer. Math. Soc.* **357** (2005), 509–534.
- [2] M. BAYER AND L. J. BILLERA, Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered set, *Invent. Math.* **79** (1985), 143–157.
- [3] M. BAYER AND A. KLAPPER, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991), 33–47.
- [4] M. BAYER AND G. HETYEI, Flag vectors of Eulerian partially ordered sets, *European J. Combin.* **22** (2001), 5–26.
- [5] M. BAYER AND G. HETYEI, Generalizations of Eulerian partially ordered sets, flag numbers, and the Möbius function, *Discrete Math.* **256** (2002), 577–593.
- [6] L. J. BILLERA AND G. HETYEI, Linear inequalities for flags in graded partially ordered sets, *J. Combin. Theory Ser. A* **89** (2000), 77–104.
- [7] L. J. BILLERA AND G. HETYEI, Decompositions of partially ordered sets, *Order* **17** (2000), 141–166.
- [8] A. BJÖRNER, Topological methods in *Handbook of combinatorics*, Vol. 1, 2, 1819–1872, Elsevier, Amsterdam, 1995.
- [9] A. BJÖRNER AND M. WACHS, Bruhat order of Coxeter groups and shellability, *Adv. in Math.* **43** (1982), 87–100.
- [10] A. BJÖRNER AND M. WACHS, On lexicographically shellable posets, *Trans. Amer. Math. Soc.* **277** (1983), 323–341.
- [11] R. EHRENBORG,  $k$ -Eulerian posets, *Order* **18** (2001), 227–236.
- [12] R. EHRENBORG AND M. READDY, Coproducts and the  $cd$ -index, *J. Algebraic Combin.* **8** (1998), 273–299.
- [13] R. EHRENBORG AND M. READDY, Homology of Newtonian coalgebras, *European J. Combin.* **23** (2002), 919–927.
- [14] R. EHRENBORG AND M. READDY, Classification of the factorial functions of Eulerian binomial and Sheffer posets, *J. Combin. Theory Ser. A* **114** (2007), 339–359.
- [15] R. FORMAN, Morse theory for cell complexes, *Adv. Math.* **134** (1998), 90–145.
- [16] J. GROSS AND T. W. TUCKER, “Topological graph theory,” Reprint of the 1987 original [Wiley, New York] with a new preface and supplementary bibliography, Dover Publications, Inc., Mineola, NY, 2001.
- [17] B. R. HEAP AND M. S. LYNN, The structure of powers of nonnegative matrices. I. The index of convergence, *SIAM J. Appl. Math.* **14** (1966), 610–639.
- [18] J. C. HOLLADAY AND R. S. VARGA, On powers of non-negative matrices, *Proc. Amer. Math. Soc.* **9** (1958), 631–634.
- [19] D. KOZLOV, General lexicographic shellability and orbit arrangements, *Ann. of Comb.* **1** (1997), 67–90.
- [20] C. LECH, A note on recurring series, *Ark. Mat.* **2** (1953), 417–421.
- [21] K. MAHLER, On the Taylor coefficients of rational functions, *Proc. Cambridge Philos. Soc.* **52** (1956), 39–48.
- [22] D. PERRIN AND M-P. SCHÜTZENBERGER, Synchronizing prefix codes and automata and the road coloring problem, in: Symbolic dynamics and its applications (New Haven, CT, 1991), 295–318, *Contemp. Math.*, **135**, Amer. Math. Soc., Providence, RI, 1992.
- [23] V. PTÁK, On a combinatorial theorem and its application to nonnegative matrices,, *Czechoslovak Math. J.* **8(83)** (1958), 487–495.
- [24] V. PTÁK AND I. SEDLAČEK, On the index of imprimitivity of nonnegative matrices, *Czechoslovak Math. J.* **8(83)** (1958), 496–501.
- [25] D. ROSENBLATT, On the graphs and asymptotic forms of finite Boolean relation matrices and stochastic matrices, *Naval. Res. Logist. Quart.* **4** (1957), 151–167.



- [26] V. N. SACHKOV AND V. E. TARAKANOV, “Combinatorics of nonnegative matrices,” *Translations of Mathematical Monographs*, vol. 213, The American Mathematical Society, Providence, RI, 2002.
- [27] T. SKOLEM, Einige Sätze über gewisse Reihenentwicklungen und exponentiale Beziehungen mit Anwendung auf diophantische Gleichungen, *Oslo Vid. Akad. Skrifter I* **6** (1933), 1–61.
- [28] R. P. STANLEY, Flag  $f$ -vectors and the  $cd$ -index, *Math. Z.* **216** (1994), 483–499.
- [29] R. P. STANLEY, “Enumerative Combinatorics, Vol. I,” Cambridge University Press, 1997.
- [30] R. P. STANLEY, “Enumerative Combinatorics, Vol. II,” Cambridge University Press, 1999.
- [31] M. WACHS, Poset topology: tools and applications, in *Geometric combinatorics* (E. Miller, V. Reiner, B. Sturmfels, eds.) *IAS/Park City Math. Series, 13* Amer. Math. Soc., Providence, RI, 2007, 497–615.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027 .  
WWW: <http://www.ms.uky.edu/~jrge/>.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNC-CHARLOTTE, CHARLOTTE NC 28223-0001.  
WWW: <http://www.math.uncc.edu/~ghetyei/>.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027 .  
WWW: <http://www.ms.uky.edu/~readdy/>.