The Tchebyshev Transforms of the First and Second Kind*

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Abstract

An in-depth study of the Tchebyshev transforms of the first and second kind of a poset is taken. The Tchebyshev transform of the first kind is shown to preserve desirable combinatorial properties, including EL-shellability and nonnegativity of the **cd**-index. When restricted to Eulerian posets, it corresponds to the Billera, Ehrenborg and Readdy omega map of oriented matroids. The Tchebyshev transform of the second kind U is a Hopf algebra endomorphism on the space of quasisymmetric functions which, when restricted to Eulerian posets, coincides with Stembridge's peak enumerator. The complete spectrum of U is determined, generalizing work of Billera, Hsiao and van Willigenburg. The type B quasisymmetric function of a poset is introduced and, like Ehrenborg's classical quasisymmetric function of a poset, it is a comodule morphism with respect to the quasisymmetric functions QSym. Finally, similarities among the omega map, Ehrenborg's r-signed Birkhoff transform, and the Tchebyshev transforms motivate a general study of chain maps which occur naturally in the setting of combinatorial Hopf algebras.

1 Introduction

The Tchebyshev transform (of the first kind) of a partially ordered set, introduced by Hetyei [22], enjoys many properties. When applied to an Eulerian poset, it preserves the Eulerian property [22]. Its name derives from the fact that when this transform is applied to the ladder poset, the **cd**-index of the resulting poset (expressed in terms of the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{e} = \mathbf{a} - \mathbf{b}$) yields a noncommutative generalization of the familiar Tchebyshev polynomial of the first kind [22, Section 8].

The **ab**-index is a noncommutative polynomial which encodes the flag f-vector of a poset. Via a change of basis, one obtains the **cd**-index, a polynomial that removes all the linear redundancies in the case of Eulerian posets [3]. The **cd**-index has proven to be an extraordinarily useful tool for studying inequalities for the face incidence structure of polytopes [6, 15, 17].

The omega map ω, discovered by Billera, Ehrenborg and Readdy [7], links the flag f-vector of the intersection lattice of a hyperplane arrangement with the corresponding zonotope, and more generally, the oriented matroid. On the chain level the omega map is the inverse of a "forgetful map" between posets. Aguiar and N. Bergeron observed the omega map is actually Stembridge's peak enumerator ϑ [29]. See [9] for details.

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$$
\begin{array}{lrcl}
\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle : & \Psi(P * Q) & = & \Psi(P) \cdot \Psi(Q) \\
\text{QSym} : & F(P \times Q) & = & F(P) \cdot F(Q) \\
\text{BQSym} : & F_B(P \diamond Q) & = & F_B(P) \cdot F_B(Q)
\end{array}
$$

Figure 1: The product structures of $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$, QSym and BQSym and their relation to poset products.

In this paper we discover new properties of the Tchebyshev transform. On the flag vector level it is a linear transformation. Surprisingly, when restricted to the class of Eulerian posets the Tchebyshev transform is equivalent to the omega map. The core idea underlying this equivalence is that the Zaslavsky's expression [31] for the number of regions in a hyperplane arrangement ($\sum_{x \in P} (-1)^{\rho(x)} \cdot \mu(\widehat{0}, x)$) applied to an Eulerian poset gives the cardinality of the poset. As a corollary, the Tchebyshev transform preserves nonnegativity of the **cd**-index.

We also show the Tchebyshev transform preserves EL-shellability. The edge labeling we give implies that on the flag vector level the Tchebyshev transform of the Cartesian product of two posets equals the dual diamond product of the transformed posets, that is,

$$\Psi(T(P \times Q)) = \Psi(T(P)) \diamond^* \Psi(T(Q)),$$

where T denotes the Tchebyshev transform. This proof is bijective for posets having R-labelings. A second proof is given in a more algebraic setting. See Sections 9 and 11.

The theory broadens when studying the Tchebyshev transform of the second kind U. Hetyei [22] observed there is another transform U which when applied to the ladder poset yields the Tchebyshev polynomials of the second kind. This Tchebyshev transform of the second kind is a Hopf algebra endomorphism on the space of quasisymmetric functions QSym. The Tchebyshev transform U and the peak enumerator ϑ coincide on the **cd**-level but differ on the **ab**-level, that is, they agree on the **cd**-index of Eulerian posets, but differ on the **ab**-index of general posets. Billera, Hsiao and van Willigenburg [9] determined the eigenvalues and eigenvectors of Stembridge's map ϑ when it acts on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ to itself. As the transform U acts on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to itself, we can extend the diagonalization of the map ϑ to this broader setting, hence deriving the complete spectrum and eigenvectors.

There are many ways to encode the flag vector of a poset. One is via the **ab**-index. Another is via quasisymmetric functions. We will now introduce a third, which we call the *type B quasisymmetric function F_B of a poset.* The type B quasisymmetric functions BQSym were introduced by Chow [13]. All three encodings behave nicely under different poset products. See Figure 1. The **ab**-index Ψ and the quasisymmetric function F of a poset are coalgebra maps. In contrast, the type B quasisymmetric function F_B of a poset is a comodule map with respect to the classical quasisymmetric function. See Figure 2.

In the study of the omega map ω relating a hyperplane arrangement to its zonotope, the r-signed Birkhoff transform BT in [16], and the Tchebyshev transforms T and U, the essential defining map has one of the following forms:

$$g(u) = \sum_{k \geq 1} \sum_{u} \kappa(u_{(1)}) \cdot \mathbf{b} \cdot \widehat{g}(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \widehat{g}(u_{(k)}), \tag{1.1}$$

$$\widetilde{g}(u) = \sum_{k \geq 1} \sum_{u} \widehat{g}(u_{(1)}) \cdot \mathbf{b} \cdot \widehat{g}(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \widehat{g}(u_{(k)}). \tag{1.2}$$

$$
\begin{array}{lrcl}
\mathbb{Z}\langle \mathbf{a},\mathbf{b}\rangle : & \Delta(\Psi(P)) & = & \displaystyle\sum_{\widehat{0}<x<\widehat{1}} \Psi([\widehat{0},x]) \otimes \Psi([x,\widehat{1}]) \\
\mathrm{QSym} : & \Delta^{\mathrm{QSym}}(F(P)) & = & \displaystyle\sum_{\widehat{0}\leq x\leq\widehat{1}} F([\widehat{0},x]) \otimes F([x,\widehat{1}]) \\
\mathrm{BQSym} : & \Delta^{\mathrm{BQSym}}(F_B(P)) & = & \displaystyle\sum_{\widehat{0}< x\leq\widehat{1}} F_B([\widehat{0},x]) \otimes F([x,\widehat{1}])
\end{array}
$$

Figure 2: The coalgebra structures of $\mathbb{Z}\langle \mathbf{a},\mathbf{b}\rangle$, QSym and BQSym and their relation to posets.

The maps ω, BT and T have the form (1.1) and the map U has the form (1.2). We therefore call these maps $\widetilde{g}$ and g the *chain maps of the first and second kind.* This phenomenon suggests a wider theory exists on the coalgebra level.

In Sections 12 and 13 we study general functions of these types. We show the chain map of the second kind $\widetilde{g}$ is a Hopf algebra endomorphism on quasisymmetric functions. This is a concrete example of Aguiar, Bergeron and Sottile's theorem that the algebra of quasisymmetric functions QSym is the terminal object in the category of combinatorial Hopf algebras [1]. Furthermore, the chain map of the first kind g is an algebra map on the type B quasisymmetric functions. See Theorems 12.5 and 13.5. The map g is also a comodule endomorphism on the type B quasisymmetric functions. See Theorem 13.7.

We end the paper with concluding remarks and many questions for further study.

2 Background Definitions

For a graded poset P with rank function ρ, minimal element $\widehat{0}$ and maximal element $\widehat{1}$, let $P \cup \{\widehat{-1}\}$ and $P \cup \{\widehat{2}\}$ denote P adjoined with a new minimal element $\widehat{-1}$, respectively a new maximal element $\widehat{2}$. For a chain $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$ in P define the *weight* of the chain c by

$$
\mathrm{wt}(c) = (\mathbf{a}-\mathbf{b})^{\rho(x_0,x_1)-1} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x_1,x_2)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x_{k-1},x_k)-1},
$$

where $\rho(x,y) = \rho(y) - \rho(x)$ and $\mathbf{a}$ and $\mathbf{b}$ are noncommutative variables. Observe that $\mathbf{k}\langle \mathbf{a},\mathbf{b}\rangle$ is spanned by the $\mathbf{ab}$-polynomials of the form $\mathrm{wt}(c)$. The $\mathbf{ab}$-*index* of the poset P is defined as

$$
\Psi(P) = \sum_{c} \mathrm{wt}(c),
$$

where the sum is over all chains c in P. For further information on posets, see Stanley [25].

A poset is *Eulerian* if every interval $[x,y]$, where $x < y$, has the same number of elements of even rank as elements of odd rank. When P is Eulerian the $\mathbf{ab}$-index of P can be written in terms of $\mathbf{c} = \mathbf{a}+\mathbf{b}$ and $\mathbf{d} = \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}$, and the resulting noncommutative polynomial is called the $\mathbf{cd}$-*index* [4]. Its importance lies in that it removes all the linear redundancies in the flag f-vector entries [3], it mirrors geometric operations on a polytope as operators on the corresponding $\mathbf{cd}$-index [19, 21], and it is amenable to algebraic techniques to derive inequalities on the flag vectors [6, 15, 17].

On the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ define a coproduct Δ on an **ab**-monomial $u_1 u_2 \cdots u_k$ by

$$\Delta(u_1 u_2 \cdots u_k) = \sum_{i=1}^{k} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_k,$$

where each u_i is either an $\mathbf{a}$ or a $\mathbf{b}$ and extend by linearity to $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. It is straightforward to verify the coproduct Δ is coassociative, that is, $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$. For convenience in what follows in later sections, define $\Delta^k : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle^{\otimes k}$ by $\Delta^1 = \mathrm{id}$ and $\Delta^{k+1} = (\mathrm{id} \otimes \Delta^k) \circ \Delta$. The coproduct Δ satisfies the Newtonian condition:

$$\Delta(u \cdot v) = \sum_{u} u_{(1)} \otimes u_{(2)} v + \sum_{v} u v_{(1)} \otimes v_{(2)}. \tag{2.1}$$

Here we are using the usual Sweedler notation [30], that is, $\Delta(u) = \sum_u u_{(1)} \otimes u_{(2)}$.

The essential property of the coproduct Δ is that it makes the **ab**-index into a coalgebra homomorphism [21].

Theorem 2.1 *For a graded poset P, the coproduct of the* **ab***-index of P is given by*

$$\Delta(\Psi(P)) = \sum_{\widehat{0} < x < \widehat{1}} \Psi([\widehat{0}, x]) \otimes \Psi([x, \widehat{1}]).$$

This allows poset computations to be translated into the coalgebra $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. See [6, 7, 16, 18, 19, 21].

Using the coassociativity, we have the following corollary.

Corollary 2.2

$$\Delta^k(\Psi(P)) = \sum_{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}} \Psi([x_0, x_1]) \otimes \Psi([x_1, x_2]) \otimes \cdots \otimes \Psi([x_{k-1}, x_k]). \tag{2.2}$$

There is an involution on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ that sends each monomial $u = u_1 u_2 \cdots u_k$ to its reverse $u^* = u_k \cdots u_2 u_1$. Directly we have $(u \cdot v)^* = v^* \cdot u^*$, $\Delta(u^*) = \sum_u u^*_{(2)} \otimes u^*_{(1)}$ and $\Psi(P^*) = \Psi(P)^*$ where P^* denotes the dual of the poset P.

3 Quasisymmetric functions

Another way to encode the flag f-vector of a poset P is by the quasisymmetric function $F(P)$ [14]. Let P be a poset of rank n, where $n \geq 0$ and $\mathbf{k}$ is an infinite field of characteristic zero. The *quasisymmetric function* of the poset P is defined as the limit

$$F(P) = \lim_{m \longrightarrow \infty} \sum_{\widehat{0} = x_0 \leq x_1 \leq \cdots \leq x_m = \widehat{1}} t_1^{\rho(x_0, x_1)} \cdot t_2^{\rho(x_1, x_2)} \cdots t_m^{\rho(x_{m-1}, x_m)}. \tag{3.1}$$

Observe that for $m = 2$ this sum is a homogeneous rank-generating function, that is, it encodes the f-vector of the poset. For larger m it encodes all the entries in the flag f-vector of cardinality less than or equal to $m - 1$.

A different way to define the quasisymmetric function of a poset is due to Stanley [27]. It is

$$F(P) = \sum_{\widehat{0}=x_0 \leq x_1 \leq \cdots \leq x_{m-1} < x_m = \widehat{1}} t_1^{\rho(x_0,x_1)} \cdot t_2^{\rho(x_1,x_2)} \cdots t_m^{\rho(x_{m-1},x_m)},$$

where the sum ranges over all such multichains with the last step $x_{m-1} < x_m$ a strict inequality. In this paper we will use the original definition in equation (3.1).

The power series $F(P)$ is homogeneous of degree n in the infinitely-many variables $t_1, t_2, \ldots$. It also enjoys the following quasisymmetry: for $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ the coefficients of $t_{i_1}^{p_1} \cdot t_{i_2}^{p_2} \cdots t_{i_k}^{p_k}$ and $t_{j_1}^{p_1} \cdot t_{j_2}^{p_2} \cdots t_{j_k}^{p_k}$ are the same. Power series in the variables $t_1, t_2, \ldots$ that exhibit quasisymmetry are called *quasisymmetric functions* and the algebra of these power series is denoted by QSym. It is straightforward to observe that a linear basis for QSym is given by the *monomial quasisymmetric function*, defined by

$$M_{(p_1,p_2,\ldots,p_k)} = \sum_{i_1 < i_2 < \cdots < i_k} t_{i_1}^{p_1} t_{i_2}^{p_2} \cdots t_{i_k}^{p_k}.$$

Since every polynomial in $\mathbf{a} = (\mathbf{a} - \mathbf{b}) + \mathbf{b}$ and $\mathbf{b}$ can be written in terms of $\mathbf{a} - \mathbf{b}$ and $\mathbf{b}$, we can define a linear map γ from $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ to QSym by

$$\gamma\left((\mathbf{a} - \mathbf{b})^{p_1 - 1} \cdot \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{p_2 - 1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a} - \mathbf{b})^{p_k - 1}\right) = M_{(p_1,\ldots,p_k)}.$$

This map is an isomorphism between $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ and quasisymmetric functions having no constant term. For a poset P of rank greater than or equal to one, we have $\gamma(\Psi(P)) = F(P)$. For the one element poset $\bullet$ of rank 0, let $F(\bullet) = 1_{\mathrm{QSym}}$. Here we write 1_{QSym} for the identity element of the quasisymmetric functions in order to distinguish it from the unit in $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. For more on the Hopf algebra structure of the quasisymmetric functions QSym, we refer the reader to [14].

Let us mention two important identities for the quasisymmetric function $F(P)$ of a graded poset P. For P and Q two graded posets, we have

$$\begin{aligned} F(P) \cdot F(Q) &= F(P \times Q), && (3.2) \\ \Delta^{\mathrm{QSym}}(F(P)) &= \sum_{\widehat{0} \leq x \leq \widehat{1}} F([\widehat{0}, x]) \otimes F([x, \widehat{1}]) \\ &= F(P) \otimes 1_{\mathrm{QSym}} + 1_{\mathrm{QSym}} \otimes F(P) + \sum_{\widehat{0} < x < \widehat{1}} F([\widehat{0}, x]) \otimes F([x, \widehat{1}]), && (3.3) \end{aligned}$$

where equation (3.3) is valid when the poset P has rank at least 1. Note the coproduct on quasisymmetric functions differs from the coproduct on **ab**-polynomials. In order to avoid confusion, we denote the coproduct on quasisymmetric functions by $\Delta^{\mathrm{QSym}}(f) = \sum_f^{\mathrm{QSym}} f_{(1)} \otimes f_{(2)}$. For proofs of these identities, see [14, Proposition 4.4].

From a poset perspective identities (3.2) and (3.3) define the algebra and coalgebra structure of QSym. Equation (3.3) also motivates the following relation between the two coproducts Δ and Δ^{QSym}:

$$\Delta^{\mathrm{QSym}}(\gamma(v)) = \gamma(v) \otimes 1_{\mathrm{QSym}} + 1_{\mathrm{QSym}} \otimes \gamma(v) + \sum_{v} \gamma(v_{(1)}) \otimes \gamma(v_{(2)}). \tag{3.4}$$

4 Enumerating flags in the Tchebyshev transform of a poset

Definition 4.1 *For a graded poset P with cover relation $\prec$ define the Tchebyshev transform (of the first kind) $T(P)$ to be the graded poset with elements given by the set*

$$T(P) = \{[x, y] \;\;:\;\; x, y \in P \cup \{\widehat{-1}\},\;\; x < y\} \cup \{\hat{1}_{T(P)}\},$$

and the cover relation given by the following three rules:

(i) $[x, y] \prec_{T(P)} [y, w]$ *if* $y \prec w$,

(ii) $[x, y] \prec_{T(P)} [x, w]$ *if* $y \prec w$, *and*

(iii) $[x, \hat{1}] \prec_{T(P)} \hat{1}_{T(P)}$.

As a remark, Hetyei's original definition of the Tchebyshev transform is in terms of the order relation rather than the cover relation of the poset. Note the rank function $\rho_{T(P)}$ on $T(P)$ satisfies $\rho_{T(P)}([x, y]) = \rho_P(y)$ and $\rho_{T(P)}(\hat{1}_{T(P)}) = \rho(P) + 1$.

Our interest in studying the Tchebyshev transform of posets arises from the following result of Hetyei [22].

Theorem 4.2 *Let P be an Eulerian poset. The Tchebyshev transform of P, $T(P)$, is also an Eulerian poset.*

For a graded poset P let $z : T(P) \to P \cup \{\hat{2}\}$ be the map $z([x, y]) = y$ and $z(\hat{1}_{T(P)}) = \hat{2}$. Observe the map z is order and rank preserving and hence preserves chains and the weight of chains.

We now prove a proposition which can be viewed as an analogue of a result of Bayer and Sturmfels [5] (see Proposition 4.6.2 in [11]) and of Proposition 4.1 in [16]. This connection will be made clearer in Sections 12 and 13.

Proposition 4.3 *For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{2}\}$ in $P \cup \{\hat{2}\}$, the cardinality of the inverse image of c is given by*

$$|z^{-1}(c)| = \prod_{i=1}^{k-1} |[x_{i-1}, x_i]|.$$

To prove this proposition we need a lemma and its corollary.

Lemma 4.4 *Given three elements $x < y < w$ in the poset P, the condition $[x, y] <_{T(P)} [z, w]$ is equivalent to either $z = x$ or $z \in [y, w)$.*

Proof: We proceed by induction on $\rho(y, w)$. If $\rho(y, w) = 1$, we have by definition that the element z is either x (condition (ii)) or y (condition (i)). Assume now that $\rho(y, w) \geq 2$ and let $[u, v]$ be an atom in the interval $[[x, y], [z, w]]$. Since $\rho(v, w) < \rho(y, w)$ we have by the induction hypothesis that either $z = u$ or $z \in [v, w)$. The union of all such intervals $[v, w)$ is the open interval (y, w). Moreover, since v covers y, we have that u is either x or y. That is, the only choices for z are $\{x\} \cup \{y\} \cup (y, w) = \{x\} \cup [y, w)$, proving the induction step. $\square$

Corollary 4.5 *Given three elements $x < y < w$ in the poset P, the number of elements z such that $[x, y] <_{T(P)} [z, w]$ equals the cardinality of the interval $[y, w]$.*

The proof of Proposition 4.3 follows by repeated use of Corollary 4.5.

Lemma 4.4 is essentially [22, Definition 2.1] and Definition 4.1 is [22, Proposition 2.3] in the case of graded posets.

5 The Tchebyshev transform on ab-polynomials

In this section we express the **ab**-index of the Tchebyshev transform in terms of the **ab**-index of the original poset.

Define two linear maps A and C from $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ to $\mathbb{Z}$ as follows. Let A be the algebra map $A(\mathbf{a}) = 1$ and $A(\mathbf{b}) = 0$ and let C be the map given by the relation

$$C(u) = 2 \cdot A(u) + \sum_{u} A(u_{(1)}) \cdot A(u_{(2)}).$$

Lemma 5.1 *A graded poset P satisfies*

$$A(\Psi(P)) = 1 \quad \textit{and} \quad C(\Psi(P)) = |P|.$$

Proof: The first identity $A(\Psi(P)) = 1$ follows since the chain $\{\widehat{0} < \widehat{1}\}$ is the only chain having non-zero weight after substituting $\mathbf{b} = 0$. The second identity follows from

$$\begin{aligned} C(\Psi(P)) &= 2 \cdot A(\Psi(P)) + \sum_{\widehat{0} < x < \widehat{1}} A(\Psi([\widehat{0}, x])) \cdot A(\Psi([x, \widehat{1}])) \\ &= 2 + \sum_{\widehat{0} < x < \widehat{1}} 1 \\ &= |P|, \end{aligned}$$

where the first step follows from the fact the **ab**-index is a coalgebra homomorphism. □

Lemma 5.2 *The linear map C satisfies the recursion*

$$\begin{aligned} C(1) &= 2, \\ C(\mathbf{a} \cdot u) &= A(u) + C(u), \\ C(\mathbf{b} \cdot u) &= A(u). \end{aligned}$$

Proof: Directly $C(1) = 2 \cdot A(1) = 2$. For the second identity, by the Newtonian condition (2.1) we have

$$\begin{aligned} C(\mathbf{a} \cdot u) &= 2 \cdot A(\mathbf{a} \cdot u) + A(1) \cdot A(u) + \sum_u A(\mathbf{a} \cdot u_{(1)}) \cdot A(u_{(2)}) \\ &= 3 \cdot A(u) + \sum_u A(u_{(1)}) \cdot A(u_{(2)}) \\ &= A(u) + C(u). \end{aligned}$$

Similarly, the third identity follows from

$$\begin{aligned} C(\mathbf{b} \cdot u) &= 2 \cdot A(\mathbf{b} \cdot u) + A(1) \cdot A(u) + \sum_u A(\mathbf{b} \cdot u_{(1)}) \cdot A(u_{(2)}) \\ &= A(u). \end{aligned}$$

□

We now consider three linear operators on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. For a homogeneous **ab**-polynomial u define κ and ν by

$$\kappa(u) = A(u) \cdot (\mathbf{a} - \mathbf{b})^{\deg(u)} \quad \text{and} \quad \nu(u) = C(u) \cdot (\mathbf{a} - \mathbf{b})^{\deg(u)},$$

and extend by linearity. Define T by the sum

$$T(u) = \sum_{k \geq 1} \sum_u \nu(u_{(1)}) \cdot \mathbf{b} \cdot \nu(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \nu(u_{(k-1)}) \cdot \mathbf{b} \cdot \kappa(u_{(k)}), \tag{5.1}$$

where the coproduct is into k parts. Observe that equation (5.1) directly implies the following proposition.

Proposition 5.3 *The operator T satisfies the functional identity*

$$T(u) = \kappa(u) + \sum_u \nu(u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)}).$$

The slight abuse of notation between the Tchebyshev transform of a graded poset and the Tchebyshev transform of **ab**-monomials is explained by the following theorem.

Theorem 5.4 *The* **ab***-index of the Tchebyshev transform of a graded poset P is given by*

$$\Psi(T(P)) = T(\Psi(P) \cdot \mathbf{a}).$$

Proof: Using the chain definition of the **ab**-index and Proposition 4.3, we have

$$\begin{aligned}
\Psi(T(P)) &= \sum_{c} |z^{-1}(c)| \cdot \mathrm{wt}(c) \\
&= \sum_{k\geq 1} \sum_{c} \prod_{i=1}^{k-1} C(\Psi([x_{i-1}, x_i])) \cdot A(\Psi([x_{k-1}, x_k])) \cdot \mathrm{wt}(c) \\
&= \sum_{k\geq 1} \sum_{c} \prod_{i=1}^{k-1} C(\Psi([x_{i-1}, x_i])) \cdot A(\Psi([x_{k-1}, x_k])) \\
&\quad \cdot \left(\prod_{i=1}^{k-1} (\mathbf{a}-\mathbf{b})^{\rho(x_{i-1},x_i)-1} \cdot \mathbf{b} \right) \cdot (\mathbf{a}-\mathbf{b})^{\rho(x_{k-1},x_k)-1} \\
&= \sum_{k\geq 1} \sum_{c} \left(\prod_{i=1}^{k-1} \nu(\Psi[x_{i-1}, x_i]) \cdot \mathbf{b} \right) \cdot \kappa(\Psi[x_{k-1}, x_k]) \\
&= \sum_{k\geq 1} \sum_{u} \left(\prod_{i=1}^{k-1} \nu(u_{(i)}) \cdot \mathbf{b} \right) \cdot \kappa(u_{(k)}) \\
&= T(u).
\end{aligned}$$

Here the second to last step uses the fact the **ab**-index is a coalgebra homomorphism and u is the **ab**-index of $P \cup \widehat{2}$, that is, $u = \Psi(P \cup \widehat{2}) = \Psi(P) \cdot \mathbf{a}$. $\square$

6 Connection with the ω operator of oriented matroids

We begin by recalling the ω map for oriented matroids [7].

Theorem 6.1 *Let $\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \to \mathbb{Z}\langle \mathbf{c}, 2\mathbf{d}\rangle$ be the linear map defined on monomials in the variables* **a** *and* **b** *by replacing each occurrence of* **ab** *by* 2**d** *and the remaining letters with* **c***'s. For an oriented matroid, let R be its lattice of regions and L be its lattice of flats. Then the* **cd***-index of R is given by*

$$\Psi(R) = \omega(\mathbf{a} \cdot \Psi(L))^*.$$

In fact, the **cd***-index of the lattice of regions R is indeed a* **c**-2**d***-index.*

Hsiao has found an analogous version of this theorem for the Birkhoff transform of a distributive lattice [24]. Ehrenborg has generalized Hsiao's work to an r-signed Birkhoff transform [16]. In this section we show the Tchebyshev transform is likewise connected to the omega map. This allows us to conclude the Tchebyshev transform preserves nonnegativity of the **cd**-index.

Theorem 6.2 *Given a* $\mathbf{cd}$*-polynomial* v*, it satisfies*

$$T(v \cdot \mathbf{a}) = \omega(\mathbf{a} \cdot v^*)^*.$$

Proof: Following [7] let η be the unique operator on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ such that

$$\eta(\Psi(P)) = \left(\sum_{\widehat{0} \leq x \leq \widehat{1}} (-1)^{\rho(x)} \cdot \mu(\widehat{0}, x) \right) \cdot (\mathbf{a} - \mathbf{b})^{\rho(P)-1},$$

for all posets P. Next, let the operator φ be defined as follows:

$$\varphi(u) = \sum_{k \geq 1} \sum_{u} \kappa(u_{(1)}) \cdot \mathbf{b} \cdot \eta(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \eta(u_{(k)}). \tag{6.1}$$

By Proposition 5.5 in [7] we have $\omega(\mathbf{a} \cdot v) = \varphi(\mathbf{a} \cdot v)$. Also observe

$$T(u^*)^* = \sum_{k \geq 1} \sum_{u} \kappa(u_{(1)}) \cdot \mathbf{b} \cdot \nu(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \nu(u_{(k)}). \tag{6.2}$$

For any Eulerian poset P we have $\eta(\Psi(P)) = \nu(\Psi(P))$ since $(-1)^{\rho(x)} \cdot \mu(\widehat{0}, x) = 1$ for all elements x in an Eulerian poset P. Since the $\mathbf{cd}$-indexes of all Eulerian posets span all $\mathbf{cd}$-polynomials, we have for all $\mathbf{cd}$-polynomials v that $\eta(v) = \nu(v)$. Now consider the coproduct Δ^k applied to $u = \mathbf{a} \cdot v$, where v is a $\mathbf{cd}$-polynomial. We obtain

$$\Delta^k(\mathbf{a} \cdot v) \in \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \otimes \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle^{\otimes (k-1)}.$$

Hence the expressions in equations (6.1) and (6.2) agree on $u = \mathbf{a} \cdot v$. $\square$

Recall that the Tchebyshev transform T preserves Eulerianness. We obtain two important corollaries.

Theorem 6.3 *If an Eulerian poset* P *has a non-negative* $\mathbf{cd}$*-index then so does the Tchebyshev transform* $T(P)$*, that is,* $\Psi(P) \geq 0$ *implies* $\Psi(T(P)) \geq 0$.

Proof: The $\mathbf{cd}$-polynomial $\Psi(P)$ has non-negative terms as an $\mathbf{ab}$-polynomial. Applying Theorems 5.4 and 6.2 and observing that ω sends an $\mathbf{ab}$-monomial to a $\mathbf{c}$-$2\mathbf{d}$-monomial, we see that non-negativity is preserved. $\square$

Corollary 6.4 *The Tchebyshev transform* $T(P)$ *of an Eulerian poset* P *has a* $\mathbf{c}$-$2\mathbf{d}$*-index, that is, the* $\mathbf{cd}$*-index* $\Psi(T(P))$ *belongs to* $\mathbb{Z}\langle \mathbf{c}, 2\mathbf{d} \rangle$.

Since a given **cd**-monomial expands into 2^k **ab**-monomials, where k is the number of **c**'s and **d**'s appearing in the monomial, we also have the following corollary.

Corollary 6.5 *Let u be a* **cd***-monomial consisting of k letters. Then the Tchebyshev transform $T(u \cdot \mathbf{a})$ is a sum of 2^k* **c***-2***d***-monomials.*

In the paper [22] Hetyei introduced a sequence of posets $T_1, T_2, \ldots$ which he called the Tchebyshev posets. The Tchebyshev poset T_{n+1} is simply the Tchebyshev transform of the rank n ladder poset L_n, where L_n is the unique poset with **cd**-index $\mathbf{c}^{n-1}$. The rank n ladder poset has two elements in each non-trivial rank and each element covers each element of rank one less. This poset is also known as the butterfly poset.

Hetyei has the following result for coefficients of the **cd**-index of T_n [22, Theorem 7.1].

Corollary 6.6 *The coefficient of the* **cd***-monomial $\mathbf{c}^{k_1}\mathbf{d}\mathbf{c}^{k_2}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{k_{r+1}}$ in the* **cd***-index of the Tchebyshev poset $T_n = T(L_n)$ is given by*

$$2^r \cdot (k_1+1)\cdots(k_r+1).$$

Proof: By Theorem 6.2 we only have to find the coefficient of $\mathbf{c}^{k_{r+1}}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{k_2}\mathbf{d}\mathbf{c}^{k_1}$ in $\omega(\mathbf{a}\cdot\mathbf{c}^n)$. When expanding $\mathbf{a}\cdot\mathbf{c}^n$ in terms of **ab**-monomials, we are only interested in those monomials where the consecutive letters **ab** appear precisely in the positions of the **d**'s in the **cd**-monomial $\mathbf{c}^{k_{r+1}}\mathbf{d}\cdots\mathbf{d}\mathbf{c}^{k_2}\mathbf{d}\mathbf{c}^{k_1}$. It remains to choose the other letters in the **ab**-monomial. The first k_{r+1} letters before the first **ab** must be all **a**'s. The next string of letters (between the first and second **ab**'s) must have the form $\mathbf{b}^i\mathbf{a}^j$ where $i+j=k_r$. Hence there are k_r+1 possibilities. The same holds for every other string of letters, giving the product $(k_1+1)\cdots(k_r+1)$. Finally, the omega map assigns a factor of 2 with each **d**. $\square$

Recall the hyperplane arrangement in $\mathbb{R}^n$ consisting of the n coordinate hyperplanes $x_i = 0$ for $1 \leq i \leq n$ has intersection lattice corresponding to the Boolean algebra B_n. The regions of this arrangement correspond to faces of the n-dimensional crosspolytope C_n. Hence another corollary of Theorem 6.2 is as follows.

Corollary 6.7 *The* **cd***-index of the Tchebyshev transform of the Boolean algebra B_n equals the* **cd***-index of the n-dimensional crosspolytope C_n, that is,*

$$\Psi(T(B_n)) = \Psi(C_n).$$

Observe that $T(B_n) \ncong C_n$ as can be seen from Figure 3.

7 Recursions for the Tchebyshev transform

In this section we develop recursions for computing the Tchebyshev transform. They are especially important for **ab**-polynomials.

Define a new operator σ on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by

$$\sigma(u) = \sum_u \kappa(u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)}),$$

where u is an **ab**-polynomial.

Proposition 7.1 *The operator T satisfies the following joint recursion with the operator σ:*

$$\begin{aligned} T(1) &= 1 && (7.1)\\ T(\mathbf{a} \cdot u) &= (\mathbf{a} + \mathbf{b}) \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(u), && (7.2)\\ T(\mathbf{b} \cdot u) &= 2\mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(u), && (7.3)\\ \sigma(1) &= 0, && (7.4)\\ \sigma(\mathbf{a} \cdot u) &= \mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(u), && (7.5)\\ \sigma(\mathbf{b} \cdot u) &= \mathbf{b} \cdot T(u). && (7.6) \end{aligned}$$

Proof: Directly $T(1) = 1$ and $\sigma(1) = 0$. Using the Newtonian condition (2.1), we have

$$\begin{aligned} T(\mathbf{a} \cdot u) &= \kappa(\mathbf{a} \cdot u) + \nu(1) \cdot \mathbf{b} \cdot T(u) + \sum_u \nu(\mathbf{a} \cdot u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= (\mathbf{a} - \mathbf{b}) \cdot \kappa(u) + 2\mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sum_u (\kappa(u_{(1)}) + \nu(u_{(1)})) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= (\mathbf{a} + \mathbf{b}) \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sum_u \kappa(u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= (\mathbf{a} + \mathbf{b}) \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(u). \end{aligned}$$

Here we have used the functional equation in Proposition 5.3 in the first and third equalities. Similarly, we obtain

$$\begin{aligned} T(\mathbf{b} \cdot u) &= \kappa(\mathbf{b} \cdot u) + \nu(1) \cdot \mathbf{b} \cdot T(u) + \sum_u \nu(\mathbf{b} \cdot u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= 2\mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sum_u \kappa(u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= 2\mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(u). \end{aligned}$$

For the operator σ we have

$$\begin{aligned} \sigma(\mathbf{a} \cdot u) &= \kappa(1) \cdot \mathbf{b} \cdot T(u) + \sum_u \kappa(\mathbf{a} \cdot u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= \mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sum_u \kappa(u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= \mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(u), \end{aligned}$$

and

$$\begin{aligned} \sigma(\mathbf{b} \cdot u) &= \kappa(1) \cdot \mathbf{b} \cdot T(u) + \sum_u \kappa(\mathbf{b} \cdot u_{(1)}) \cdot \mathbf{b} \cdot T(u_{(2)})\\ &= \mathbf{b} \cdot T(u). \end{aligned}$$

□

Corollary 7.2 *For an* **ab**-*polynomial* u,

$$T((\mathbf{a}-\mathbf{b})\cdot u) = (\mathbf{a}-\mathbf{b})\cdot T(u).$$

As a consequence, the following identity holds:

$$T((\mathbf{c}^2-2\mathbf{d})\cdot u) = (\mathbf{c}^2-2\mathbf{d})\cdot T(u).$$

Proof: The first part follows from subtracting equation (7.3) from equation (7.2). The second part follows from $(\mathbf{a}-\mathbf{b})^2 = \mathbf{c}^2 - 2\mathbf{d}$. $\square$

Define the operator π on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ by

$$\pi(u) = 2\mathbf{b}\cdot T(u) + 2(\mathbf{a}-\mathbf{b})\cdot\sigma(u).$$

We now restrict our attention to the subalgebra $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$.

Proposition 7.3 *The operator* T *satisfies the following joint recursion with the operator* π*:*

$$\begin{aligned}
T(\mathbf{a}) &= \mathbf{c} && (7.7)\\
T(\mathbf{c}\cdot u) &= \mathbf{c}\cdot T(u) + \pi(u), && (7.8)\\
T(\mathbf{d}\cdot u) &= 2\mathbf{d}\cdot T(u) + \mathbf{c}\cdot\pi(u), && (7.9)\\
\pi(\mathbf{a}) &= 2\mathbf{d}, && (7.10)\\
\pi(\mathbf{c}\cdot u) &= 2\mathbf{d}\cdot T(u) + \mathbf{c}\cdot\pi(u), && (7.11)\\
\pi(\mathbf{d}\cdot u) &= \mathbf{c}\cdot 2\mathbf{d}\cdot T(u) + 2\mathbf{d}\cdot\pi(u). && (7.12)
\end{aligned}$$

Proof: By Proposition 7.1 we have

$$\begin{aligned}
T(\mathbf{c}\cdot u) &= T(\mathbf{a}\cdot u) + T(\mathbf{b}\cdot u)\\
&= (\mathbf{a}+\mathbf{b})\cdot T(u) + 2\mathbf{b}\cdot T(u) + 2(\mathbf{a}-\mathbf{b})\cdot\sigma(u) && (7.13)\\
&= \mathbf{c}\cdot T(u) + \pi(u). && (7.14)
\end{aligned}$$

Iterating Proposition 7.1 twice yields

$$\begin{aligned}
T(\mathbf{d}\cdot u) &= T(\mathbf{ab}\cdot u) + T(\mathbf{ba}\cdot u)\\
&= (\mathbf{a}+\mathbf{b})\cdot T(\mathbf{b}\cdot u) + (\mathbf{a}-\mathbf{b})\cdot\sigma(\mathbf{b}\cdot u) + 2\mathbf{b}\cdot T(\mathbf{a}\cdot u) + (\mathbf{a}-\mathbf{b})\cdot\sigma(\mathbf{a}\cdot u)\\
&= [2\mathbf{cb} + (\mathbf{a}-\mathbf{b})\cdot\mathbf{b} + 2\mathbf{b}\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot\mathbf{b}]\cdot T(u)\\
&\quad + [\mathbf{c}\cdot(\mathbf{a}-\mathbf{b}) + 2\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})]\cdot\sigma(u)\\
&= (2\mathbf{d} + 2\mathbf{c}\cdot\mathbf{b})\cdot T(u) + 2\mathbf{c}\cdot(\mathbf{a}-\mathbf{b})\cdot\sigma(u) && (7.15)\\
&= 2\mathbf{d}\cdot T(u) + \mathbf{c}\cdot\pi(u). && (7.16)
\end{aligned}$$

For the operator π we have

$$\begin{aligned}
\pi(\mathbf{c} \cdot u) &= 2\mathbf{b} \cdot T(\mathbf{c} \cdot u) + 2(\mathbf{a} - \mathbf{b}) \cdot \sigma(\mathbf{c} \cdot u) \\
&= 2\mathbf{b} \cdot [(\mathbf{a} + \mathbf{b}) \cdot T(u) + 2\mathbf{b} \cdot T(u) + 2(\mathbf{a} - \mathbf{b}) \cdot \sigma(u)] \\
&\quad + 2(\mathbf{a} - \mathbf{b}) \cdot [2\mathbf{b} \cdot T(u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(u)] \\
&= (2\mathbf{d} + 2\mathbf{cb}) \cdot T(u) + \mathbf{c} \cdot 2(\mathbf{a} - \mathbf{b}) \cdot \sigma(u) \\
&= 2\mathbf{d} \cdot T(u) + \mathbf{c} \cdot \pi(u).
\end{aligned}$$

Here for the second equality we have applied (7.13) and the sum of (7.5) and (7.6).

A straightforward double iteration of Proposition 7.1 yields

$$\begin{aligned}
\sigma(\mathbf{d} \cdot u) &= \sigma(\mathbf{ab} \cdot u) + \sigma(\mathbf{ba} \cdot u) \\
&= \mathbf{b} \cdot T(\mathbf{b} \cdot u) + (\mathbf{a} - \mathbf{b}) \cdot \sigma(\mathbf{b} \cdot u) + \mathbf{b} \cdot T(\mathbf{a} \cdot u) \\
&= \left[2\mathbf{b}^2 + (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} + \mathbf{b} \cdot (\mathbf{a} + \mathbf{b})\right] \cdot T(u) + 2\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \cdot \sigma(u) \\
&= (\mathbf{d} + 2\mathbf{b}^2) \cdot T(u) + 2\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \cdot \sigma(u). \qquad (7.17)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\pi(\mathbf{d} \cdot u) &= 2\mathbf{b} \cdot T(\mathbf{d} \cdot u) + 2(\mathbf{a} - \mathbf{b}) \cdot \sigma(\mathbf{d} \cdot u) \\
&= 2\mathbf{b} \cdot [(2\mathbf{d} + 2\mathbf{c} \cdot \mathbf{b}) \cdot T(u) + 2\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) \cdot \sigma(u)] \\
&\quad + 2(\mathbf{a} - \mathbf{b}) \cdot \left[(\mathbf{d} + 2\mathbf{b}^2) \cdot T(u) + 2\mathbf{b} \cdot (\mathbf{a} - \mathbf{b}) \cdot \sigma(u)\right] \\
&= \left[4\mathbf{bd} + 4\mathbf{bcb} + 2(\mathbf{a} - \mathbf{b})\mathbf{d} + 4(\mathbf{a} - \mathbf{b})\mathbf{b}^2\right] \cdot T(u) \\
&\quad + [4\mathbf{b} \cdot \mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) + 4(\mathbf{a} - \mathbf{b})\mathbf{b}(\mathbf{a} - \mathbf{b})] \cdot \sigma(u) \\
&= (2\mathbf{cd} + 2\mathbf{d} \cdot 2\mathbf{b}) \cdot T(u) + 2\mathbf{d} \cdot 2(\mathbf{a} - \mathbf{b}) \cdot \sigma(u) \\
&= \mathbf{c} \cdot 2\mathbf{d} \cdot T(u) + 2\mathbf{d} \cdot \pi(u),
\end{aligned}$$

where the second equality follows from (7.15) and (7.17). □

Note that Proposition 7.3 offers different proofs for Theorem 6.3, Corollaries 6.4 and 6.5.

The next proposition relates the operators T and π with the operator ω.

Proposition 7.4 *For* $\mathbf{cd}$*-polynomials* v*, the following two identities hold:*

$$\begin{aligned}
T(v \cdot \mathbf{a}) &= \omega(\mathbf{a} \cdot v^*)^*, \\
\pi(v \cdot \mathbf{a}) &= \omega(\mathbf{a} \cdot v^* \cdot \mathbf{b})^*.
\end{aligned}$$

This proposition is straightforward to prove using induction, and hence we omit the proof. Notice this argument offers a second proof of Theorem 6.2.

The next relation extends a result from [22] where the special case of the ladder poset was considered.

Corollary 7.5 *For all* **ab**-*polynomials* u*, the following recursion holds:*

$$T(\mathbf{c}^2 \cdot u) = 2\mathbf{c} \cdot T(\mathbf{c} \cdot u) + (2\mathbf{d} - \mathbf{c}^2) \cdot T(u).$$

Proof: From Proposition 7.3, we have

$$\begin{aligned} T(\mathbf{c}^2 \cdot u) &= \mathbf{c} \cdot T(\mathbf{c} \cdot u) + \pi(\mathbf{c} \cdot u) \\ &= \mathbf{c} \cdot T(\mathbf{c} \cdot u) + 2\mathbf{d} \cdot T(u) + \mathbf{c} \cdot \pi(u) \\ &= \mathbf{c} \cdot T(\mathbf{c} \cdot u) + 2\mathbf{d} \cdot T(u) + \mathbf{c} \cdot (T(\mathbf{c} \cdot u) - \mathbf{c} \cdot T(u)) \\ &= 2\mathbf{c} \cdot T(\mathbf{c} \cdot u) + (2\mathbf{d} - \mathbf{c}^2) \cdot T(u). \end{aligned}$$

□

As a corollary to this recursion, we can now explain the name Tchebyshev. This result is originally due to Hetyei [22].

Corollary 7.6 *The substitution* x *for* $\mathbf{c}$ *and* $(x^2 - 1)/2$ *for* $\mathbf{d}$ *in* $T(\mathbf{c}^{n-1} \cdot \mathbf{a})$ *yields the Tchebyshev polynomial of the first kind* $T_n(x)$*.*

Under this substitution the recurrence in Corollary 7.5 becomes the recurrence for the Tchebyshev polynomials. The substitution for $\mathbf{c}$ and $\mathbf{d}$ takes $T(\mathbf{a})$ and $T(\mathbf{c} \cdot \mathbf{a})$ to $T_1(x) = x$ and $T_2(x) = 2x^2 - 1$, respectively.

8 EL-shellability

For a poset P let $\mathcal{H}(P)$ be the set of edges in the Hasse diagram of P, that is, $\mathcal{H}(P) = \{(x, y) : x, y \in P,\ x \prec y\}$, where $\prec$ denotes the cover relation in the poset P. An *R-labeling* of a poset P is a map λ from $\mathcal{H}(P)$ to Λ, a linearly ordered set of labels, such that in every interval $[x, y]$ there is a unique maximal (saturated) chain $x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ having increasing labels $\lambda(x_0, x_1) \leq_\Lambda \lambda(x_1, x_2) \leq_\Lambda \cdots \leq_\Lambda \lambda(x_{k-1}, x_k)$. Such a chain is called *rising*. Furthermore an R-labeling is an *EL-labeling* if the unique rising chain in every interval is also the maximal chain with the lexicographically least labels. A poset having an EL-labeling is said to be *EL-shellable*. For further information regarding EL-labelings and their topological consequences, see [12].

Recall the Jordan-Hölder set $JH(x, y)$ of an interval $[x, y]$ is the collection of all strings of labels occurring from the maximal chains in the interval, that is,

$$JH(x, y) = \{(\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k)) \ :\ x = x_0 \prec x_1 \prec \cdots \prec x_k = y\}.$$

Theorem 8.1 *Let* P *be an* EL*-shellable poset. Then the Tchebyshev transform* $T(P)$ *is an* EL*-shellable poset.*

Proof: Suppose the poset P has label set $\Lambda = \{\lambda_1 < \cdots < \lambda_k\}$. Define the new label set $\Gamma = \{\lambda_1^s < \cdots < \lambda_k^s < 0 < \lambda_1^b < \cdots < \lambda_k^b\}$. Here one should think of the superscript s as denoting "small" labels and the superscript b as denoting "big" labels. In the Tchebyshev poset $T(P)$ label the edges in the Hasse diagram by the following rule:

$$\begin{cases} \lambda([x,y],[y,w]) & = & \lambda(y,w)^s, \\ \lambda([x,y],[x,w]) & = & \lambda(y,w)^b, \\ \lambda([x,y],\widehat{1}_{T(P)}) & = & 0. \end{cases}$$

We claim this is an EL-labeling of the Tchebyshev poset $T(P)$. For a set X of strings of labels from the set Λ, let X^s and X^b denoted the set of strings where each label has been signed with s, respectively b. Similarly, let X^{sb} denote the set of strings where each label has been arbitrarily signed s or b.

There are three types of intervals to consider.

(i) An interval of the form $I = [[x,y],[x,w]]$ in $T(P)$ is isomorphic to the interval $[y,w]$ in the original poset P. In this case, the edge labels are from the set $\{\lambda_1^b, \ldots, \lambda_k^b\}$ and the Jordan-Hölder set of the interval I is described by

$$JH(y,w)^b.$$

Hence the lexicographically least maximal chain in the interval I is to take the lexicographically least maximal chain in the interval $[y,w]$ and change the labels λ_i to λ_i^b.

(ii) Let I be an interval of the form $[[x,y],[z,w]]$, where z is an element of rank j from the half-open interval $[y,w)$ in the poset P. Observe that $0 \leq j < k$. Any maximal chain $\{[x,y] = [x_0,y_0] \prec [x_1,y_1] \prec \cdots \prec [x_k,y_k] = [z,w]\}$ in the interval I satisfies $\{y = y_0 \prec y_1 \prec \cdots \prec y_k = w\}$ is a maximal chain in the interval $[y,w]$ and $z = y_j = x_{j+1} = \cdots = x_k$. Thus the Jordan-Hölder set of the interval I is described by

$$\bigcup_{z \prec y_{j+1}} JH(y,z)^{sb} \circ \lambda(z,y_{j+1})^s \circ JH(y_{j+1},w)^b,$$

where $\circ$ denotes concatenation. To obtain a rising chain in the interval I, let $m = \{y = y_0 \prec y_1 \prec \cdots \prec y_j = z\}$ be the unique rising chain in the interval $[y,z]$ and let $m' = \{z = y_j \prec y_{j+1} \prec \cdots \prec y_k = w\}$ be the unique rising chain in the interval $[z,w]$. Set $x_i = y_{i-1}$ for $0 < i \leq j$ and $x_i = z$ for $j+1 \leq i \leq k$. The string of labels of this maximal chain is given by

$$(\lambda(y_0,y_1)^s, \lambda(y_1,y_2)^s, \ldots, \lambda(y_{j-1},y_j)^s, \lambda(y_j,y_{j+1})^s, \lambda(y_{j+1},y_{j+2})^b, \ldots, \lambda(y_{k-1},y_k)^b).$$

It is straightforward to see that this chain is the unique rising and lexicographic least maximal chain in the interval.

(iii) Let I be the interval of the form $[[x,y],\widehat{1}_{T(P)}]$. Any maximal chain $\{[x,y] = [x_0,y_0] \prec [x_1,y_1] \prec \cdots \prec [x_k,y_k] \prec \widehat{1}_{T(P)}\}$ in the interval I satisfies $\{y = y_0 \prec y_1 \prec \cdots \prec y_k = \widehat{1}_P\}$ is a maximal chain in the interval $[y,\widehat{1}_P]$. Thus the Jordan-Hölder set of the interval I is described by

$$JH(y,\widehat{1}_P)^{sb} \circ 0.$$

Since all the labels signed with s are smaller than 0, a rising chain can only have these "small" labels. The unique rising chain in the interval $[y,\widehat{1}_P]$ is $\{y = y_0 \prec y_1 \prec \cdots \prec y_k = \widehat{1}_P\}$. To obtain the desired maximal chain in the interval I with the correct labels, let $x_i = y_{i-1}$ for $0 < i \leq k$. This rising chain is also the lexicographic least.

Hence we conclude $T(P)$ has an EL-labeling. □

As a corollary to Theorem 8.1 and its proof we have:

Corollary 8.2 *Let P be a poset with an R-labeling using the label set Λ. Then the Tchebyshev transform $T(P)$ has an R-labeling with the label set given by $\Lambda^{sb} \cup \{0\}$ and the Jordan-Hölder set given by $JH(T(P)) = JH(P)^{sb} \circ 0$.*

Recall that the Tchebyshev poset T_n is simply the Tchebyshev transform of the ladder poset L_{n-1}. Since the ladder poset is EL-shellable, we obtain the next corollary, originally due to Hetyei [22].

Corollary 8.3 *The Tchebyshev poset T_n is EL-shellable.*

9 The Tchebyshev transform of Cartesian products

In the papers [18, 21], Ehrenborg-Fox and Ehrenborg-Readdy studied the behavior of the **cd**-index under the *Cartesian product* $P \times Q$ and the *diamond product* $P \diamond Q$, where P and Q are posets. This latter product is defined as $P \diamond Q = (P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$. For our purposes, we need to consider the dual of the diamond product, namely

$$P \diamond^* Q = (P - \{\hat{1}\}) \times (Q - \{\hat{1}\}) \cup \{\hat{1}\}.$$

In other words, $P \diamond^* Q = (P^* \diamond Q^*)^*$.

We have the following result.

Theorem 9.1 *Given two posets P and Q, the flag f-vector of the Tchebyshev transform of the Cartesian product $P \times Q$ is equal to the flag f-vector of the dual diamond product of the two Tchebyshev transforms $T(P)$ and $T(Q)$, that is,*

$$\Psi(T(P \times Q)) = \Psi(T(P) \diamond^* T(Q)).$$

In general, it is not true that the two posets $T(P \times Q)$ and $T(P) \diamond^* T(Q)$ are isomorphic. A counterexample is to take $P = B_2$ and $Q = B_1$. The Tchebyshev transform of B_2 is isomorphic to the face lattice of a square and the Tchebyshev transform of B_1 is the face lattice of a line segment. Hence $T(B_2) \diamond^* T(B_1)$ is the face lattice of the 3-dimensional crosspolytope. However the Tchebyshev transform of $B_2 \times B_1 = B_3$ is not a lattice. It is the face poset of the CW-complex displayed in Figure 3.

Observe that an alternate proof of Corollary 6.7 follows directly from Theorem 9.1 by considering the Boolean algebra $B_n = B_1^n$.

To prove Theorem 9.1 we need the following result from [21].

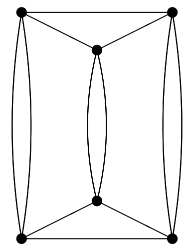

Figure 3: A CW-decomposition of the 2-sphere with three 2-gons, two triangles and three squares. The face lattice is the Tchebyshev transform of B_3.

Theorem 9.2 *There exist two bilinear operators M and N on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ that correspond to the Cartesian and diamond product of graded posets. In other words, two graded posets P and Q satisfy*

$$\Psi(P \times Q) = M(\Psi(P), \Psi(Q)), \tag{9.1}$$
$$\Psi(P \diamond Q) = N(\Psi(P), \Psi(Q)). \tag{9.2}$$

Recursions for the two bilinear operators M and N have been developed in [18]. Defining N^* by $N^*(u, v) = N(u^*, v^*)^*$, we have

$$\Psi(P \diamond^* Q) = N^*(\Psi(P), \Psi(Q)). \tag{9.3}$$

Theorem 9.2 states that on the flag f-vector level the Cartesian product and the dual diamond product are bilinear. Hence Theorem 9.1 can be reformulated as follows.

Theorem 9.3 *Any two* **ab***-polynomials u and v satisfy*

$$T(M(u, v) \cdot \mathbf{a}) = N^*(T(u \cdot \mathbf{a}), T(v \cdot \mathbf{a})).$$

Proposition 9.4 *Any two* **ab***-polynomials u and v satisfy*

$$\omega(\mathbf{a} \cdot M(u, v)) = N(\omega(\mathbf{a} \cdot u), \omega(\mathbf{a} \cdot v)).$$

Proof: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two central hyperplane arrangements. Let L_i be the intersection lattice of $\mathcal{H}_i$ and let Z_i be the associated zonotope. Then the intersection lattice L of $\mathcal{H}_1 \times \mathcal{H}_2$ is the Cartesian product of the intersection lattices L_1 and L_2. The associated zonotope Z of $\mathcal{H}_1 \times \mathcal{H}_2$ is the Cartesian product of the zonotopes Z_1 and Z_2. Especially, the face lattice of Z is the diamond product of the face lattices of Z_1 and Z_2. Hence we have that $\omega(\mathbf{a} \cdot M(\Psi(L_1), \Psi(L_2))) = \omega(\mathbf{a} \cdot \Psi(L)) = \Psi(Z) =$

$N(\Psi(Z_1), \Psi(Z_2)) = N(\omega(\mathbf{a} \cdot \Psi(L_1)), \omega(\mathbf{a} \cdot \Psi(L_1)))$. Since the **ab**-indexes of the intersection lattices of hyperplane arrangements span $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ (see [8]), the desired identity is proved. $\square$

Combining Theorem 6.2 and Proposition 9.4, we obtain that Theorem 9.1 is true for **cd**-polynomials.

To prove Theorem 9.1 it is enough to prove the identity for a class of posets having **ab**-indexes which span $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. We will prove the identity for posets that admit R-labelings.

Proposition 9.5 *Let P_1 and P_2 be two posets such that each has an R-labeling. Then we have*

$$\Psi(T(P_1 \times P_2)) = \Psi(T(P_1) \diamond^* T(P_2)).$$

Let P be a graded poset of rank $n+1$ that has an R-labeling. The strings of labels in the Jordan-Hölder set $JH(P)$ have length $n+1$. For such a string $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$, define its *decent word* to be $u_\lambda = u_1 u_2 \cdots u_n$ by letting $u_i = \mathbf{a}$ if $\lambda_i \leq \lambda_{i+1}$ and $u_{i+1} = \mathbf{b}$ otherwise. Then we have the following result which expresses the **ab**-index of the poset P in terms of the Jordan-Hölder set $JH(P)$.

Proposition 9.6 *Let P be a poset with an R-labeling. Then the* **ab***-index of P is given by*

$$\Psi(P) = \sum_{\lambda \in JH(P)} u_\lambda.$$

The original formulation of this result is due to Björner-Stanley [10]. The reformulation in Proposition 9.6 can be found in [7].

Given two strings $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$, define their *shuffle product* $\mathbf{x} \star \mathbf{y}$ to be the set of all $\binom{n+m}{n}$ shuffles of them, that is,

$$\begin{aligned} \mathbf{x} \star \mathbf{y} \quad = \quad & \{(z_1, \ldots, z_{n+m}) \ : \ z_{i_p} = x_p, z_{j_q} = y_q, \\ & \text{where } \{i_1 < \cdots < i_n\} \cup \{j_1 < \cdots < j_m\} = \{1, \ldots, n+m\}\}. \end{aligned}$$

For two sets of strings X and Y, define their shuffle product by

$$X \star Y = \bigcup_{\mathbf{x} \in X, \, \mathbf{y} \in Y} \mathbf{x} \star \mathbf{y}.$$

Lemma 9.7 *For $i = 1, 2$ let P_i be a poset of rank n_i with R-labeling λ^i and linearly ordered label poset Λ^i. Without loss of generality assume that Λ^1 and Λ^2 are disjoint. Let Γ be a linear extension of the union $\Lambda^1 \cup \Lambda^2$. Then $P_1 \times P_2$ has an R-labeling γ described by*

$$\left\{ \begin{array}{lcl} \gamma((x, y), (z, y)) & = & \lambda^1(x, z), \\ \gamma((x, y), (x, w)) & = & \lambda^2(y, w). \end{array} \right.$$

Moreover, the Jordan-Hölder set $JH(P_1 \times P_2)$ is given by all shuffle products of the strings from $JH(P_1)$ and $JH(P_2)$, that is,

$$JH(P_1 \times P_2) = JH(P_1) \star JH(P_2).$$

Proof: Every maximal chain in the product $P_1 \times P_2$ comes from one maximal chain in P_1 and one maximal chain in P_2. Conversely, for each pair (c_1, c_2) of maximal chains, where c_i is a maximal chain in P_i, there are $\binom{n_1+n_2}{n_1}$ maximal chains in $P_1 \times P_2$. Moreover, the labels of these $\binom{n_1+n_2}{n_1}$ maximal chains are the shuffle product of the labels of c_1 and the labels of c_2. Hence the Jordan-Hölder set of the Cartesian product $P_1 \times P_2$ has the desired form.

Consider an interval $I = [(x_1, x_2), (y_1, y_2)]$ in the product $P_1 \times P_2$. Let m^i be the string of labels of a maximal chain in the interval $[x_i, y_i]$ in the poset P_i. If m^1 or m^2 has a descent then all the strings of labels in the shuffle product $m^1 \star m^2$ have at least one descent. Now let λ^i be the string of labels of the unique rising chain in the interval $[x_i, y_i]$. Then there is exactly one shuffle among $\lambda^1 \star \lambda^2$ that is a rising string. Hence the interval I has a unique rising chain, proving γ is an R-labeling. $\square$

Lemma 9.8 *For $i = 1, 2$ let P_i be a poset with an R-labeling λ^i and linearly ordered label poset Λ^i. Assume that each edge in the Hasse diagram between a coatom of P_i and the maximal element $\widehat{1}_{P_i}$ is labeled 0 and no other labels are equal to 0. This condition can be expressed as*

$$\lambda^i(x, y) = 0 \iff x \prec y = \widehat{1}_{P_i}.$$

Without loss of generality assume that $\Lambda^1 \cap \Lambda^2 = \{0\}$. Let Γ be a linear extension of the union $\Lambda^1 \cup \Lambda^2$. Then $P_1 \diamond^* P_2$ has an R-labeling γ described by

$$\begin{cases} \gamma((x, y), (z, y)) & = & \lambda^1(x, z), \\ \gamma((x, y), (x, w)) & = & \lambda^2(y, w), \\ \gamma((x, y), \widehat{1}_{P_1 \diamond^* P_2}) & = & 0. \end{cases}$$

Moreover, the Jordan-Hölder set is given by

$$JH(P_1 \diamond^* P_2) = (JH_0(P_1) \star JH_0(P_2)) \circ 0,$$

where $JH_0(P_i)$ is the set of all the strings in the Jordan Hölder set $JH(P_i)$ with the element 0 at the end removed.

Proof: Directly from the identity $(P_1 - \{\widehat{1}_{P_1}\}) \times (P_2 - \{\widehat{1}_{P_2}\}) = (P_1 \diamond^* P_2) - \{\widehat{1}_{P_1 \diamond^* P_2}\}$ it follows that $JH_0(P_1 \diamond^* P_2) = JH_0(P_1) \star JH_0(P_2)$, thus verifying the Jordan Hölder set of the dual diamond product is as described.

It remains to observe that γ is an R-labeling. By the same reasoning as in the proof of Lemma 9.7, each interval of the form $[(x, y), (z, w)]$ has a unique rising chain. Hence it is enough to show each interval of the form $I = [(x, y), \widehat{1}_{P_1 \diamond^* P_2}]$ has a unique rising chain.

Let $m^i \circ 0$ be the string of labels of a maximal chain in the interval $[x_i, \widehat{1}_{P_i}]$ in the poset P_i. If $m^1 \circ 0$ or $m^2 \circ 0$ has a descent then all the strings of labels in the shuffle product $(m^1 \star m^2) \circ 0$ have at least one descent. Now let $\lambda^i \circ 0$ be the string of labels of the unique rising chain in the interval $[x_i, \widehat{1}_{P_i}]$. Then there is exactly one shuffle among $(\lambda^1 \star \lambda^2) \circ 0$ that is a rising string. Hence the interval I has a unique rising chain, proving γ is an R-labeling. □

Proof of Proposition 9.5: Let the R-labeling of the poset P_i have label set Λ_i, where we assume Λ_1 and Λ_2 are disjoint. Then the Cartesian product $P_1 \times P_2$ has an R-labeling with the label set $\Lambda_1 \cup \Lambda_2$ and the Jordan-Hölder set $JH(P_1 \times P_2) = JH(P_1) \star JH(P_2)$. By Corollary 8.2, the Tchebyshev transform of the product $P_1 \times P_2$ has an R-labeling with label set $(\Lambda_1 \cup \Lambda_2)^{sb} \cup \{0\}$ and Jordan-Hölder set $(JH(P_1) \star JH(P_2))^{sb} \circ 0$.

Similarly, by Corollary 8.2 the Tchebyshev transform of the poset P_i has an R-labeling with label set $\Lambda_i^{sb} \cup \{0\}$ and Jordan-Hölder set $JH(P_i)^{sb} \circ 0$. Now by Lemma 9.8 the diamond product $T(P_1) \diamond^* T(P_2)$ has an R-labeling with label set $\Lambda_1^{sb} \cup \Lambda_2^{sb} \cup \{0\}$ and Jordan Hölder set $(JH(P_1)^{sb} \star JH(P_2)^{sb}) \circ 0$.

As sets, the two label sets agree:

$$(\Lambda_1 \cup \Lambda_2)^{sb} \cup \{0\} = \Lambda_1^{sb} \cup \Lambda_2^{sb} \cup \{0\}.$$

Additionally, as linearly ordered sets they are also equal, since we can first choose the linear extension of $\Lambda_1 \cup \Lambda_2$ to be the unique linear order where all the labels from Λ_1 is an initial segment. Moreover, choose the linear extension of $(\Lambda_1^{sb} \cup \{0\}) \cup (\Lambda_2^{sb} \cup \{0\})$ to be as $\Lambda_1^s, \Lambda_2^s, \{0\}, \Lambda_1^b, \Lambda_2^b$, in that order.

Finally, observe that the Jordan-Hölder sets of the two posets $T(P_1 \times P_2)$ and $T(P_1) \diamond^* T(P_2)$ are also equal, namely,

$$(JH(P_1) \star JH(P_2))^{sb} \circ 0 = (JH(P_1)^{sb} \star JH(P_2)^{sb}) \circ 0.$$

Hence, by Proposition 9.6, the posets have the same **ab**-index. □

10 The Tchebyshev operator of the second kind

Following Hetyei, we will now define the Tchebyshev operator of the second kind. We demonstrate it is an algebra map with respect to the mixing operator M and a coalgebra map with respect to the coproduct Δ. Moreover, we find the spectrum of this operator, generalizing work in [9].

Define the two linear maps $H, H^* : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by $H(1) = H^*(1) = 0$ and $H(\mathbf{a} \cdot u) = H(\mathbf{b} \cdot u) = H^*(u \cdot \mathbf{a}) = H^*(u \cdot \mathbf{b}) = u$. The map H appears in [7]. We have the following result from the same reference.

Lemma 10.1 *A poset P of rank at least 2 satisfies*

$$\begin{aligned} H(\Psi(P)) &= \sum_{a} \Psi([a, \widehat{1}]), \\ H^*(\Psi(P)) &= \sum_{c} \Psi([\widehat{0}, c]), \end{aligned}$$

where the first sum ranges over all atoms a of the poset P and the second sum ranges over all coatoms c of the poset P.

Observe both H and H^* restrict to $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ by $H(\mathbf{c} \cdot u) = H^*(u \cdot \mathbf{c}) = 2u$, $H(\mathbf{d} \cdot u) = \mathbf{c} \cdot u$ and $H^*(u \cdot \mathbf{d}) = u \cdot \mathbf{c}$.

Definition 10.2 *The* Tchebyshev transform of the second kind *is the linear map* $U : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ *defined by*

$$U(u) = H^*(T(u \cdot \mathbf{a})).$$

The explanation for this name is given by the next corollary. This result is originally due to Hetyei.

Corollary 10.3 *Substituting* $\mathbf{c}$ *to be* x *and* $\mathbf{d}$ *to be* $(x^2 - 1)/2$ *in* $1/2 \cdot U(\mathbf{c}^n)$ *yields the Tchebyshev polynomial of the second kind* $U_n(x)$*.*

Proof: First, under the substitution the expressions $1/2 \cdot U(1) = 1$ and $1/2 \cdot U(\mathbf{c}) = 2\mathbf{c}$ become $U_0(x) = 1$ and $U_1(x) = 2x$. Second, the recursion in Corollary 7.5 transforms into $U(\mathbf{c}^2 \cdot u) = 2\mathbf{c} \cdot U(\mathbf{c} \cdot u) + (2\mathbf{d} - \mathbf{c}^2) \cdot U(u)$. Under the given substitution this becomes the recursion for the Tchebyshev polynomials of the second kind. $\square$

Proposition 10.4 *The Tchebyshev transform of the second kind has the following expression:*

$$U(u) = \sum_{k \geq 1} \sum_{u} \nu(u_{(1)}) \cdot \mathbf{b} \cdot \nu(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \nu(u_{(k)}).$$

Proof: By applying the definition of the Tchebyshev transform appearing in equation (5.1), we have

$$\begin{aligned} T(u \cdot \mathbf{a}) \quad = \quad & \sum_{k \geq 1} \sum_{u} \nu(u_{(1)}) \cdot \mathbf{b} \cdot \nu(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \nu(u_{(k-1)}) \cdot \mathbf{b} \cdot \kappa(u_{(k)} \cdot \mathbf{a}) \\ & + \sum_{k \geq 1} \sum_{u} \nu(u_{(1)}) \cdot \mathbf{b} \cdot \nu(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \nu(u_{(k)}) \cdot \mathbf{b} \cdot \kappa(1). \end{aligned}$$

Since $\kappa(u_{(k)} \cdot \mathbf{a}) = \kappa(u_{(k)}) \cdot (\mathbf{a} - \mathbf{b})$ the first sum is equal to $T(u) \cdot (\mathbf{a} - \mathbf{b})$. Now apply H^* and by noticing that $H^*(T(u) \cdot (\mathbf{a} - \mathbf{b})) = 0$, the result follows. $\square$

Corollary 10.5 *The Tchebyshev transform of the second kind is invariant under duality, that is,* $U(u^*) = U(u)^*$*.*

Theorem 10.6 *The Tchebyshev transform of the second kind is a coalgebra homomorphism, that is,*

$$\Delta(U(u)) = \sum_u U(u_{(1)}) \otimes U(u_{(2)}).$$

Proof: Recall that $\Delta(\nu(u)) = 0$. By applying Proposition 10.4, we obtain

$$\begin{aligned}
\Delta(U(u)) &= \sum_{k\geq 1} \sum_u \sum_{i=1}^{k-1} \nu(u_{(1)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \nu(u_{(i)}) \otimes \nu(u_{(i+1)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \nu(u_{(k)}) \\
&= \sum_{i,j\geq 1} \sum_u \sum_{u_{(1)}} \sum_{u_{(2)}} \nu(u_{(1,1)}) \cdot \mathbf{b} \cdot \cdots \nu(u_{(1,i)}) \otimes \nu(u_{(2,1)}) \cdot \mathbf{b} \cdot \cdots \nu(u_{(2,j)}) \\
&= \sum_u U(u_{(1)}) \otimes U(u_{(2)}).
\end{aligned}$$

□

Theorem 10.7 *The Tchebyshev transform of the second kind is an algebra map under the product M. In other words, the following identity holds for any two* **ab***-polynomials u and v:*

$$U(M(u, v)) = M(U(u), U(v)). \tag{10.1}$$

Proof: By Lemma 2.3 in [16], we have

$$H^*(N^*(u, v)) = M(H^*(u), H^*(v)).$$

Applying H^* to Theorem 9.3, we obtain

$$\begin{aligned}
U(M(u, v)) &= H^*(T(M(u, v) \cdot \mathbf{a})) \\
&= H^*(N^*(T(u \cdot \mathbf{a}), T(v \cdot \mathbf{a}))) \\
&= M(H^*(T(u \cdot \mathbf{a})), H^*(T(v \cdot \mathbf{a}))) \\
&= M(U(u), U(v)).
\end{aligned}$$

□

Proposition 10.8 *Assume u_i is an eigenvector with eigenvalue λ_i of the Tchebyshev transform of the second kind U for $i = 1, 2$. Then $M(u_1, u_2)$ is an eigenvector with eigenvalue $\lambda_1 \cdot \lambda_2$.*

Proof: Directly $U(M(u_1, u_2)) = M(U(u_1), U(u_2)) = M(\lambda_1 \cdot u_1, \lambda_2 \cdot u_2) = \lambda_1 \cdot \lambda_2 \cdot M(u_1, u_2)$. □

Proposition 10.9 *Assume u is an eigenvector with eigenvalue λ of the Tchebyshev transform of the second kind U. Then $(\mathbf{a} - \mathbf{b}) \cdot u$ is an eigenvector with eigenvalue λ.*

Proof: Observe that $U((\mathbf{a}-\mathbf{b})\cdot u) = H^*(T((\mathbf{a}-\mathbf{b})\cdot u\cdot \mathbf{a})) = (\mathbf{a}-\mathbf{b})\cdot H^*(T(u\cdot\mathbf{a})) = (\mathbf{a}-\mathbf{b})\cdot U(u)$, where the second step is by Corollary 7.2. □

Let $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle_n$ denote the set of all homogeneous **ab**-polynomials of degree n with coefficients in the field $\mathbf{k}$. Hence the dimension of $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle_n$ is 2^n and U_n is an endomorphism on $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle_n$.

Theorem 10.10 *Let U_n denote the restriction of U to* **ab***-polynomials of degree n, that is, $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle_n$. Then the linear operator U_n is diagonalizable and has the eigenvalue 2^{i+1} of multiplicity $\binom{n}{i}$ for $0 \leq i \leq n$. Furthermore, a complete set of eigenvectors can be obtained by starting with 1 and repeatedly iterating the two operations*

$$\begin{aligned} u &\longmapsto \mathrm{Pyr}(u) = M(u,1), \\ u &\longmapsto L(u) = (\mathbf{a}-\mathbf{b})\cdot u, \end{aligned}$$

n times.

Proof: Observe that 1 is an eigenvector with eigenvalue 2. By iterating Propositions 10.8 and 10.9 n times, we obtain 2^n eigenvectors of degree n. By Proposition 3.4 in [9] we know

$$\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle_{n+1} = \mathrm{Pyr}(\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle_n) \oplus L(\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle_n).$$

Hence this set of eigenvectors is a complete set of eigenvectors, that is, there are no linear dependencies among them.

Also since the pyramid operation Pyr multiplies an eigenvalue by 2 and the second operation L preserves the eigenvalue, we may conclude the distribution of the eigenvalues of U_n is precisely the binomial distribution. □

11 A Hopf-algebra endomorphism on quasisymmetric functions

The main result of this section is prove the Tchebyshev transform of the second kind is a Hopf algebra endomorphism.

Define the map U on a quasisymmetric function f (where we intentionally use the same symbol as the Tchebyshev transform of the second kind) by

$$U(f) = \gamma(U(\gamma^{-1}(f))),$$

where $f \in \mathrm{QSym}$ does not have a constant term. Extend linearly to all quasisymmetric functions by setting $U(1_{\mathrm{QSym}}) = 1_{\mathrm{QSym}}$.

Theorems 10.7 and 10.6 imply the following result.

Theorem 11.1 *The map U is a Hopf algebra endomorphism on the Hopf algebra of quasisymmetric functions.*

Sketch of proof: We leave it to the reader to verify that U behaves well with the unit and the counit of quasisymmetric functions. Since the mixing operator M on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ corresponds to the Cartesian product on graded posets (equation (9.1)) and the Cartesian product corresponds to the product of quasisymmetric functions (equation (3.2)), Theorem 10.7 implies the map U is algebra endomorphism on the quasisymmetric functions.

Now for a quasisymmetric polynomial $f = \gamma(v)$, we have

$$\begin{aligned}\Delta^{\mathrm{QSym}}(U(f)) &= \Delta^{\mathrm{QSym}}(\gamma(U(v)))\\ &= \gamma(U(v)) \otimes 1_{\mathrm{QSym}} + 1_{\mathrm{QSym}} \otimes \gamma(U(v)) + \sum_{v} \gamma(U(v_{(1)})) \otimes \gamma(U(v_{(2)}))\\ &= U(f) \otimes 1_{\mathrm{QSym}} + 1_{\mathrm{QSym}} \otimes U(f) + \sum_{v} U(\gamma(v_{(1)})) \otimes U(\gamma(v_{(2)}))\\ &= (U \otimes U) \circ \Delta^{\mathrm{QSym}}(f),\end{aligned}$$

where the second step is that U is a coalgebra endomorphism on $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$. $\square$

12 Chain maps of the first and second kind

The results in Sections 5, 10 and 11 motivate us to consider two general classes of maps. In this section, we show one such class, the chain map of the second kind, is a Hopf algebra endomorphism of quasisymmetric functions.

Definition 12.1 *A character G on $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle$ is a functional $G : \mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle \longrightarrow \mathbf{k}$ which is multiplicative with respect to Cartesian product of posets, that is,*

$$G(\Psi(P \times Q)) = G(\Psi(P)) \cdot G(\Psi(Q)),$$

for all posets P and Q of rank greater than or equal to 1.

Theorem 11.1 can be extended in the following manner. Let G be a character on $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle$. Define the functions $\widehat{g}$, $\widetilde{g}$ and g on $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle$ by

$$\begin{aligned}\widehat{g}(u) &= G(u) \cdot (\mathbf{a}-\mathbf{b})^{\deg(u)},\\ \widetilde{g}(u) &= \sum_{k\geq 1} \sum_{u} \widehat{g}(u_{(1)}) \cdot \mathbf{b} \cdot \widehat{g}(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \widehat{g}(u_{(k)}),\\ g(u) &= \sum_{k\geq 1} \sum_{u} \kappa(u_{(1)}) \cdot \mathbf{b} \cdot \widehat{g}(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \widehat{g}(u_{(k)}).\end{aligned}$$

We call the maps g and $\tilde{g}$, respectively, *the chain maps of the first and second kind.*

Examples 12.2 (i) If G satisfies $G(\Psi(P)) = 1$ for all posets P, then $\widehat{g} = \kappa$ and the two maps g and $\widetilde{g}$ are both equal to the identity map.

(ii) $G(\Psi(P)) = \sum_{x \in P} (-1)^{\rho(x)} \cdot \mu(\widehat{0}, x)$. Then g is the φ map of oriented matroids (see equation (6.1)) and $\widetilde{g}$ is the Stembridge ϑ map.

(iii) An extension of the previous example is to take $G(\Psi(P)) = \sum_{x \in P} (1 - r)^{\rho(x)} \cdot \mu(\widehat{0}, x)$. In this case, g corresponds to φ_r of the r-signed Birkhoff transform and $\widetilde{g}$ is the r-signed analogue of the Stembridge map, ϑ_r, see [16].

(iv) $G(\Psi(P))$ is the cardinality of the poset P. In this case we have $g(u^*)^*$ is the Tchebyshev transform of the first kind and $\widetilde{g}(u^*)^*$ is the Tchebyshev transform of the second kind.

Proposition 12.3 *The following relations hold between the functions $\widetilde{g}$ and g:*

$$\begin{aligned} \Delta(\widetilde{g}(u)) &= \sum_u \widetilde{g}(u_{(1)}) \otimes \widetilde{g}(u_{(2)}), && (12.1) \\ \Delta(g(u)) &= \sum_u g(u_{(1)}) \otimes \widetilde{g}(u_{(2)}), && (12.2) \\ g(\mathbf{a} \cdot u) &= (\mathbf{a} - \mathbf{b}) \cdot g(u) + \mathbf{b} \cdot \widetilde{g}(u), && (12.3) \\ g(\mathbf{b} \cdot u) &= \mathbf{b} \cdot \widetilde{g}(u). && (12.4) \end{aligned}$$

Proof: The proof that $\widetilde{g}$ is a coalgebra endomorphism follows exactly along the same lines as the proofs of Theorems 10.6 and 11.1. The same proof idea also establishes equation (12.2). Identity (12.3) follows from

$$\begin{aligned} g(\mathbf{a} \cdot u) &= \sum_{k \geq 1} \sum_u \kappa(\mathbf{a} \cdot u_{(1)}) \cdot \mathbf{b} \cdot \widehat{g}(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \widehat{g}(u_{(k)}) \\ &\quad + \sum_{k \geq 1} \sum_u \kappa(1) \cdot \mathbf{b} \cdot \widehat{g}(u_{(1)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \widehat{g}(u_{(k)}) \\ &= (\mathbf{a} - \mathbf{b}) \cdot g(u) + \mathbf{b} \cdot \widetilde{g}(u). \end{aligned}$$

Identity (12.4) follows in a similar manner. □

Extend the composition $G \circ \Psi$ by letting $G(\Psi(\bullet)) = 1$, where $\bullet$ denotes the one element poset.

Proposition 12.4 *The chain map of the second kind $\widetilde{g}$ has the following form when applied to the* **ab***-index, respectively, the quasisymmetric function of a poset P:*

$$\begin{aligned} \widetilde{g}(\Psi(P)) &= \sum_c G(\Psi([x_0, x_1])) \cdots G(\Psi([x_{k-1}, x_k])) \cdot \mathrm{wt}(c), && (12.5) \\ \widetilde{g}(F(P)) &= \sum_c G(\Psi([x_0, x_1])) \cdots G(\Psi([x_{k-1}, x_k])) \cdot M_{(\rho(x_0,x_1), \ldots, \rho(x_{k-1},x_k))}, && (12.6) \\ \widetilde{g}(F(P)) &= \lim_{j \longrightarrow \infty} \sum_m G(\Psi([x_0, x_1])) \cdots G(\Psi([x_{j-1}, x_j])) \cdot t_1^{\rho(x_0,x_1)} \cdots t_j^{\rho(x_{j-1},x_j)}, && (12.7) \end{aligned}$$

where the first two sums are over all chains $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$ in the poset P and the third sum is over all multichains $m = \{\widehat{0} = x_0 \leq x_1 \leq \cdots \leq x_j = \widehat{1}\}$ in P.

Proof: The first identity follows by using the definition of $\widetilde{g}$ and the fact that the **ab**-index is a coalgebra homomorphism. The second identity follows from the first by applying the map γ.

To prove the third identity, let $t_{j+1} = t_{j+2} = \cdots = 0$ in identity (12.6). This restricts the sum to chains having at most j steps, that is, $k \leq j$. Such chains can be expressed in terms of multichains with j steps. By the definition of the monomial quasisymmetric function, we then have

$$\begin{aligned} & \widetilde{g}(F(P))|_{t_{j+1}=t_{j+2}=\cdots=0} \\ = & \sum_{\widehat{0}=x_0 \leq x_1 \leq \cdots \leq x_j = \widehat{1}} G(\Psi([x_0, x_1])) \cdots G(\Psi([x_{j-1}, x_j])) \cdot t_1^{\rho(x_0,x_1)} \cdots t_j^{\rho(x_{j-1},x_j)}. \end{aligned}$$

Letting j tend to infinity yields the desired identity. $\square$

Observe in the proofs of Propositions 12.3 and 12.4 we only used the fact that G is functional on $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$, not that G is a character on $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$.

Theorem 12.5 *Let G be a character on $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$. The chain map of the second kind $\widetilde{g}$ is an algebra map under the product M. In other words, the following identity holds for any **ab**-polynomials u and v:*

$$\widetilde{g}(M(u, v)) \quad = \quad M(\widetilde{g}(u), \widetilde{g}(v)).$$

Equivalently, the chain map of the second kind $\widetilde{g}$ is an algebra map on quasisymmetric functions QSym*. That is, for any quasisymmetric functions f_1 and f_2 the following identity holds:*

$$\widetilde{g}(f_1 \cdot f_2) \quad = \quad \widetilde{g}(f_1) \cdot \widetilde{g}(f_2).$$

Proof: A multichain of length m in the Cartesian product $P \times Q$ corresponds to two multichains of length m, with one coming from the poset P and the other from the poset Q. By applying equation (12.7) three times, we have

$$\begin{aligned} & \widetilde{g}(F(P \times Q)) \\ = & \lim_{m \longrightarrow \infty} \sum_{\widehat{0}=(x_0,y_0) \leq (x_1,y_1) \leq \cdots \leq (x_m,y_m) = \widehat{1}} \cdots G(\Psi([(x_{i-1}, y_{i-1}), (x_i, y_i)])) \cdots t_i^{\rho((x_{i-1},y_{i-1}),(x_i,y_i))} \cdots \\ = & \lim_{m \longrightarrow \infty} \left(\sum_{\widehat{0}=x_0 \leq x_1 \leq \cdots \leq x_m = \widehat{1}} \cdots G(\Psi([x_{i-1}, x_i])) \cdots t_i^{\rho(x_{i-1},x_i)} \cdots \right) \\ & \qquad \cdot \left(\sum_{\widehat{0}=y_0 \leq y_1 \leq \cdots \leq y_m = \widehat{1}} \cdots G(\Psi([y_{i-1}, y_i])) \cdots t_i^{\rho(y_{i-1},y_i)} \cdots \right) \\ = & \widetilde{g}(F(P)) \cdot \widetilde{g}(F(Q)), \end{aligned}$$

where we only write the generic factor in each term. $\square$

Combining equation (12.1) in Proposition 12.3 with Theorem 12.5, we obtain:

Theorem 12.6 *Let G be a character on $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle$. Then the associated function $\widetilde{g}$ is a Hopf algebra endomorphism on the quasisymmetric functions* QSym*.*

This theorem is a special case of a more general theorem due to Aguiar, Bergeron and Sottile [1]. They proved that in the category of combinatorial Hopf algebras the quasisymmetric functions QSym is a terminal object. A combinatorial Hopf algebra is a Hopf algebra H together with a character G. Their results then state that given a combinatorial Hopf algebra H with character G, there exists a Hopf algebra homomorphism $\psi : H \longrightarrow \mathrm{QSym}$ such that $G = \zeta \circ \psi$, where ζ is the character on QSym defined by $\zeta(f) = A(\gamma^{-1}(f))$ and $\zeta(1_{\mathrm{QSym}}) = 1$.

13 Type B quasisymmetric functions

We now turn our attention to the chain map of the first kind. In this section we will assume the underlying map G is multiplicative with respect to the Cartesian product of posets. The purpose of this section is to prove the chain map of the first kind g is an algebra map under the product N, and moreover, to prove g is a comodule map.

Theorem 13.1 *The chain map of the first kind g is an algebra map under the product N. That is, for any* **ab***-polynomials u and v the following identity holds:*

$$g(N(u,v)) \quad = \quad N(g(u), g(v)).$$

By observing $N(\mathbf{a}\cdot u, \mathbf{a}\cdot v) = \mathbf{a}\cdot M(u,v)$ (see Proposition 7.8 in [18]), we have the corollary:

Corollary 13.2 *For all* **ab***-polynomials u and v the following identity holds:*

$$g(\mathbf{a}\cdot M(u,v)) \quad = \quad N(g(\mathbf{a}\cdot u), g(\mathbf{a}\cdot v)).$$

This corollary implies Theorem 12.5 by applying the H map.

In order to prove Theorem 13.1, we introduce the type B quasisymmetric functions due to Chow [13]. Let BQSym denote the algebra $\mathbf{k}[s] \otimes \mathrm{QSym}$. We view BQSym as a subalgebra of $\mathbf{k}[s, t_1, t_2, \ldots] \cong \mathbf{k}[s] \otimes \mathbf{k}[t_1, t_2, \ldots]$.

Define the *type B quasisymmetric function* of a poset P by

$$\begin{aligned} F_B(P) &= \sum_{\widehat{0}<x\leq\widehat{1}} s^{\rho(x)-1}\cdot F([x,\widehat{1}]) \\ &= \lim_{m\longrightarrow\infty} \sum_{\widehat{0}<x_0\leq x_1\leq\cdots\leq x_m=\widehat{1}} s^{\rho(\widehat{0},x_0)-1}\cdot t_1^{\rho(x_0,x_1)}\cdot t_2^{\rho(x_1,x_2)}\cdots t_m^{\rho(x_{m-1},x_m)}. \end{aligned}$$

Theorem 13.3 *The type B quasisymmetric function of a poset is an algebra map taking the diamond product on posets into the product of type B quasisymmetric functions* BQSym*. That is, two graded posets P and Q satisfy*

$$F_B(P\diamond Q) = F_B(P)\cdot F_B(Q).$$

Proof: Applying the definition of F_B to the diamond product $P\diamond Q$ yields

$$\begin{aligned} F_B(P\diamond Q) &= \sum_{\widehat{0}<(x,y)\leq\widehat{1}_{P\diamond Q}} s^{\rho_{P\diamond Q}((x,y))-1}\cdot F([(x,y),\widehat{1}_{P\diamond Q}]) \\ &= \left(\sum_{\widehat{0}<x\leq\widehat{1}_P} s^{\rho_P(x)-1}\cdot F([x,\widehat{1}_P])\right)\cdot\left(\sum_{\widehat{0}<y\leq\widehat{1}_Q} s^{\rho_Q(y)-1}\cdot F([y,\widehat{1}_Q])\right) \\ &= F_B(P)\cdot F_B(Q). \end{aligned}$$

Here we are using $\rho_{P\diamond Q}((x,y)) = \rho_P(x)+\rho_Q(y)-1$ and that the quasisymmetric function F is multiplicative on posets. $\square$

Let γ_B be the isomorphism between $\mathbf{k}\langle\mathbf{a},\mathbf{b}\rangle$ and BQSym defined by

$$\gamma_B\left((\mathbf{a}-\mathbf{b})^p\cdot\mathbf{b}\cdot(\mathbf{a}-\mathbf{b})^{p_1-1}\cdot\mathbf{b}\cdots\mathbf{b}\cdot(\mathbf{a}-\mathbf{b})^{p_k-1}\right) = s^p\cdot M_{(p_1,\ldots,p_k)},$$

where $p\geq 0$ and $p_1,\ldots,p_k\geq 1$, that is, $\gamma_B(\Psi(P)) = F_B(P)$. Define the linear map g on BQSym by $g(f) = \gamma_B(g(\gamma_B^{-1}(f)))$. Hence Theorem 13.3 states

$$\gamma_B(N(u,v)) = \gamma_B(u)\cdot\gamma_B(v). \tag{13.1}$$

Proposition 13.4 *The chain map of the first kind g has the following form when applied to the* **ab***-index, respectively, the type B quasisymmetric function of a poset P:*

$$g(\Psi(P)) = \sum_c G(\Psi([x_0,x_1]))\cdots G(\Psi([x_{k-1},x_k]))\cdot\mathrm{wt}(c), \tag{13.2}$$

$$g(F_B(P)) = \sum_c G(\Psi([x_0,x_1]))\cdots G(\Psi([x_{k-1},x_k]))\cdot s^{\rho(\widehat{0},x_0)-1}\cdot M_{(\rho(x_0,x_1),\ldots,\rho(x_{k-1},x_k))}, \tag{13.3}$$

$$g(F_B(P)) = \sum_{\widehat{0}<x\leq\widehat{1}} s^{\rho(x)-1}\cdot\widetilde{g}(F([x,\widehat{1}])), \tag{13.4}$$

$$\begin{aligned} g(F_B(P)) = \lim_{j\longrightarrow\infty}\sum_m G(\Psi([x_0,x_1]))\cdots G(\Psi([x_{j-1},x_j])) \\ \cdot s^{\rho(x_0)-1}\cdot t_1^{\rho(x_0,x_1)}\cdots t_j^{\rho(x_{j-1},x_j)}, \end{aligned} \tag{13.5}$$

where the first two sums are over all chains $c = \{\widehat{0} = x_0 < x_1 < \cdots < x_k = \widehat{1}\}$ in the poset P and the fourth sum is over all chains satisfying $m = \{\widehat{0} < x_0 \leq x_1 \leq \cdots \leq x_j = \widehat{1}\}$ in P.

Proof: By the definition of g and $\widetilde{g}$, we have

$$g(u) \quad = \quad \kappa(u) + \sum_u \kappa(u_{(1)}) \cdot \mathbf{b} \cdot \widetilde{g}(u_{(2)}).$$

Apply this identity to the **ab**-index of a poset P and use equation (12.5) to expand the factor $\widetilde{g}(\Psi([x, \widehat{1}]))$. We then obtain

$$\begin{aligned} g(\Psi(P)) &= (\mathbf{a}-\mathbf{b})^{\rho(P)-1} + \sum_{\widehat{0}<x<\widehat{1}} (\mathbf{a}-\mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \widetilde{g}(\Psi([x,\widehat{1}])) \\ &= (\mathbf{a}-\mathbf{b})^{\rho(P)-1} + \sum_{\widehat{0}<x<\widehat{1}} \sum_{k\geq 1} \sum_{x=x_0<x_1<\cdots<x_k=\widehat{1}} G(\Psi([x_0,x_1]))\cdots G(\Psi([x_{k-1},x_k])) \\ &\qquad \cdot (\mathbf{a}-\mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x_0,x_1)-1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(x_{k-1},x_k)-1}, \end{aligned}$$

which is equivalent to the first identity of the proposition.

To prove the second identity, apply the isomorphism γ_B to the first identity. The third identity follows from the second and equation (12.6). Similarly, the fourth identity follows by the third and equation (12.7) $\square$

As a remark, we did not use the fact that G is a character on $\mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ in the proof of Proposition 13.4.

Theorem 13.5 *The linear map g is an algebra homomorphism on the type B quasisymmetric functions* BQSym*.*

Proof: By equation (13.4) we have

$$\begin{aligned} g(F_B(P \diamond Q)) &= \sum_{\widehat{0}<(x,y)\leq \widehat{1}_{P\diamond Q}} s^{\rho_{P\diamond Q}((x,y))-1} \cdot \widetilde{g}(F([(x,y), \widehat{1}_{P\diamond Q}])) \\ &= \left(\sum_{\widehat{0}<x\leq\widehat{1}_P} s^{\rho_P(x)-1} \cdot \widetilde{g}(F([x,\widehat{1}_P])) \right) \cdot \left(\sum_{\widehat{0}<y\leq\widehat{1}_Q} s^{\rho_Q(y)-1} \cdot \widetilde{g}(F([y,\widehat{1}_Q])) \right) \\ &= g(F_B(P)) \cdot g(F_B(Q)). \end{aligned}$$

Since the type B quasisymmetric function of posets spans the space BQSym, the result follows. $\square$

We remark that Theorems 13.1 and 13.5 are equivalent via the isomorphism γ_B.

Define the coproduct $\Delta^{\mathrm{BQSym}} : \mathrm{BQSym} \longrightarrow \mathrm{BQSym} \otimes \mathrm{QSym}$ by

$$\Delta^{\mathrm{BQSym}}(s^p \cdot f) = \sum_f^{\mathrm{QSym}} s^p \cdot f_{(1)} \otimes f_{(2)},$$

where $f \in \mathrm{QSym}$. We then have the following result.

Theorem 13.6 *The type B quasisymmetric function of a poset is a comodule map taking the comodule structure of posets into the coproduct Δ^{BQSym} on type B quasisymmetric functions* BQSym*. More specifically, a graded poset P satisfies the following identity:*

$$\Delta^{\mathrm{BQSym}}(F_B(P)) = \sum_{\widehat{0}<x\leq\widehat{1}} F_B([\widehat{0},x]) \otimes F([x,\widehat{1}]).$$

Proof: By applying the definition of F_B, we obtain

$$\begin{aligned}
\Delta^{\mathrm{BQSym}}(F_B(P)) &= \Delta^{\mathrm{BQSym}}\left(\sum_{\widehat{0}<y\leq\widehat{1}} s^{\rho(y)-1} \cdot F([y,\widehat{1}])\right) \\
&= \sum_{\widehat{0}<y\leq\widehat{1}} \sum_{y\leq x\leq\widehat{1}} s^{\rho(y)-1} \cdot F([y,x]) \otimes F([x,\widehat{1}]) \\
&= \sum_{\widehat{0}<x\leq\widehat{1}} \left(\sum_{0<y\leq x} s^{\rho(y)-1} \cdot F([y,x])\right) \otimes F([x,\widehat{1}]) \\
&= \sum_{\widehat{0}<x\leq\widehat{1}} F_B([\widehat{0},x]) \otimes F([x,\widehat{1}]).
\end{aligned}$$

□

Theorem 13.7 *The linear map g is a comodule endomorphism on* BQSym*, that is,*

$$\Delta^{\mathrm{BQSym}} \circ g = (g \otimes \widetilde{g}) \circ \Delta^{\mathrm{BQSym}}.$$

Proof: By applying equation (13.4) twice and Theorem 13.6, we obtain

$$\begin{aligned}
\Delta^{\mathrm{BQSym}}(g(F_B(P))) &= \Delta^{\mathrm{BQSym}}\left(\sum_{\widehat{0}<x\leq\widehat{1}} s^{\rho(x)-1} \cdot \widetilde{g}(F([x,\widehat{1}]))\right) \\
&= \sum_{\widehat{0}<x\leq\widehat{1}} \sum_{x\leq y\leq\widehat{1}} s^{\rho(x)-1} \cdot \widetilde{g}(F([x,y])) \otimes \widetilde{g}(F([y,\widehat{1}])) \\
&= \sum_{\widehat{0}<y\leq\widehat{1}} \left(\sum_{\widehat{0}<x\leq y} s^{\rho(x)-1} \cdot \widetilde{g}(F([x,y]))\right) \otimes \widetilde{g}(F([y,\widehat{1}])) \\
&= \sum_{\widehat{0}<y\leq\widehat{1}} g(F_B([\widehat{0},y])) \otimes \widetilde{g}(F([y,\widehat{1}])) \\
&= (g \otimes \widetilde{g})\left(\sum_{\widehat{0}<y\leq\widehat{1}} F_B([\widehat{0},y]) \otimes F([y,\widehat{1}])\right) \\
&= (g \otimes \widetilde{g})\left(\Delta^{\mathrm{BQSym}}(F_B(P))\right).
\end{aligned}$$

The result follows since the type B quasisymmetric functions $F_B(P)$ span BQSym as P ranges over all posets. $\Box$

14 Concluding remarks

Recall the following theorem of Hetyei [23, Theorem 1.10].

Theorem 14.1 *The order complex of $T(P) - \{\widehat{0}, \widehat{1}\}$ triangulates the suspension of the order complex of $P - \{\widehat{0}, \widehat{1}\}$.*

Corollary 14.2 *If P is the face poset of a spherical CW-complex then the Tchebyshev transform $T(P)$ is also the face poset of a spherical CW-complex.*

A natural conjecture to make is the following.

Conjecture 14.3 *If P is a spherical and shellable poset then the Tchebyshev transform $T(P)$ is also a shellable poset.*

See the related conjecture [23, Conjecture A.2].

A Gorenstein* poset is an Eulerian poset which is Cohen-Macaulay. Furthermore, a Gorenstein* lattice is a Gorenstein* poset that also is a lattice. A recent result about Gorenstein* lattices is that their **cd**-index are coefficientwise minimized by the Boolean algebra. This was conjectured by Stanley [26] and recently proved by Ehrenborg and Karu [20]. In the special case of face lattices of convex polytopes this was settled earlier by Billera and Ehrenborg [6].

Hetyei has proved that the Tchebyshev transform will give a host of new examples of Gorenstein* posets.

Theorem 14.4 *For a graded poset P if P is Gorenstein then the Tchebyshev transform $T(P)$ is Gorenstein. Especially, for an Eulerian poset P we have if P is Gorenstein* then the Tchebyshev transform $T(P)$ is Gorenstein*.*

Proof: It is enough to prove the first statement. Recall the equivalences (a) and (c) of Theorem 5.1 in Chapter II in [28]: A triangulation of a topological space X is Gorenstein if and only if the homology of X and all the local homology (that is, the relative homology $H(X, X - p)$ for $p \in X$) is 0 except for being one-dimensional in the top dimension. Thus Gorenstein is a topological condition. Since the order complex of Tchebyshev transform $T(P)$ triangulates the suspension of the order complex of P and that the Gorenstein property is preserved under suspension, the statement holds. $\Box$

One could also ask to study the behavior of a Cohen-Macaulay poset P under the Tchebyshev transform T. For example, can a system of parameters and a basis for $T(P)$ be determined from the original system of parameters and basis of P.

Since the classical quasisymmetric functions correspond to the symmetric group, that is, the Weyl group of type A, the following two questions are natural. Are there analogues of the quasisymmetric functions for other Weyl groups other than type A and B? Similarly, are there analogues of the two maps F and F_B on posets for the other Weyl groups?

For instance, one can introduce another extension of the quasisymmetric function of a poset. Namely, define

$$F'(P) = \sum_{\widehat{0} < x \leq y < \widehat{1}} s^{\rho(\widehat{0},x)-1} \cdot F([x,y]) \cdot u^{\rho(y,\widehat{1})-1}.$$

This poset invariant is multiplicative with respect to the product $(P - \{\widehat{0},\widehat{1}\}) \times (Q - \{\widehat{0},\widehat{1}\}) \cup \{\widehat{0},\widehat{1}\}$ and has a bi-comodule structure. Also, it behaves nicely with the map defined by

$$g'(u) = \sum_{k \geq 2} \sum_{u} \kappa(u_{(1)}) \cdot \mathbf{b} \cdot \widehat{g}(u_{(2)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \widehat{g}(u_{(k-1)}) \cdot \mathbf{b} \cdot \kappa(u_{(k)}).$$

The essential question to answer is if these maps naturally appear in geometry or combinatorics. Also, is this a $\widetilde{B}$ analogue of the quasisymmetric function of a poset?

Is there a notion of a type B combinatorial Hopf algebra? Moreover, is there an Aguiar, Bergeron and Sottile type theorem, that is, that the pair $(\mathrm{QSym}, \mathrm{BQSym})$ is the terminal object in this category? These two question also extend to the other Weyl groups.

Another result due to Aguiar, Bergeron and Sottile [1] is that every character G of a Hopf algebra factors into an even character G_+ and an odd character G_-. In a recent paper Aguiar and Hsiao [2] described this factorization explicitly for the character ζ. The character ζ is the character underlying Example 12.2 (i). Are there similar explicit factorizations into even and odd characters for Examples 12.2 (ii) through (iv)?

We end with three open questions about the chain maps g and $\tilde{g}$. Find other examples of poset transformations so that the resulting linear transformation on the **ab**-index has the form of g or $\widetilde{g}$. Find the general theorem which determines the spectrum of the maps g and $\widetilde{g}$. Alternatively, find subclasses of multiplicative maps where this is possible. Recall that for the Tchebyshev transform of the second kind U we were able to do this.

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