BALANCED AND BRUHAT GRAPHS

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Abstract. We generalize chain enumeration in graded partially ordered sets by relaxing the graded, poset and Eulerian requirements. The resulting balanced digraphs, which include the classical Eulerian posets having an R-labeling, imply the existence of the (non-homogeneous) **cd**-index, a key invariant for studying inequalities for the flag vector of polytopes. Mirroring Alexander duality for Eulerian posets, we show an analogue of Alexander duality for bounded balanced digraphs. For Bruhat graphs of Coxeter groups, an important family of balanced graphs, our theory gives elementary proofs of the existence of the complete **cd**-index and its properties. We also introduce the rising and falling quasisymmetric functions of a labeled acyclic digraph and show they are Hopf algebra homomorphisms mapping balanced digraphs to the Stembridge peak algebra. We conjecture non-negativity of the **cd**-index for acyclic digraphs having a balanced linear edge labeling.

1. Introduction

The **cd**-index is an important invariant for studying face incidence data of polytopes, and more generally, chain enumeration of Eulerian posets. It is a non-commutative polynomial which removes all the linear redundancies which hold among the flag vector entries [4] as described by the generalized Dehn–Sommerville relations [1]. Ehrenborg and Readdy's discovery of the inherent coalgebraic structure of the **cd**-index and the techniques developed in [29] have been applied to settle many fundamental problems, including giving compact proofs of old results [1, 10], transparent techniques to compute flag vectors of oriented matroids [9], explicit formulas for the toric h-vector, versions of Stanley's Gorenstein* conjecture [8, 10] leading up to a proof of the conjecture itself [26], new nontrivial inequalities among the face incidence data of polytopes [22, 23] and extending classical subspace arrangement results to other manifolds [24, 34].

There are two new developments in this area. The first is work of Ehrenborg, Goresky and Readdy, who extend flag vector enumeration ideas to Whitney stratified spaces and quasi-graded posets [24, 32]. The very notion of enumeration is replaced with the topologically meaningful Euler-enumeration in the case of Whitney stratified-spaces, and weighted zeta functions in the case of quasi-graded posets. The Eulerian condition becomes a natural condition involving the Euler characteristic and weighted zeta function, respectively. Unlike the case of polytopes and regular decompositions of spheres, the coefficients of the **cd**-index can be negative, expanding the nature of questions in the field.

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The second development is Billera and Brenti's work on the "complete" **cd**-index, a nonhomogeneous extension of the **cd**-index [7]. It is known that the (strong) Bruhat order on a Coxeter group forms an Eulerian poset [53], hence any interval has a **cd**-index. Using quasisymmetric function theory, they prove the Bruhat graph, a directed graph which includes the cover relations of the Bruhat order as well as "algebraic shortcuts" between elements, has a *non-homogeneous* **cd**-index. Furthermore, they show one can compute the Kazhdan–Lusztig polynomials via this complete **cd**-index of Bruhat intervals. It is exactly this paper which motivated the present authors to look for a general setting to guarantee the existence of this non-homogeneous **cd**-index.

Recall a partially ordered set (poset) is *graded* if its elements have a well-defined distance from the minimal element of the poset. Björner and Stanley [12, Theorem 2.7] showed that if a graded poset has a combinatorial labeling of its cover relations known as an R-labeling, one can determine the flag f-vector in terms of the labeling inherited by the maximal chains. When the poset is Eulerian, that is, every interval satisfies the Euler–Poincaré relation, one can reduce this information to the classical **cd**-index.

By relaxing the graded, poset and Eulerian requirements, we study a general class of labeled directed graphs which satisfy a *balanced condition.* Recall a poset having an R-labeling demands that there be exactly one rising chain in each interval of the poset and, if the poset is Eulerian, exactly one falling chain in each interval. Our balanced condition states the number of rising paths of length k must equal the number of falling paths of length k. This allows us to directly prove the existence of the **cd**-index for balanced graphs and capture the results for Bruhat graphs as an important special case.

The presentation we give is self-contained. To underscore the connection with posets, results which also hold for the **ab**- and **cd**-index of graded posets will be stated as separate remarks.

An overview of the paper is as follows. In Section 2 we introduce the notion of a labeled acyclic digraph in order to model poset structure in this more general setting. An interpretation of its chain enumeration is given in terms of directed paths in the graph. In Section 3 we then set the coalgebraic groundwork for flag enumeration in labeled acyclic digraphs. We show the **ab**-index of a labeled acyclic digraph is a coalgebra homomorphism from the linear span of bounded labeled acyclic digraphs to the polynomial ring $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$; see Corollary 3.4.

We introduce the $\tilde{r}$ and $\tilde{f}$ polynomials in Section 4 to q-enumerate the rising and falling chains in the intervals of a labeled digraph. These polynomials hark back to the theory of Coxeter groups and Kazhdan–Lusztig polynomials; see [13, Chapter 5] as well as [42, 43]. The main result (Theorem 4.7) gives three equivalent statements which imply the (non-homogeneous) **ab**-index of an acyclic digraph can be written as a (non-homogeneous) **cd**-index. The key condition is that the number of rising paths of length k equals the number of falling paths of length k. We include a second proof of one of the implications in Theorem 4.7 which uses Hochschild cohomology.

A theory that mimics the notion of an Eulerian poset would not be complete without Alexander duality. Recall that for an Eulerian poset P with decomposition $S \dot\cup T \dot\cup \{\hat{0}, \hat{1}\}$ and rank function ρ, the celebrated Alexander duality states that the Möbius function values $\mu(\hat{0}, \hat{1})$ of each of the two posets $S \cup \{\hat{0}, \hat{1}\}$ and $T \cup \{\hat{0}, \hat{1}\}$ are equal up to the sign $(-1)^{\rho(P)-1}$. In Section 5 we introduce the notion of a restricted digraph. We state Alexander duality where the Möbius function is replaced by a signed sum over falling chains.

In Section 6 we apply our results to the important family of Bruhat graphs. Using the existence of a reflection ordering, introduced by Dyer [20], the existence of the **cd**-index of the Bruhat graph and its properties follow.

In Section 7 we review the basic set-up surrounding the ring of quasisymmetric functions. For a bounded labeled digraph we introduce the *rising* and *falling quasisymmetric functions* and relate these with a shift of the aforementioned rising and falling polynomials. We show the rising and falling quasisymmetric functions are Hopf algebra homomorphisms from the Hopf algebra formed by the linear span of bounded labeled acyclic digraphs to the quasisymmetric functions. We reformulate Theorem 4.7 in terms of Stembridge's peak algebra [51].

Section 8 begins with the result that given any polynomial in the ring $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ having non-negative coefficients, we show how to build an Eulerian poset having this **cd**-polynomial.

Recall that the classical **cd**-index of the face lattice of a polytope, and more generally, any spherically-shellable poset, has non-negative coefficients [50]. Non-negativity also holds for Gorenstein* posets [39]. These results form two cornerstones for the research program of classifying all the linear inequalities satisfied by the **cd**-index. We conjecture non-negativity for the **cd**-index of a bounded labeled acyclic digraph equipped with a balanced edge labeling that is linear; see Conjecture 8.4.

In the concluding remarks we end with open questions and research directions we are pursuing.

2. Labeled graphs

We begin by introducing a class of directed graphs in order to relax the notion of grading in a graded partially ordered set (poset). For further details about posets, see [49, Chapter 3].

Let $G = (V, E)$ be a directed, acyclic and locally finite graph with multiple edges allowed. Recall that an *acyclic graph* does not have any directed cycles and the property of a graph being *locally finite* requires that there are a finite number of paths between any two vertices. Each directed edge e has a tail and a head, denoted respectively by $\mathrm{tail}(e)$ and $\mathrm{head}(e)$. View each directed edge as an arrow from its tail to its head. A directed path p of length k from a vertex x to a vertex y is a list of k directed edges $(e_1, e_2, \ldots, e_k)$ such that $\mathrm{tail}(e_1) = x$, $\mathrm{head}(e_k) = y$ and $\mathrm{head}(e_i) = \mathrm{tail}(e_{i+1})$ for $i = 1, \ldots, k-1$. We denote the length of a path p by $\ell(p)$.

Since the graph is acyclic, it does not have any loops. Furthermore, the acyclicity condition implies there is a natural partial order on the vertices of G by defining the order relation $x \leq y$ if there is a directed path from the vertex x to the vertex y. It is straightforward to verify that this relation is reflexive, antisymmetric and transitive. Furthermore, it allows us to define the *interval* $[x, y]$ to be

$$[x, y] = \{z \in V(G) \ : \ \text{there is a directed path from } x \text{ to } z \text{ and a directed path from } z \text{ to } y\}.$$

We view the interval $[x, y]$ as the vertex-induced subgraph of the digraph G, where the edges have the same labels as in the digraph G. The locally finite condition is now equivalent to that every interval $[x, y]$ in the graph has finite cardinality.

Example 2.1. Consider a (locally finite) poset P and let the directed edges be the cover relations of the poset, in other words, the Hasse diagram of P is the digraph. When we draw the Hasse diagram of a poset we view its edges as being directed upward. Moreover, the fact the poset is locally finite implies that the associated digraph is locally finite. Hence this is an acyclic digraph.

A relaxed notion of edge labeling is needed which will enable us to define the **ab**-index, and ultimately, the **cd**-index. Let Λ be a set with a relation $\sim$, that is, there is a subset $R \subseteq \Lambda \times \Lambda$ such that for $i, j \in \Lambda$ we have $i \sim j$ if and only if $(i, j) \in R$. A *labeling of* G is a function λ from the set of edges of G to the set Λ. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commutative variables each of degree one. For a path $p = (e_1, \ldots, e_k)$ of length k, where $k \geq 1$, we define the *descent word* $u(p)$ to be the **ab**-monomial $u(p) = u_1 u_2 \cdots u_{k-1}$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } \lambda(e_i) \sim \lambda(e_{i+1}), \\ \mathbf{b} & \text{if } \lambda(e_i) \not\sim \lambda(e_{i+1}). \end{cases}$$

Observe that the descent word $u(p)$ has degree $k - 1$, that is, one less than the length of the path p. The **ab**-*index* of an interval $[x, y]$ is defined to be

$$\Psi([x, y]) = \sum_{p} u(p), \tag{2.1}$$

where the sum is over all directed paths p from x to y.

Example 2.2. In the case when the relation on Λ is a linear order, the digraph is the Hasse diagram of a graded poset and every interval has a unique rising chain. This condition is the classical notion of R-labeling introduced by Björner and Stanley [12].

In keeping with the poset motivation, we will continue to use the terminology rising and falling in our more general setting. See the paper [14], where Björner and Wachs weakened the condition that Λ is a linear order to a partial order. As a remark, one can further loosen the condition on the relation on Λ so that the only labels which need to be compared are pairs of elements $(\lambda(e), \lambda(f))$ such that $\text{head}(e) = \text{tail}(f)$.

For graded posets with an R-labeling equation (2.1) gives a different definition of the notion of the **ab**-index of a poset. See [28, Lemma 3.1] for more details.

Given a labeled directed graph G, define the graph G^* by reversing all the edges, keeping the edge labeling the same, and reversing the relation $\sim$ on Λ, that is, for $e \in E(G)$ we have $\text{head}_{G^*}(e) = \text{tail}_G(e)$ and $\text{tail}_{G^*}(e) = \text{head}_G(e)$. The labeling is given by $\lambda_{G^*}(e) = \lambda_G(e)$. Finally, the new relation Λ^* is given by $i \sim^* j$ if and only if $j \sim i$ for $i, j \in \Lambda$. For an **ab**-monomial $u = u_1 u_2 \cdots u_k$ define the reverse monomial by $u^* = u_k \cdots u_2 u_1$ and extend linearly to an involution on the non-commutative polynomial ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. Observe that a path p from x to y in G corresponds to a path p^* from y to x in G^*. Moreover, the descent word of the path satisfies $u(p^*) = u(p)^*$. Finally this relation extends to the **ab**-index of the entire interval $[x, y]$, that is, $\Psi([x, y]^*) = \Psi([y, x]) = \Psi([x, y])^*$.

3. Coalgebras

In this section we develop the underlying coalgebraic structure of labeled acyclic digraphs.

Let $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ be the non-commutative polynomial ring in the degree 1 variables $\mathbf{a}$ and $\mathbf{b}$ with integer coefficients. On the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ define a coproduct Δ by defining it on an $\mathbf{ab}$-monomial $u_1 u_2 \cdots u_n$ by

$$\Delta(u_1 u_2 \cdots u_n) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n,$$

and extend by linearity to $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. This coproduct, together with the usual multiplication, does not form a bialgebra. Instead the *Newtonian condition* is satisfied:

$$\Delta(v \cdot w) = \sum_{w} v \cdot w_{(1)} \otimes w_{(2)} + \sum_{v} v_{(1)} \otimes v_{(2)} \cdot w. \tag{3.1}$$

Here we use the Sweedler notation for the coproduct [38, 52]. This gives the ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ a Newtonian coalgebra structure.

Theorem 3.1. *For a labeled acyclic digraph G with two vertices x and y, the following holds:*

$$\Delta(\Psi([x, y])) = \sum_{x<z<y} \Psi([x, z]) \otimes \Psi([z, y]).$$

Proof. For a path $p = (e_1, \ldots, e_k)$ let $i(p)$ denote the set of interior vertices on the path, that is, $i(p) = \{\text{head}(e_1), \ldots, \text{head}(e_{k-1})\}$. Furthermore, for a path p from x to y and $x \leq z < w \leq y$, let $p|_{[z,w]}$ denote the path restricted to the interval $[z, w]$. Now we have

$$\begin{aligned}
\Delta(\Psi([x, y])) &= \sum_{p} \Delta(u(p)) \\
&= \sum_{p} \sum_{i=1}^{\ell(p)-1} u_1(p) \cdots u_{i-1}(p) \otimes u_{i+1}(p) \cdots u_{\ell(p)-1}(p) \\
&= \sum_{p} \sum_{z \in i(p)} u(p|_{[x,z]}) \otimes u(p|_{[z,y]}) \\
&= \sum_{x<z<y} \sum_{p : z \in i(p)} u(p|_{[x,z]}) \otimes u(p|_{[z,y]}) \\
&= \sum_{x<z<y} \left(\sum_{p_1} u(p_1) \right) \otimes \left(\sum_{p_2} u(p_2) \right) \\
&= \sum_{x<z<y} \Psi([x, z]) \otimes \Psi([z, y]).
\end{aligned}$$

Here we are summing over all maximal paths p in the interval $[x, y]$. In the second to last equality p_1 and p_2 are paths in $[x, z]$, respectively $[z, y]$. $\square$

Remark 3.2. In the case of Bruhat graphs, Theorem 3.1 was stated in [7, Proposition 2.11].

An acyclic digraph G is *bounded* if has a unique source and a unique sink. Following poset notation, we denote the unique source by $\hat{0}$ and the unique sink by $\hat{1}$. For brevity, we let $\Psi(G)$ denote $\Psi([\hat{0}, \hat{1}])$.

For two bounded labeled acyclic digraphs G and H we define the product $G * H$ as follows. We tacitly assume that $V(G), V(H), E(G), E(H), \Lambda_G$ and Λ_H are disjoint. Let the vertex set of $G * H$ be the disjoint union of $V(G) - \{\hat{1}\}$ and $V(H) - \{\hat{0}\}$, that is, $V(G*H) = (V(G) - \{\hat{1}_G\}) \cup (V(H) - \{\hat{0}_H\})$. Let the edge set be

$$\begin{aligned} E(G * H) = \{e \in E(G) &\ : \ \text{head}(e) \neq \hat{1}_G\} \\ \cup \ \ \{f \in E(H) &\ : \ \text{tail}(f) \neq \hat{0}_H\} \\ \cup \ \ \{(e, f) \in E(G) \times E(H) &\ : \ \text{head}(e) = \hat{1}_G, \text{tail}(f) = \hat{0}_H\}, \end{aligned}$$

where the new edge (e, f) is defined by $\text{tail}((e, f)) = \text{tail}(e)$ and $\text{head}((e, f)) = \text{head}(f)$. Let the label set Λ be defined by $\Lambda = \Lambda_G \cup \Lambda_H \cup \Lambda_G \times \Lambda_H$, with the relation on Λ given by the following four cases:

$$\begin{cases} \lambda \sim \mu & \text{if } \lambda, \mu \in \Lambda_G, \lambda \sim_{\Lambda_G} \mu, \\ \lambda \sim (\mu_1, \mu_2) & \text{if } \lambda, \mu_1 \in \Lambda_G, \mu_2 \in \Lambda_H, \lambda \sim_{\Lambda_G} \mu_1, \\ (\lambda_1, \lambda_2) \sim \mu & \text{if } \lambda_1 \in \Lambda_G, \lambda_2, \mu \in \Lambda_H, \lambda_2 \sim_{\Lambda_H} \mu, \\ \lambda \sim \mu & \text{if } \lambda, \mu \in \Lambda_H, \lambda \sim_{\Lambda_H} \mu. \end{cases}$$

Finally, define the labeling $\lambda : E(G * H) \longrightarrow \Lambda$ by the three cases

$$\begin{cases} \lambda(e) = \lambda_G(e) & \text{if } e \in E(G), \\ \lambda((e, f)) = (\lambda_G(e), \lambda_H(f)) & \text{if } (e, f) \in E(G) \times E(H), \\ \lambda(f) = \lambda_H(f) & \text{if } f \in E(H). \end{cases}$$

This product is the labeled analogue of the Stanley product of posets; see [50].

Theorem 3.3. *Let G and H be two bounded labeled acyclic digraphs, where each has a unique source and unique sink. Then the* **ab**-*index satisfies*

$$\Psi(G * H) = \Psi(G) \cdot \Psi(H),$$

where $*$ is the labeled analogue of the Stanley product of posets.

Proof. Each directed path p from $\hat{0}$ to $\hat{1}$ in $G * H$ has the form $p = (e_1, \ldots, e_{i-1}, (e_i, f_1), f_2, \ldots, f_j)$, which factors into the two paths $p_1 = (e_1, \ldots, e_{i-1}, e_i)$ and $p_2 = (f_1, f_2, \ldots, f_j)$ in G, respectively H. The descent word also factors as $u(p) = u(p_1) \cdot u(p_2)$. By summing over all paths, the result follows. □

Let $\mathcal{G}$ be the linear span of bounded labeled acyclic digraphs with $\hat{0} \neq \hat{1}$. The space $\mathcal{G}$ is a Newtonian coalgebra with the product $*$ and the coproduct

$$\Delta(G) = \sum_{\hat{0} < z < \hat{1}} [\hat{0}, z] \otimes [z, \hat{1}].$$

Theorems 3.1 and 3.3 imply the following corollary.

Corollary 3.4. *The* **ab**-*index is a coalgebra homomorphism from* $\mathcal{G}$*, the linear span of bounded labeled acyclic digraphs with* $\hat{0} \neq \hat{1}$*, to the coalgebra* $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$.

On the coalgebra $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ define an involution $u \longmapsto \overline{u}$ by uniformly exchanging **a**'s and **b**'s. Observe this involution is a Newtonian coalgebra automorphism, that is, the product and the coproduct satisfy

$$\overline{u \cdot v} = \overline{u} \cdot \overline{v} \quad \text{and} \quad \Delta(\overline{u}) = \sum_{u} \overline{u_{(1)}} \cdot \overline{u_{(2)}}.$$

Define $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Observe that $\deg(\mathbf{c}) = 1$ and $\deg(\mathbf{d}) = 2$. In what follows we need to consider a linear order on **cd**-monomials of degree n. Let u and v be two **cd**-monomials $u = \mathbf{c}^{i_0}\mathbf{d}\mathbf{c}^{i_1}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{i_p}$ and $v = \mathbf{c}^{j_0}\mathbf{d}\mathbf{c}^{j_1}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{j_q}$. If u contains fewer occurrences of the variable **d** than v (that is, $p < q$), then set $u < v$. If u contains the same number of occurrences of the variable **d** as v ($p = q$), and the vector $(i_0, i_1, \ldots, i_p) <_{lex} (j_0, j_1, \ldots, j_p)$ in lexicographic order $<_{lex}$, then set $u < v$.

Lemma 3.5. *Let* R *be a ring and let* S *be a subring of* R*. Then the following intersection holds:*

$$R\langle \mathbf{c}, \mathbf{d}\rangle \cap S\langle \mathbf{a}, \mathbf{b}\rangle = S\langle \mathbf{c}, \mathbf{d}\rangle.$$

In other words, when any **cd**-*polynomial* w *with coefficients in* R *is expanded as an* **ab**-*polynomial and has coefficients in the subring* S*, then all the coefficients of* w*, written as a* **cd**-*polynomial, already belong to the subring* S.

Proof. The containment $R\langle \mathbf{c}, \mathbf{d}\rangle \cap S\langle \mathbf{a}, \mathbf{b}\rangle \supseteq S\langle \mathbf{c}, \mathbf{d}\rangle$ is clear. It is enough to prove the reverse containment for homogeneous **cd**-polynomials of degree n. To derive a contradiction, assume that there is a **cd**-polynomial w belonging to $R\langle \mathbf{c}, \mathbf{d}\rangle \cap S\langle \mathbf{a}, \mathbf{b}\rangle$ but not to $S\langle \mathbf{c}, \mathbf{d}\rangle$. This means there is a **cd**-monomial in w whose coefficient does not lie in the subring S. Let $u = \mathbf{c}^{i_0}\mathbf{d}\mathbf{c}^{i_1}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{i_p}$ be the first such **cd**-monomial in w with respect to the previously described linear order. Consider the **ab**-monomial $z = \mathbf{a}^{i_0}\mathbf{b}\mathbf{a}\mathbf{a}^{i_1}\mathbf{b}\mathbf{a} \cdots \mathbf{b}\mathbf{a}\mathbf{a}^{i_p}$. The **ab**-monomial z occurs when expanding u into an **ab**-polynomial. Observe that any other **cd**-monomial v that has z occurring in its **ab**-expansion must satisfy $v < u$ in the linear order. Note that the coefficient of z in the **ab**-polynomial w lies in the subring S. This coefficient is the sum of certain coefficients of the **cd**-polynomial w where all but one (the coefficient of u) belong to the subring S. This contradicts the assumption that the coefficient of u does not belong to the subring. Hence the intersection holds. ∎

4. The **cd**-index

It is now natural to ask when the **ab**-index of a directed graph can be written as a **cd**-index, that is, when is it an element of $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ with $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. To do this, we introduce the *rising* and *falling polynomial* of an interval $[x, y]$ of a directed graph, denoted $\widetilde{r}_{x,y}(q)$ and $\widetilde{f}_{x,y}(q)$. After investigating their properties, we relate these polynomials with the aforementioned ability to express the non-homogeneous **ab**-index of as a non-homogeneous **cd**-index.

A directed path $p = (e_1, e_2, \ldots, e_k)$ in a labeled digraph G is called *rising* if $\lambda(e_i) \sim \lambda(e_{i+1})$ for all $i = 1, \ldots, k-1$. Similarly, a path p is called *falling* if $\lambda(e_i) \not\sim \lambda(e_{i+1})$ for all $i = 1, \ldots, k-1$. For $x < y$ let $\widetilde{r}_{x,y}(q)$ be the polynomial

$$\widetilde{r}_{x,y}(q) = \sum_{p \in \mathcal{R}(x,y)} q^{\ell(p)-1},$$

where the sum ranges over all rising paths p from x to y. Similarly, let $\widetilde{f}_{x,y}(q)$ be the polynomial

$$\widetilde{f}_{x,y}(q) = \sum_{p \in \mathcal{F}(x,y)} q^{\ell(p)-1},$$

where the sum ranges over all falling paths p from x to y.

Define two algebra maps κ and λ on $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ by letting

$$\begin{array}{lll} \kappa(\mathbf{a}) = \mathbf{a} - \mathbf{b}, & \kappa(\mathbf{b}) = 0, & \kappa(1) = 1, \\ \lambda(\mathbf{a}) = 0, & \lambda(\mathbf{b}) = \mathbf{b} - \mathbf{a}, & \lambda(1) = 1. \end{array}$$

The map κ appeared first in the paper [29, Section 10], whereas the λ map is more recent; see [47, Section 2.2.2]. Observe these two maps are related by $\overline{\kappa(u)} = \lambda(\overline{u})$. The κ and λ maps allows one to recapture the $\widetilde{r}$- and $\widetilde{f}$-polynomials from the **ab**-index $\Psi([x, y])$ as follows.

Lemma 4.1. *For an interval $[x, y]$ in a labeled digraph G,*

$$\kappa(\Psi([x, y])) = \widetilde{r}_{x,y}(\mathbf{a} - \mathbf{b}), \tag{4.1}$$

$$\lambda(\Psi([x, y])) = \widetilde{f}_{x,y}(\mathbf{b} - \mathbf{a}). \tag{4.2}$$

Proof. Since $\kappa(\mathbf{b}) = 0$, the algebra map κ applied to an **ab**-monomial only preserves the pure **a**-terms, and then replaces each **a** with $\mathbf{a} - \mathbf{b}$. Hence $\kappa(\Psi([x, y]))$ enumerates the rising chains. A symmetric argument proves the second identity. $\square$

Lemma 4.2. *For any* **ab***-polynomial u the following two identities hold:*

$$u = \kappa(u) + \sum_{u} \kappa(u_{(1)}) \cdot \mathbf{b} \cdot u_{(2)}, \tag{4.3}$$

$$u = \lambda(u) + \sum_{u} \lambda(u_{(1)}) \cdot \mathbf{a} \cdot u_{(2)}. \tag{4.4}$$

Proof. Since equation (4.3) is linear in u, it is enough to prove it for **ab**-monomials. We proceed by induction on the degree of an **ab**-monomial. The three base cases $u = 1$, $u = \mathbf{a}$ and $u = \mathbf{b}$ are straightforward to verify. Assume now that equation (4.3) holds for the **ab**-monomials v and w. Then it also holds for the product $v \cdot w$ by the following calculation. The right-hand side of the identity (4.3) in the case $u = v \cdot w$ is equal to

$$\begin{aligned}
\kappa(v \cdot w) &+ \sum_w \kappa(v \cdot w_{(1)}) \cdot \mathbf{b} \cdot w_{(2)} + \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdot v_{(2)} \cdot w \\
&= \kappa(v) \cdot \left(\kappa(w) + \sum_w \kappa(w_{(1)}) \cdot \mathbf{b} \cdot w_{(2)} \right) + \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdot v_{(2)} \cdot w \\
&= \kappa(v) \cdot w + \sum_v \kappa(v_{(1)}) \cdot \mathbf{b} \cdot v_{(2)} \cdot w \\
&= v \cdot w.
\end{aligned}$$

The second identity (4.4) follows by applying the involution $u \longmapsto \overline{u}$ and the relation $\overline{\kappa(u)} = \lambda(\overline{u})$. $\square$

Remark 4.3. Equation (4.3) is really Stanley's recursion [50] for the **ab**-index of a graded poset with rank function ρ, that is,

$$\Psi([x, y]) = (\mathbf{a} - \mathbf{b})^{\rho(x,y)-1} + \sum_{x<z<y} (\mathbf{a} - \mathbf{b})^{\rho(x,z)-1} \cdot \mathbf{b} \cdot \Psi([z, y]).$$

This recursion follows directly by conditioning on the first non-zero element in a chain. Equation (4.3) can be proven by using the fact that $\kappa(\Psi([x, y])) = (\mathbf{a} - \mathbf{b})^{\rho(x,y)-1}$, the **ab**-index is a coalgebra homomorphism [29], and the **ab**-indexes of graded posets span $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$.

Remark 4.4. The coalgebra $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ does not have a counit. Philosophically speaking, the two identities (4.3) and (4.4) are a replacement for the defining relation of the counit since they both allow us to recapture the polynomial u after applying the coproduct Δ.

Recall that the two non-commutative variables $\mathbf{c}$ and $\mathbf{d}$ are defined by $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$, with $\mathbf{c}$ of degree 1 and $\mathbf{d}$ of degree 2. The next lemma shows that **ab**-polynomials of a certain form are indeed **cd**-polynomials.

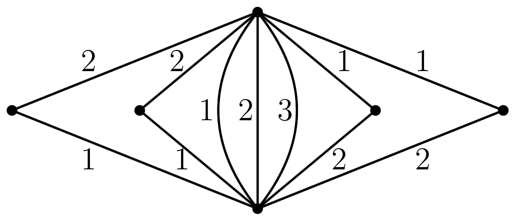
Lemma 4.5. *Let $p(x)$ and $q(x)$ be two polynomials in $\mathbb{Z}[x]$ such that their odd degree terms agree, that is, $p(x) - p(-x) = q(x) - q(-x)$. Then*

$$p(\mathbf{a} - \mathbf{b}) + q(\mathbf{b} - \mathbf{a}) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle, \tag{4.5}$$

$$p(\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} + p(\mathbf{b} - \mathbf{a}) \cdot \mathbf{a} \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle. \tag{4.6}$$

Proof. First note that $(\mathbf{a} - \mathbf{b})^{2 \cdot k} = (\mathbf{b} - \mathbf{a})^{2 \cdot k} = (\mathbf{c}^2 - 2 \cdot \mathbf{d})^k$ and $(\mathbf{a} - \mathbf{b})^{2 \cdot k+1} + (\mathbf{b} - \mathbf{a})^{2 \cdot k+1} = 0$. Hence by linearity it follows that the polynomial in (4.5) is a **cd**-polynomial for all polynomials $p(x)$ and $q(x)$ satisfying the condition of the lemma. The fact that the polynomial in (4.6) is a **cd**-polynomial follows again by linearity and by considering the parity of the power of the monomial x^n. For even powers we have $(\mathbf{a} - \mathbf{b})^{2 \cdot k} \cdot \mathbf{b} + (\mathbf{b} - \mathbf{a})^{2 \cdot k} \cdot \mathbf{a} = (\mathbf{c}^2 - 2 \cdot \mathbf{d})^k \cdot \mathbf{c}$ and for odd powers we have $(\mathbf{a} - \mathbf{b})^{2 \cdot k+1} \cdot \mathbf{b} + (\mathbf{b} - \mathbf{a})^{2 \cdot k+1} \cdot \mathbf{a} = -(\mathbf{c}^2 - 2 \cdot \mathbf{d})^{k+1}$. $\square$

Remark 4.6. Equations (4.5) and (4.6) in Lemma 4.5 can be viewed as linearizations of statements due to Stanley [50].



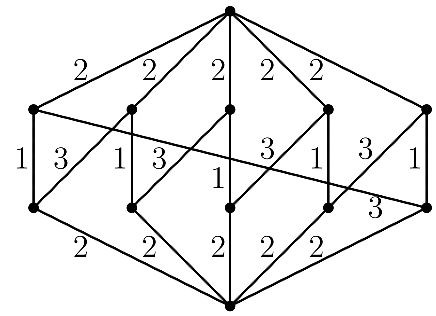


FIGURE 1. Two balanced directed graphs where the relation on the labeled set $\Lambda = \{1, 2, 3\}$ is the natural linear order. Their respective **cd**-indexes are $2 \cdot \mathbf{c} + 3$ and $5 \cdot \mathbf{d}$. These two examples show that the **cd**-index of a graph is not necessarily homogeneous and that the coefficient of the **c**-power term is not necessarily 1.

We now come to the main result of this section.

Theorem 4.7. *For a labeled acyclic digraph G, the following three statements are equivalent:*

(i) *For every interval $[x, y]$ in the digraph G and for every non-negative integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(ii) *For every interval $[x, y]$ in the digraph G and for every* even *positive integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(iii) *The* **ab**-*index of every interval $[x, y]$ in the digraph G, where $x < y$, is a polynomial in $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.*

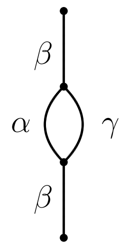
Definition 4.8. *A labeled acyclic digraph G is said to be* balanced *if it satisfies condition (i) in Theorem 4.7. Such a labeling is called a* balanced labeling*.*

A different way to express the balanced condition is that $\widetilde{r}_{x,y}(q) = \widetilde{f}_{x,y}(q)$ for all pairs of elements x and y such that $x < y$.

Example 4.9. See Figure 1 for two examples of balanced digraphs and their corresponding **cd**-indexes. In Figure 2 we give a labeled digraph with two different relations on the underlying label set. Each yields a balanced digraph. Note the resulting **cd**-indexes differ, with the second relation yielding a negative coefficient.

Remark 4.10. Condition (ii), that is, it suffices to check the balanced condition for paths of even length, has a corresponding statement for graded posets. A graded poset P of odd rank satisfying that every proper interval of P is Eulerian is also an Eulerian poset. See [49, Chapter 3, Exercise 69(c)].

Remark 4.11. Consider graded posets with an R-labeling. In this case, the balanced condition implies that the number of rising chains (namely 1) in an interval $[x, y]$ of rank $k + 1$ is equal to the number of falling chains, that is, $1 = h_{\emptyset}([x, y]) = h_{\{1,\ldots,k\}}([x, y])$. By Hall's theorem on the Möbius function, this can be stated as $\mu(x, y) = (-1)^{\rho(x,y)}$. Since this relation holds for all intervals $[x, y]$, this



Relation (i): $\alpha \sim \beta \sim \gamma$
$\gamma \not\sim \beta \not\sim \alpha$

Relation (ii): $\alpha \sim \beta \sim \alpha$
$\gamma \not\sim \beta \not\sim \gamma$

FIGURE 2. A labeled directed graph and two different relations on the label set $\Lambda = \{\alpha, \beta, \gamma\}$. Both relations yield a balanced graph. Relation (i) is the linear order and the **cd**-index is $\mathbf{ab} + \mathbf{ba} = \mathbf{d}$, whereas relation (ii) gives the **cd**-index $\mathbf{aa} + \mathbf{bb} = \mathbf{c}^2 - \mathbf{d}$.

implies that the poset is Eulerian and hence the **cd**-index exists. This result is classical; see [4, 50]. This is reminiscent of the work in [11], where it was shown that if the Euler–Poincaré relation holds for every interval in a poset then the poset satisfies the generalized Dehn–Sommerville relations and has a **cd**-index.

Proof of Theorem 4.7. The implication that $(i) \Longrightarrow (ii)$ is clear. For $(iii) \Longrightarrow (i)$, observe that the variables $\mathbf{c}$ and $\mathbf{d}$ are symmetric in $\mathbf{a}$ and $\mathbf{b}$. Hence $\widetilde{r}_{x,y}(q) = \Psi([x,y])|_{\mathbf{a}=q,\mathbf{b}=0} = \Psi([x,y])|_{\mathbf{c}=q,\mathbf{d}=0} = \Psi([x,y])|_{\mathbf{a}=0,\mathbf{b}=q} = \widetilde{f}_{x,y}(q)$.

Finally, assume that (ii) is true and we will prove the existence of the **cd**-index (iii). The proof is by induction on the longest path in the interval $[x,y]$. The base case is when the length of the longest path is 1. In this case the **cd**-index is just the number of edges between x and y. Assume now that the **cd**-index exists for all subintervals in $[x,y]$. Add equations (4.3) and (4.4) to obtain

$$2 \cdot u = \kappa(u) + \lambda(u) + \sum_{u} \big(\kappa(u_{(1)}) \cdot \mathbf{b} + \lambda(u_{(1)}) \cdot \mathbf{a}\big) \cdot u_{(2)}.$$

Now apply this equation to $u = \Psi([x,y])$, the **ab**-index of the entire interval $[x,y]$. Since the **ab**-index is a coalgebra homomorphism, we have that

$$2 \cdot \Psi([x,y]) = \widetilde{r}_{x,y}(\mathbf{a}-\mathbf{b}) + \widetilde{f}_{x,y}(\mathbf{b}-\mathbf{a}) + \sum_{x<z<y} \Big(\widetilde{r}_{x,z}(\mathbf{a}-\mathbf{b}) \cdot \mathbf{b} + \widetilde{f}_{x,z}(\mathbf{b}-\mathbf{a}) \cdot \mathbf{a}\Big) \cdot \Psi([z,y]).$$

By the induction hypothesis we know that $\Psi([x,z])$ and $\Psi([z,y])$ are both **cd**-polynomials with integer coefficients. By the implication $(iii) \Longrightarrow (i)$, we have that $\widetilde{r}_{x,z}(q) = \widetilde{f}_{x,z}(q)$. Thus by equation (4.6) of Lemma 4.5, we know that the term inside the summation sign is a **cd**-polynomial with integer coefficients. Similarly, by equation (4.5) of the same lemma, the sum of the two terms outside the summation sign is a **cd**-polynomial with integer coefficients. Hence the expression $2 \cdot \Psi([x,y])$ is a **cd**-polynomial with integer coefficients. By Lemma 3.5 with $R = \mathbb{Q}$ and $S = \mathbb{Z}$, we conclude that the **cd**-polynomial $\Psi([x,y])$ has integer coefficients. $\square$

Remark 4.12. The existence of the **cd**-index for Eulerian posets can be proved in a similar manner. Observe that for any interval $[x,y]$ in a graded poset we have that $\lambda(\Psi([x,y])) = (-1)^{\rho(x,y)} \cdot \mu(x,y) \cdot$

$(\mathbf{b}-\mathbf{a})^{\rho(x,y)-1}$. Hence for an interval $[x,y]$ which satisfies the Eulerian condition we have that

$$\kappa(\Psi([x,y])) = (\mathbf{a}-\mathbf{b})^{\rho(x,y)-1},$$
$$\lambda(\Psi([x,y])) = (\mathbf{b}-\mathbf{a})^{\rho(x,y)-1}.$$

Now proceed as in the proof of Theorem 4.7.

A different way to prove the implication $(ii) \Longrightarrow (iii)$ in Theorem 4.7 is to use the homology techniques developed in [30]. Let R be a commutative ring with a unit and let A be an R-module with a coassociative coproduct Δ. Let $d_n : A^{\otimes n} \longrightarrow A^{\otimes(n+1)}$ denote the map

$$d_n = \sum_{i+j=n-1} (-1)^i \cdot \mathrm{id}^{\otimes i} \cdot \Delta \cdot \mathrm{id}^{\otimes j}.$$

The coassociativity of the coproduct Δ implies that $d_n \circ d_{n+1} = 0$, that is, d_n is the boundary map of a chain complex. In [30] the Hochschild cohomology is computed for the chain complex

$$0 \longrightarrow A \xrightarrow{d_1} A^{\otimes 2} \xrightarrow{d_2} A^{\otimes 3} \longrightarrow \cdots \tag{4.7}$$

when A is the Newtonian coalgebra $R\langle\mathbf{c},\mathbf{d}\rangle$. Theorem 4.1 in [30] states when the ring R has 2 as a unit, the cohomology vanishes everywhere except in the bottom cohomology. Armed with this result, we can give a different proof.

Second proof of the implication $(ii) \Longrightarrow (iii)$ in Theorem 4.7. Let R be a ring that contains the integers and has 2 as a unit. (One such example is $R = \mathbb{Q}$.) The proof is again by induction on the longest path in the interval $[x,y]$. Since the **ab**-index is a coalgebra homomorphism and by the induction hypothesis, we have that

$$\Delta(\Psi([x,y])) = \sum_{x<z<y} \Psi([x,z]) \otimes \Psi([z,y]) \in R\langle\mathbf{c},\mathbf{d}\rangle \otimes R\langle\mathbf{c},\mathbf{d}\rangle.$$

Since Δ is coassociative, we also have $d_2(\Delta(\Psi([x,y])) = 0$, that is, the element $\Delta(\Psi([x,y])) = d_1(\Psi([x,y]))$ lies in the kernel of the map d_2. The chain complex (4.7) is exact at this point, so there is an element $w \in R\langle\mathbf{c},\mathbf{d}\rangle$ such that $\Delta(w) = d_1(w) = \Delta(\Psi([x,y]))$. Hence w and $\Psi([x,y]))$ differ by an element in the kernel of $\Delta : R\langle\mathbf{a},\mathbf{b}\rangle \longrightarrow R\langle\mathbf{a},\mathbf{b}\rangle \otimes R\langle\mathbf{a},\mathbf{b}\rangle$. The kernel of Δ is $R[\mathbf{a}-\mathbf{b}]$, so we have that $\Psi([x,y]) - w = p(\mathbf{a}-\mathbf{b})$ for some polynomial $p(x)$.

Let n be an odd positive integer. Condition (ii) states that the number of rising paths from x to y of length $n+1$ is equal to the number of falling paths from x to y of length $n+1$. This is equivalent to the condition that the coefficients of $\mathbf{a}^n$ and $\mathbf{b}^n$ in $\Psi([x,y])$ are identical. Since w is a **cd**-polynomial, the coefficients of $\mathbf{a}^n$ and $\mathbf{b}^n$ are also the same. Hence the coefficients of $\mathbf{a}^n$ and $\mathbf{b}^n$ in $p(\mathbf{a}-\mathbf{b})$ are the same, proving that the polynomial p only has even degree terms, that is, $p(\mathbf{a}-\mathbf{b})$ is a polynomial in the variable $(\mathbf{a}-\mathbf{b})^2 = \mathbf{c}^2 - 2\cdot\mathbf{d}$. Hence $\Psi([x,y])$ belongs to $R\langle\mathbf{c},\mathbf{d}\rangle$. Again by Lemma 3.5 we have $\Psi([x,y]) \in \mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$. ∎

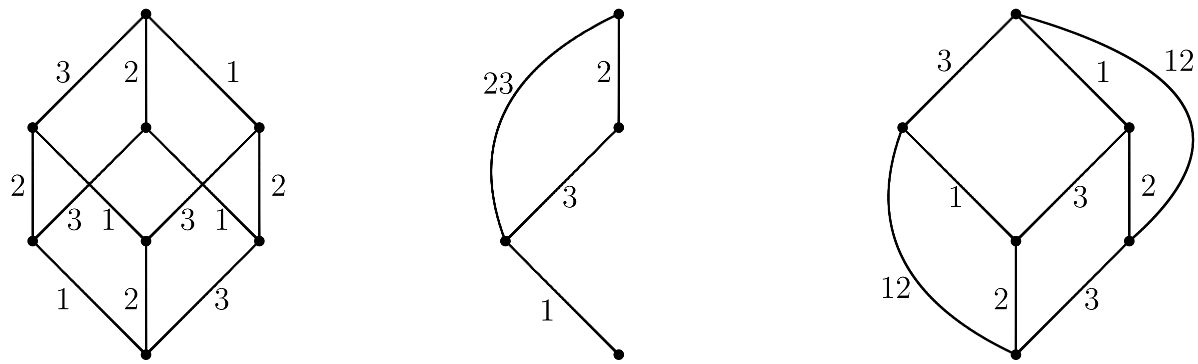


FIGURE 3. The Boolean algebra $G = B_3$ with its classical R-labeling $\lambda(I \to I \dot\cup \{i\}) = i$, and the two restricted digraphs G_S and G_T, where $S = \{\{1\}, \{1,3\}\}$ and $T = \{\{2\}, \{3\}, \{1,2\}, \{2,3\}\}$. Observe that G_S has no falling paths, whereas G_T has two falling paths of lengths 2 and 3. We have $\tilde{f}_{G_S}(q) = 0$ and $\tilde{f}_{G_T}(q) = q + q^2$, implying $\tilde{f}_{G_S}(-1) = 0$ and $\tilde{f}_{G_R}(-1) = 0$.

5. ALEXANDER DUALITY

In this section we assume all the digraphs under consideration are bounded with minimal element $\hat{0}$ and maximal element $\hat{1}$. For a labeled acyclic digraph G with vertex set V, we define the restricted digraph G_S where S is a subset of $V - \{\hat{0}, \hat{1}\}$. The edge label set is given by

$$\Lambda^+ = \bigcup_{n \geq 1} \Lambda^n,$$

and the relation $\sim$ on Λ^+ is $(\lambda_1, \ldots, \lambda_m) \sim (\mu_1, \ldots, \mu_n)$ if and only if $\lambda_m \sim \mu_1$. The vertex set of G_S is $S \cup \{\hat{0}, \hat{1}\}$. For every rising path $p = (e_1, \ldots, e_k)$ in the digraph G which starts and ends in $S \cup \{\hat{0}, \hat{1}\}$ but none of the intermediate vertices are in S, that is, $\mathrm{tail}(e_1), \mathrm{head}(e_k) \in S \cup \{\hat{0}, \hat{1}\}$ but $\mathrm{head}(e_2), \ldots, \mathrm{head}(e_{k-1}) \notin S$, let there be a directed edge in G_S from $\mathrm{tail}(e_1)$ to $\mathrm{head}(e_k)$ with the label $(\lambda(e_1), \ldots, \lambda(e_k))$ in Λ^+.

For two vertices x and y in the restricted graph G_S observe that the number of rising paths from x to y is the same as the number of rising paths in the graph G. This follows since a path in G_S corresponds to a path in G as follows. Let $p' = (e'_1, \ldots, e'_j)$ be a path in G_S. We obtain a path p in G by concatenating the rising paths that are associated with the edges e'_i. Furthermore the condition that the path p' is rising in G_S is exactly that the path p is rising in G, since the only condition that needs to be verified is that p is rising at the gluing vertices $\mathrm{head}(e'_1), \ldots, \mathrm{head}(e'_{j-1})$.

Let $\ell(G)$ denote the length of the longest path in the digraph G. We say that an acyclic digraph has the *parity condition* if the length of every path from the source $\hat{0}$ to the sink $\hat{1}$ has the same parity. Then in a digraph which has the parity condition, the length of any path from $\hat{0}$ to $\hat{1}$ is congruent to $\ell(G)$ modulo 2.

We can now formulate Alexander duality for balanced digraphs. See Figure 3 for an illustration of this theorem.

Theorem 5.1 (Alexander duality for balanced digraphs)**.** *Let G be a balanced acyclic digraph that satisfies the parity condition. Let the vertex set have the partition $V = S \dot\cup T \dot\cup \{\hat{0}, \hat{1}\}$. Then the falling paths in the two restricted digraph G_S and G_T satisfy the identity*

$$\widetilde{f}_{G_S}(-1) = (-1)^{\ell(G)-1} \cdot \widetilde{f}_{G_T}(-1).$$

Before proving this theorem, we must establish one more result. For a path $p = (e_1, \ldots, e_k)$ in the digraph G, recall that $i(p)$ is the set of all interior vertices of the path, that is, $i(p) = \{\text{head}(e_1), \ldots, \text{head}(e_{k-1})\}$. Note that $|i(p)| = \ell(p) - 1$. Furthermore, let $\text{Asc}(p)$ and $\text{Des}(p)$ denote the set of vertices where the path p has ascents, respectively, descents, that is,

$$\begin{aligned} \text{Asc}(p) &= \{\text{head}(e_i) \;:\; \lambda(e_i) \sim \lambda(e_{i+1})\}, \\ \text{Des}(p) &= \{\text{head}(e_i) \;:\; \lambda(e_i) \not\sim \lambda(e_{i+1})\}. \end{aligned}$$

Directly $i(p)$ is the disjoint union of $\text{Asc}(p)$ and $\text{Des}(p)$.

Proposition 5.2. *Let G be a bounded labeled acyclic digraph such that in every interval the number of rising paths equals the number of falling paths. Let the vertex set of G have the partition $V = S \dot\cup T \dot\cup \{\hat{0}, \hat{1}\}$. Then the following two sums are equal:*

$$\sum_{\substack{p \\ \text{Asc}(p) \subseteq T \\ \text{Des}(p) \subseteq S}} (-1)^{|i(p) \cap S|} = \sum_{\substack{p \\ \text{Asc}(p) \subseteq S \\ \text{Des}(p) \subseteq T}} (-1)^{|i(p) \cap S|}.$$

Proof. Let $A(S)$ and $B(S)$ denoted the left-hand side of the identity, respectively, the right-hand side. The proof is by double induction. First we induct over the longest path in the digraph. Here the induction base is $\ell(G) = 1$, that is, the graph consists only of the source and the sink. Each path has length 1 and is both rising and falling. Thus the statement is immediate.

Now assume that the statement holds for all digraphs of length less than $\ell(G)$. We induct on the set S. The induction basis is when S is empty. Then $A(\emptyset)$ and $B(\emptyset)$ are the number of rising, respectively, falling chains in the graph G. The balanced condition implies that they are equal, completing the induction basis.

For the induction step, assume that $A(S) = B(S)$ for a set S. We will prove it for $S \cup \{x\}$ where x is an element not in S. Observe that

$$\begin{aligned} A(S \cup \{x\}) - A(S) &= \sum_{\substack{p \\ \mathrm{Asc}(p) \subseteq T - \{x\} \\ x \in \mathrm{Des}(p) \subseteq S \cup \{x\}}} (-1)^{|i(p) \cap (S \cup \{x\})|} - \sum_{\substack{p \\ x \in \mathrm{Asc}(p) \subseteq T \\ \mathrm{Des}(p) \subseteq S}} (-1)^{|i(p) \cap S|} \\ &= - \sum_{\substack{p \\ \mathrm{Asc}(p) \subseteq T - \{x\} \\ x \in \mathrm{Des}(p) \subseteq S \cup \{x\}}} (-1)^{|i(p) \cap S|} - \sum_{\substack{p \\ x \in \mathrm{Asc}(p) \subseteq T \\ \mathrm{Des}(p) \subseteq S}} (-1)^{|i(p) \cap S|}. \end{aligned}$$

Combining these two sums we obtain a sum over all paths p through the vertex x such that $\mathrm{Asc}(p) - \{x\} \subseteq T$ and $\mathrm{Des}(p) - \{x\} \subseteq S$. That is, there is no condition on the path at the vertex x. Hence any such path is the concatenation of a path p_1 in $[\hat{0}, x]$ and a path p_2 in $[x, \hat{1}]$. Using that $|i(p) \cap S| = |i(p_1) \cap S| + |i(p_2) \cap S|$, the difference $A(S \cup \{x\}) - A(S)$ is given by

$$\begin{aligned} A(S \cup \{x\}) - A(S) &= - \sum_{\substack{p_1 \\ \mathrm{Asc}(p_1) \subseteq T \\ \mathrm{Des}(p_1) \subseteq S}} (-1)^{|i(p_1) \cap S|} \cdot \sum_{\substack{p_2 \\ \mathrm{Asc}(p_2) \subseteq T \\ \mathrm{Des}(p_2) \subseteq S}} (-1)^{|i(p_2) \cap S|} \\ &= -A_{[\hat{0}, x]}(S \cap (\hat{0}, x)) \cdot A_{[x, \hat{1}]}(S \cap (x, \hat{1})), \end{aligned}$$

where the first sum is over paths p_1 in $[\hat{0}, x]$ and the second sum is over paths p_2 in $[x, \hat{1}]$. By applying the first induction hypothesis to the smaller digraphs $[\hat{0}, x]$ and $[x, \hat{1}]$ we have

$$\begin{aligned} A(S \cup \{x\}) - A(S) &= -B_{[\hat{0}, x]}(S \cap (\hat{0}, x)) \cdot B_{[x, \hat{1}]}(S \cap (x, \hat{1})) \\ &= - \sum_{\substack{p_1 \\ a(p_1) \subseteq S \\ d(p_1) \subseteq T}} (-1)^{|i(p_1) \cap S|} \cdot \sum_{\substack{p_2 \\ a(p_2) \subseteq S \\ d(p_2) \subseteq T}} (-1)^{|i(p_2) \cap S|} \\ &= - \sum_{\substack{p \\ x \in i(p) \\ \mathrm{Asc}(p) - \{x\} \subseteq S \\ \mathrm{Des}(p) - \{x\} \subseteq T}} (-1)^{|i(p) \cap S|} \\ &= \sum_{\substack{p \\ x \in \mathrm{Asc}(p) \subseteq S \cup \{x\} \\ \mathrm{Des}(p) \subseteq T - \{x\}}} (-1)^{|i(p) \cap (S \cup \{x\})|} - \sum_{\substack{p \\ \mathrm{Asc}(p) \subseteq S \\ x \in \mathrm{Des}(p) \subseteq T}} (-1)^{|i(p) \cap S|} \\ &= B(S \cup \{x\}) - B(S). \end{aligned}$$

Hence $A(S \cup \{x\}) = B(S \cup \{x\})$ completing the induction. □

The statement of Proposition 5.2 is not symmetric in S as the following corollary illustrates. Also note the assumptions in Proposition 5.2 are not as strict as the balanced condition.

Corollary 5.3. *Let G be a labeled acyclic digraph such that in every interval the number of rising paths equals the number of falling paths. Then following two alternating sums agree:*

$$\sum_{p \text{ rising}} (-1)^{\ell(p)} = \sum_{p \text{ falling}} (-1)^{\ell(p)}.$$

Proof. Take $T = \emptyset$ in Proposition 5.2. $\square$

Proof of Theorem 5.1. Expanding $\widetilde{f}_{G_S}(-1)$ we have

$$\widetilde{f}_{G_S}(-1) = \sum_{p'} (-1)^{\ell(p')-1},$$

where the sum is over all falling paths p' in G_S. By replacing each edge in the path p' with the associated rising path in G, we obtain a path p in the digraph G such that $\mathrm{Asc}(p) \subseteq T$ and $\mathrm{Des}(p) \subseteq S$. Hence

$$\widetilde{f}_{G_S}(-1) = \sum_{\substack{p \\ \mathrm{Asc}(p) \subseteq T \\ \mathrm{Des}(p) \subseteq S}} (-1)^{|i(p) \cap S|}.$$

By a symmetric argument we have

$$(-1)^{\ell(G)-1} \cdot \widetilde{f}_{G_T}(-1) = (-1)^{\ell(G)-1} \cdot \sum_{\substack{p \\ \mathrm{Asc}(p) \subseteq S \\ \mathrm{Des}(p) \subseteq T}} (-1)^{|i(p) \cap T|} = \sum_{\substack{p \\ \mathrm{Asc}(p) \subseteq S \\ \mathrm{Des}(p) \subseteq T}} (-1)^{|i(p) \cap S|},$$

where we used the parity condition that $|i(p) \cap S| + |i(p) \cap T| = i(p) \equiv \ell(G) - 1 \bmod 2$. The duality result now follows from Proposition 5.2. $\square$

6. Application to Bruhat graphs

An important application of balanced labeled graphs is to the family of Bruhat graphs. In this section we give a brief overview of Bruhat graphs. For a more complete description of Coxeter systems, we refer the reader to the book of Björner and Brenti [13].

Let (W, S) be a Coxeter system, where W denotes a (finite or infinite) Coxeter group with generators S and $\ell(u)$ denotes the length of a group element u. Let T be the set of reflections, that is, $T = \{w \cdot s \cdot w^{-1} \; : \; s \in S, w \in W\}$. The Bruhat graph has the group W as its vertex set and its set of labels Λ is the set of reflections T. The edges and their labeling are defined as follows. There is a directed edge from u to v labeled t if $u \cdot t = v$ and $\ell(u) < \ell(v)$. The underlying poset of the Bruhat graph is called the (strong) Bruhat order. It is important to note that every interval of the Bruhat order is Eulerian, that is, every interval $[x, y]$ has Möbius function given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$, where ρ denotes the rank function.

The motivation for studying the **cd**-index of Bruhat graphs is that the **cd**-index of the interval $[u, v]$ determines the Kazhdan–Lusztig polynomial $P_{u,v}(q)$. See [7, Section 3]. The first step is to define the R-polynomials $R_{u,v}(q)$. See [13, Theorem 5.1.1] for further details.

Theorem 6.1. *There is a unique family of polynomials $\{R_{u,v}(q)\}_{u,v \in W}$ with integer coefficients satisfying the following conditions:*

(i) $R_{u,v} = 0$ *if* $u \not\leq v$,
(ii) $R_{u,v} = 1$ *if* $u = v$, *and*
(iii) *If* $s \in S$ *and* $\ell(v \cdot s) < \ell(v)$ *then*

$$R_{u,v}(q) = \begin{cases} R_{us,vs}(q) & \text{if } \ell(u \cdot s) < \ell(u), \\ q \cdot R_{us,vs}(q) + (q-1) \cdot R_{u,vs}(q) & \text{if } \ell(u \cdot s) > \ell(u). \end{cases}$$

A combinatorial interpretation of the R-polynomials is given by Dyer [20]. See also [13, Proposition 5.3.1 and Theorem 5.3.4]. On the set of reflections there exist conditions for a total ordering. An ordering satisfying these conditions is called a reflection ordering. The fact that a reflection ordering exists follows from [13, Proposition 5.2.1]. Dyer's interpretation is

$$R_{u,v}(q) = q^{\ell(u,v)/2} \cdot \widetilde{R}_{u,v}\left(q^{1/2} - q^{-1/2}\right),$$

where the $\widetilde{R}$-polynomials are defined in equation (7.2) with respect to a reflection ordering of the set of reflections T.

We can now state and give a concise proof of the first main result from [7], namely the existence of the complete **cd**-index of the Bruhat order. We prefer to call it *the* **cd**-*index of the Bruhat graph* to distinguish it from the **cd**-index of the Bruhat order.

Theorem 6.2 (Billera–Brenti)**.** *For an interval $[u, v]$ in the Bruhat order, where $u < v$, the following three conditions hold:*

(i) *The interval $[u, v]$ in the Bruhat graph has a* **cd**-*index* $\Psi([u, v])$.
(ii) *Restricting the* **cd**-*index* $\Psi([u, v])$ *to those terms of degree* $\ell(v) - \ell(u) - 1$ *equals the* **cd**-*index of the graded poset* $[u, v]$.
(iii) *The degree of a term in the* **cd**-*index* $\Psi([u, v])$ *is less than or equal to* $\ell(v) - \ell(u) - 1$ *and has the same parity as* $\ell(v) - \ell(u) - 1$.

Proof. The reverse of a reflection ordering is also reflection ordering. Hence the number of rising chains of length k is equal to the number of falling chains of the same length. Thus part (i) follows from Theorem 4.7. Part (ii) follows from the fact that when one restricts the labeling to the poset structure of one interval $[u, v]$, that is, only considering the cover relations, the reflection ordering is an R-labeling. Part (iii) follows from the fact that the Bruhat graph is bipartite. $\square$

7. Quasisymmetric functions

In this section we review the basic set-up surrounding the ring of quasisymmetric functions. For a bounded labeled digraph we introduce the *rising* and *falling quasisymmetric functions* and relate these with a shift of the aforementioned rising and falling polynomials. We show the rising and falling quasisymmetric functions are Hopf algebra homomorphisms from the Hopf algebra formed by the linear span of bounded labeled acyclic digraphs to the quasisymmetric functions. We then reformulate Theorem 4.7 in terms of the peak algebra.

The connection between flag f-vectors of graded posets and quasisymmetric functions was developed by Ehrenborg [21]. The companion theory for edge labeled posets and quasisymmetric functions is due to Bergeron and Sottile [6]. The peak algebra was introduced by Stembridge [51]. The link between the peak algebra and the quasisymmetric functions of Eulerian posets was made by Bergeron, Mykytiuk, Sottile and van Willigenburg in [5].

Let Σ_n denote the set of all compositions of n, that is, all sequences $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of positive integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = n$. We form Σ_n into a poset by defining the cover relation $(\alpha_1, \ldots, \alpha_i + \alpha_{i+1}, \ldots, \alpha_m) \prec (\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_m)$. Observe that the minimal element is the composition (n) and the maximal element is the composition $(1, 1, \ldots, 1)$. In fact, for $n \geq 1$, the poset Σ_n is isomorphic to the Boolean algebra B_{n-1}. Note also that Σ_0 consists of the unique composition of the integer 0. Especially, note that each composition α in the poset Σ_n has a unique complement that we denote by α^c. To find the complement, write the composition using commas, plus signs and 1's, and exchange the commas and plus signs. As an example, the complement of $(3, 1, 2) = (1 + 1 + 1, 1, 1 + 1)$ is $(1, 1, 1 + 1 + 1, 1) = (1, 1, 3, 1)$. Finally, let $\Sigma = \cup_{n \geq 0} \Sigma_n$.

A function f in the ring $\mathbb{Z}[[w_1, w_2, \ldots]]$ of power series with bounded degree is called *quasisymmetric* if for any sequence of positive integers $\alpha_1, \alpha_2, \ldots, \alpha_m$ we have

$$\left[w_{i_1}^{\alpha_1} \cdots w_{i_k}^{\alpha_m}\right] f = \left[w_{j_1}^{\alpha_1} \cdots w_{j_k}^{\alpha_m}\right] f$$

whenever $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$, and where $[w^\alpha]f$ denotes the coefficient of w^α in f. Denote by $\mathrm{QSym} \subseteq \mathbb{Z}[[w_1, w_2, \ldots]]$ the ring of quasisymmetric functions.

For a composition $\alpha = (\alpha_1, \ldots, \alpha_m)$ the *monomial quasisymmetric function* M_α is given by

$$M_\alpha = \sum_{i_1 < \cdots < i_m} w_{i_1}^{\alpha_1} \cdots w_{i_m}^{\alpha_m}.$$

The monomial quasisymmetric functions M_α indexed by the compositions α in Σ form a basis for the quasisymmetric functions. A different basis is given by the *fundamental quasisymmetric functions* L_α. For fixed composition α, the quasisymmetric function L_α is defined by the sum

$$L_\alpha = \sum_{\alpha \leq \beta} M_\beta.$$

The quasisymmetric functions also form a Hopf algebra where the coproduct is given by

$$\Delta(M_{(\alpha_1,\ldots,\alpha_m)}) = \sum_{i=0}^{m} M_{(\alpha_1,\ldots,\alpha_i)} \otimes M_{(\alpha_{i+1},\ldots,\alpha_m)}.$$

A different way to view this coproduct is that it is equivalent to the substitution

$$\Delta(f(w_1, w_2, \ldots)) = f(w_1 \otimes 1, w_2 \otimes 1, \ldots, 1 \otimes w_1, 1 \otimes w_2, \ldots).$$

Malvenuto and Reutenauer [44] defined an automorphism ω on quasisymmetric functions by the relation

$$\omega(L_\alpha) = L_{\alpha^c}.$$

The involution ω on QSym corresponds to the involution $u \longmapsto \overline{u}$ in $\mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$. The antipode on the Hopf algebra on quasisymmetric functions is given by

$$S(M_\alpha) = (-1)^n \cdot \omega(M_{\alpha^*}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a composition of n and α^* denotes the reverse composition $(\alpha_m, \ldots, \alpha_1)$.

For a sequence of labels $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of length n, we define two compositions $\rho^R(\lambda)$ and $\rho^F(\lambda)$ of n. The composition $\rho^R(\lambda)$ records the rising runs in the sequence λ, that is, $\rho^R(\lambda) = (\rho_1, \rho_2, \ldots, \rho_m)$ if

$$\begin{aligned} \lambda_1 \sim \cdots \sim \lambda_{\rho_1} &\not\sim \lambda_{\rho_1+1} \sim \cdots \sim \lambda_{\rho_1+\rho_2} \\ &\not\sim \lambda_{\rho_1+\rho_2+1} \sim \cdots \sim \lambda_{\rho_1+\cdots+\rho_{m-1}} \\ &\not\sim \lambda_{\rho_1+\cdots+\rho_{m-1}+1} \sim \cdots \sim \lambda_n, \end{aligned}$$

where $\sum_{i=1}^{m} \rho_i = n$. Similarly, let $\rho^F(\lambda)$ record the falling runs in the sequence, that is, if $\rho^F = (\rho_1, \rho_2, \ldots, \rho_m)$ we have

$$\begin{aligned} \lambda_1 \not\sim \cdots \not\sim \lambda_{\rho_1} &\sim \lambda_{\rho_1+1} \not\sim \cdots \not\sim \lambda_{\rho_1+\rho_2} \\ &\sim \lambda_{\rho_1+\rho_2+1} \not\sim \cdots \not\sim \lambda_{\rho_1+\cdots+\rho_{m-1}} \\ &\sim \lambda_{\rho_1+\cdots+\rho_{m-1}+1} \not\sim \cdots \not\sim \lambda_n. \end{aligned}$$

Observe that in the poset Σ_n the two compositions $\rho^R(\lambda)$ and $\rho^F(\lambda)$ are complements of each other, that is, $(\rho^R(\lambda))^c = \rho^F(\lambda)$.

For a bounded labeled digraph G define the *rising* and *falling quasisymmetric functions* by

$$F^R(G) = \sum_p L_{\rho^R(\lambda(p))} \quad \text{and} \quad F^F(G) = \sum_p L_{\rho^F(\lambda(p))}, \tag{7.1}$$

where each sum is over all paths p from $\hat{0}$ to $\hat{1}$ in the digraph G. Since the two compositions $\rho^R(\lambda)$ and $\rho^F(\lambda)$ are complements, directly we have that the two quasisymmetric functions are related by the automorphism ω, that is,

$$\omega(F^R(G)) = F^F(G)).$$

Similar to the notion of the two polynomials $\widetilde{r}_{x,y}(q)$ and $\widetilde{f}_{x,y}(q)$, for $x \leq y$ define the two polynomials $\widetilde{R}_{x,y}(q)$ and $\widetilde{F}_{x,y}(q)$ by

$$\widetilde{R}_{x,y}(q) = \sum_{p} q^{\ell(p)} \quad \text{and} \quad \widetilde{F}_{x,y}(q) = \sum_{p} q^{\ell(p)}, \tag{7.2}$$

where the sum ranges over all rising, respectively falling, paths from x to y. Directly we have the relations $\widetilde{R}_{x,y}(q) = q \cdot \widetilde{r}_{x,y}(q)$ and $\widetilde{F}_{x,y}(q) = q \cdot \widetilde{f}_{x,y}(q)$ for $x < y$.

Proposition 7.1. *For a bounded labeled digraph G the two identities hold:*

$$F^R(G)\big|_{w_{m+1}=w_{m+2}=\cdots=0} = \sum_{c} \widetilde{R}_{x_0,x_1}(w_1) \cdot \widetilde{R}_{x_1,x_2}(w_2) \cdots \widetilde{R}_{x_{m-1},x_m}(w_m), \tag{7.3}$$

$$F^F(G)\big|_{w_{m+1}=w_{m+2}=\cdots=0} = \sum_{c} \widetilde{F}_{x_0,x_1}(w_1) \cdot \widetilde{F}_{x_1,x_2}(w_2) \cdots \widetilde{F}_{x_{m-1},x_m}(w_m), \tag{7.4}$$

where each sum is over all multichains $c = \{\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_m = \hat{1}\}$ of length m in the digraph G.

Proof. Each side of equation (7.3) is a polynomial in $w_1, w_2, \ldots, w_n$, where n is the length of the longest chain in the digraph G. Consider the coefficient of the monomial $w_{i_1}^{\alpha_1} \cdot w_{i_2}^{\alpha_2} \cdots w_{i_m}^{\alpha_m}$ on the right-hand side of equation (7.3), where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a composition with $m \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_m \leq n$. This counts the number of paths $p = (e_1, e_2, \ldots, e_m)$ in the digraph G such that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = n$ and

$$\begin{gathered}
\lambda(e_1) \sim \cdots \sim \lambda(e_{\alpha_1}), \\
\lambda(e_{\alpha_1+1}) \sim \cdots \sim \lambda(e_{\alpha_1+\alpha_2}), \\
\vdots \\
\lambda(e_{\alpha_1+\cdots+\alpha_{m-1}+1}) \sim \cdots \sim \lambda(e_{\alpha_1+\cdots+\alpha_m}),
\end{gathered}$$

and where the relation between $\lambda(e_{\alpha_1+\cdots+\alpha_i})$ and $\lambda(e_{\alpha_1+\cdots+\alpha_i+1})$ is not known. In other words, this coefficient enumerates the number of paths p such that $\rho^R(\lambda(p)) \leq \alpha$.

The coefficient of $w_{i_1}^{\alpha_1} \cdot w_{i_2}^{\alpha_2} \cdots w_{i_m}^{\alpha_m}$ in the left-hand side of equation (7.3) is the coefficient of M_α in $F^R(G)$. This coefficient is given by

$$\begin{aligned}
[M_\alpha]F^R(G) &= [M_\alpha] \sum_{p} L_{\rho^R(\lambda(p))} \\
&= [M_\alpha] \sum_{p} \sum_{\rho^R(\lambda(p)) \leq \alpha} M_\alpha.
\end{aligned}$$

This is the number of paths p such that $\rho^R(\lambda(p)) \leq \alpha$, proving the first identity. The second identity (7.4) follows by a symmetric argument. $\square$

Proposition 7.1 can be reformulated as follows.

Proposition 7.2. *For a bounded labeled digraph G the two identities hold:*

$$F^R(G) = \lim_{m \longrightarrow \infty} \sum_c \widetilde{R}_{x_0,x_1}(w_1) \cdot \widetilde{R}_{x_1,x_2}(w_2) \cdots \widetilde{R}_{x_{m-1},x_m}(w_m), \tag{7.5}$$

$$F^F(G) = \lim_{m \longrightarrow \infty} \sum_c \widetilde{F}_{x_0,x_1}(w_1) \cdot \widetilde{F}_{x_1,x_2}(w_2) \cdots \widetilde{F}_{x_{m-1},x_m}(w_m). \tag{7.6}$$

Define the Cartesian product $G \times H$ of two digraphs G and H to be the digraph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H) = V(G) \times E(H) \cup E(G) \times V(H)$, where the edges are defined by $\mathrm{tail}_{G \times H}((e, y)) = (\mathrm{tail}_G(e), y)$, $\mathrm{head}_{G \times H}((e, y)) = (\mathrm{head}_G(e), y)$, $\mathrm{tail}_{G \times H}((x, e)) = (x, \mathrm{tail}_H(e))$ and $\mathrm{head}_{G \times H}((x, e)) = (x, \mathrm{head}_H(e))$. Furthermore, for the Cartesian product of labeled digraphs, set $\Lambda_{G \times H} = \Lambda_G \cup \Lambda_H$, where the relation is defined by $\lambda \sim \mu$ if and only if one of the following cases hold: (i) $\lambda, \mu \in \Lambda_G, \lambda \sim_{\Lambda_G} \mu$, (ii) $\lambda \in \Lambda_G, \mu \in \Lambda_H$, (iii) $\lambda, \mu \in \Lambda_H, \lambda \sim_{\Lambda_H} \mu$. Finally, the labels of the Cartesian product are defined by $\lambda_{G \times H}((e, y)) = \lambda_G(e)$ and $\lambda_{G \times H}((x, e)) = \lambda_H(e)$.

Observe that if both of the digraphs G and H are acyclic then their Cartesian product is acyclic. Similarly, if both digraphs are locally finite, then so is their product.

Lemma 7.3. *For two labeled acyclic digraphs G and H, the $\widetilde{R}$- and $\widetilde{F}$-polynomials of the Cartesian product $G \times H$ are given by*

$$\widetilde{R}_{(x,z),(y,w)}(q) = \widetilde{R}_{x,y}(q) \cdot \widetilde{R}_{z,w}(q),$$
$$\widetilde{F}_{(x,z),(y,w)}(q) = \widetilde{F}_{x,y}(q) \cdot \widetilde{F}_{z,w}(q).$$

Proof. A rising chain in $G \times H$ must first have labels from Λ_G and then labels from Λ_H. Thus the only way to have a rising chain in the interval $[(x, z), (y, w)]$ is to first have a rising chain in $[(x, z), (y, z)] \cong [x, y]$ and then a rising chain in $[(y, z), (y, w)] \cong [z, w]$. Similarly, a falling chain must have the labels from Λ_H first and then from Λ_G. $\square$

Proposition 7.4. *For two labeled acyclic digraphs G and H, the F^R and F^F quasisymmetric functions of the interval $[(x, z), (y, w)])$ in the Cartesian product $G \times H$ are given by*

$$F^R([(x, z), (y, w)]) = F^R([x, y]) \cdot F^R([z, w]),$$
$$F^F([(x, z), (y, w)]) = F^F([x, y]) \cdot F^F([z, w]).$$

Proof. By equation (7.5) we have

$$\begin{aligned}
F^R([(x,z),(y,w)]) &= \lim_{m\longrightarrow\infty} \sum \widetilde{R}_{(x_0,z_0),(x_1,z_1)}(w_1)\cdots\widetilde{R}_{(x_{m-1},z_{m-1}),(x_m,z_m)}(w_m) \\
&= \lim_{m\longrightarrow\infty} \sum \widetilde{R}_{x_0,x_1}(w_1)\cdots\widetilde{R}_{x_{m-1},x_m}(w_m) \\
&\qquad\qquad \cdot \widetilde{R}_{z_0,z_1}(w_1)\cdots\widetilde{R}_{z_{m-1},z_m}(w_m) \\
&= \lim_{m\longrightarrow\infty} \left(\sum \widetilde{R}_{x_0,x_1}(w_1)\cdots\widetilde{R}_{x_{m-1},x_m}(w_m)\right) \\
&\qquad\qquad \cdot \left(\sum \widetilde{R}_{z_0,z_1}(w_1)\cdots\widetilde{R}_{z_{m-1},z_m}(w_m)\right) \\
&= \left(\lim_{m\longrightarrow\infty} \sum \widetilde{R}_{x_0,x_1}(w_1)\cdots\widetilde{R}_{x_{m-1},x_m}(w_m)\right) \\
&\cdot \left(\lim_{m\longrightarrow\infty} \sum \widetilde{R}_{z_0,z_1}(w_1)\cdots\widetilde{R}_{z_{m-1},z_m}(w_m)\right) \\
&= F^R([x,y]) \cdot F^R([z,w]),
\end{aligned}$$

where the two first sums are over all multichains $(x,z) = (x_0,z_0) \leq (x_1,z_1) \leq (x_2,z_2) \leq \cdots \leq (x_m,z_m) = (y,w)$ and the remaining sums are over the multichains $x = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_m = y$ and $z = z_0 \leq z_1 \leq z_2 \leq \cdots \leq z_m = w$. The dual argument gives the second identity. □

Proposition 7.5. *For $[x,y]$ an interval in a labeled acyclic digraph G,*

$$\begin{aligned}
\Delta(F^R([x,y])) &= \sum_{x\leq z\leq y} F^R([x,z]) \otimes F^R([z,y])), \\
\Delta(F^F([x,y])) &= \sum_{x\leq z\leq y} F^F([x,z]) \otimes F^F([z,y])).
\end{aligned}$$

Proof. Using equation (7.5) we have

$$\begin{aligned}
\Delta(F^R([x,y])) &= \lim_{m\longrightarrow\infty} \sum_{c} \widetilde{R}_{x_0,x_1}(w_1\otimes 1)\cdots \widetilde{R}_{x_{m-1},x_m}(w_m\otimes 1) \\
&\qquad\qquad \cdot \widetilde{R}_{x_m,x_{m+1}}(1\otimes w_1)\cdots \widetilde{R}_{x_{2m-1},x_{2m}}(1\otimes w_m) \\
&= \lim_{m\longrightarrow\infty} \sum_{x\le z\le y} \left(\sum_{c_1} \widetilde{R}_{x_0,x_1}(w_1\otimes 1)\cdots \widetilde{R}_{x_{m-1},x_m}(w_m\otimes 1)\right) \\
&\qquad\qquad \cdot \left(\sum_{c_2} \widetilde{R}_{x_m,x_{m+1}}(1\otimes w_1)\cdots \widetilde{R}_{x_{2m-1},x_{2m}}(1\otimes w_m)\right) \\
&= \lim_{m\longrightarrow\infty} \sum_{x\le z\le y} \left(\sum_{c_1} \widetilde{R}_{x_0,x_1}(w_1)\cdots \widetilde{R}_{x_{m-1},x_m}(w_m)\right) \\
&\qquad\qquad \otimes \left(\sum_{c_2} \widetilde{R}_{x_m,x_{m+1}}(w_1)\cdots \widetilde{R}_{x_{2m-1},x_{2m}}(w_m)\right) \\
&= \sum_{x\le z\le y} F^R([x,z])) \otimes F^R([z,y])),
\end{aligned}$$

where in the first sum the chain c is $\{x = x_0 \le x_1 \le \cdots \le x_m \le \cdots \le x_{2m} = y\}$, z is the element x_m in the chain c and the chains c_1 and c_2 are the two chains $\{x = x_0 \le x_1 \le \cdots \le x_m = z\}$, respectively $\{z = x_m \le \cdots \le x_{2m} = y\}$. A symmetric argument gives the second identity. □

Let $\mathcal{H}$ be the linear span of bounded labeled acyclic digraphs. The space $\mathcal{H}$ is a Hopf algebra with the product given by the Cartesian product and the coproduct given by

$$\Delta(G) = \sum_{\hat{0}\le z\le \hat{1}} [\hat{0}, z] \otimes [z, \hat{1}].$$

We have the following corollary.

Corollary 7.6. *The two quasisymmetric functions F^R and F^F are Hopf algebra homomorphisms from $\mathcal{H}$ to the quasisymmetric functions* QSym.

Proof. Follows directly from Proposition 7.5. □

Generalizing [27, Lemma 5.1], we have the following lemma.

Lemma 7.7. *For a labeled acyclic graph G,*

$$\sum_{x\le z\le y} \widetilde{R}_{x,z}(q) \cdot \widetilde{F}_{z,y}(-q) = \delta_{x,y}.$$

Proof. Using the defining relation for the antipode S, we have that

$$\begin{aligned}\delta_{x,y} &= \sum_{x \leq z \leq y} F^R([x,z]) \cdot S(F^R([z,y])) \\ &= \sum_{x \leq z \leq y} F^R([x,z]) \cdot \left(\omega(F^R([y,z]^*))\big|_{w_1=-w_1, w_2=-w_2, \ldots}\right) \\ &= \sum_{x \leq z \leq y} F^R([x,z]) \cdot \left(F^F([y,z]^*)\big|_{w_1=-w_1, w_2=-w_2, \ldots}\right).\end{aligned}$$

Setting $w_1 = q$ and $w_2 = w_3 = \cdots = 0$ the result follows by Proposition 7.1. □

This lemma also has a direct bijective proof.

Second proof of Lemma 7.7. Let $\mathcal{R}_{x,y}$ and $\mathcal{F}_{x,y}$ be the set of all rising, respectively falling, paths from x to y. Consider the disjoint union

$$\mathcal{U}_{x,y} = \bigcup_{x \leq z \leq y} \mathcal{R}_{x,z} \cdot \mathcal{F}_{z,y}.$$

In other words, $\mathcal{U}_{x,y}$ is the set of all pair of paths (p_1, p_2) such that p_1 is rising, p_2 is falling, and p_1 ends where p_2 starts. We would like to prove that

$$\sum_{(p_1,p_2) \in \mathcal{U}_{x,y}} (-1)^{\ell(p_2)} \cdot q^{\ell(p_1)+\ell(p_2)} = \delta_{x,y}.$$

When $x = y$ the result is immediate. We prove the case when $x < y$ by a sign-reversing involution σ.

Given a pair of paths (p_1, p_2) in $\mathcal{U}_{x,y}$ with $p_1 = (e_1, \ldots, e_i)$, $p_2 = (f_1, \ldots, f_j)$ and i and j not both equal to 0, define another pair of paths $\sigma(p_1, p_2) = (q_1, q_2)$ by the following four cases. Case (i): if $i = 0$, that is, $x = z$, let $q_1 = (f_1)$ and $q_2 = (f_2, \ldots, f_j)$. Case (ii): if $j = 0$, that is, $z = y$, let $q_1 = (e_1, \ldots, e_{i-1})$ and $q_2 = (e_i)$. Cases (iii) and (iv) are both when i and j are greater than 0. Compare the two labels $\lambda(e_i)$ and $\lambda(f_1)$ with the relation on Λ. Case (iii): if $\lambda(e_i) \sim \lambda(f_1)$ let the pair of paths be $q_1 = (e_1, \ldots, e_i, f_1)$ and $q_2 = (f_2, \ldots, f_j)$. Case (iv): otherwise, that is, $\lambda(e_i) \not\sim \lambda(f_1)$ let the pair of paths be $q_1 = (e_1, \ldots, e_{i-1})$ and $q_2 = (e_i, f_1, \ldots, f_j)$.

It is direct to verify that σ is an involution. Furthermore, one has that $\ell(p_1) + \ell(p_2) = \ell(q_1) + \ell(q_2)$ and that the lengths of p_2 and q_2 have different parity. Hence σ is a sign-reversing involution, proving the lemma. □

As corollary to Lemma 7.7 we have the following result. Compare with Exercise 5.11 in [13].

Corollary 7.8. *For a balanced labeled acyclic graph G,*

$$\sum_{x \leq z \leq y} \widetilde{R}_{x,z}(q) \cdot \widetilde{R}_{z,y}(-q) = \delta_{x,y}.$$

For a bipartite balanced labeled acyclic graph G,

$$\sum_{x \leq z \leq y} (-1)^{\ell(z,y)} \cdot \widetilde{R}_{x,z}(q) \cdot \widetilde{R}_{z,y}(-q) = \delta_{x,y}.$$

The quasi-symmetric functions F^R encode the same information of the labeled digraph G as the **ab**-index $\Psi(G)$. To make this more explicit, define the linear map $\gamma : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathrm{QSym}$ by

$$\gamma\left((\mathbf{a}-\mathbf{b})^{\alpha_1 - 1} \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\alpha_2 - 1} \cdot \mathbf{b} \cdots \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\alpha_k - 1}\right) = M_\alpha;$$

see [33, Section 3]. The map γ is a vector space isomorphism between $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$ and the quasisymmetric function without a constant term. Now we have for a digraph G the identity $\gamma(\Psi(G)) = F^R(G)$.

Stembridge [51] introduced a sub-Hopf algebra of the quasisymmetric functions QSym known as the peak algebra Π. It plays the same role as the subalgebra $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ of $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$. Concretely, the peak algebra is the span of the constant quasisymmetric function 1 with the image of $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ under the map γ. Hence Theorem 4.7 can be reformulated as follows.

Theorem 7.9. *For a labeled acyclic digraph G, the following are equivalent:*

(i) *For every interval $[x, y]$ in the digraph G and for every non-negative integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(ii) *For every interval $[x, y]$ in the digraph G and for every* even *positive integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(iii) *The F^R quasisymmetric function of every interval $[x, y]$ in the digraph G belongs to the peak algebra Π.*

8. Balanced linear edge labelings

We call an edge labeling *linear* if the underlying relation $(\Lambda, \sim)$ is that of a linear order.

Theorem 8.1. *Let u be a non-zero* **cd***-polynomial with non-negative coefficients. Then there exists a bounded balanced labeled acyclic digraph G with linear edge labeling which satisfies $\Psi(G) = u$.*

In order to prove this theorem, we first need the following two lemmas.

Lemma 8.2. *Let G_1 and G_2 be two bounded digraphs with balanced linear edge labeling. Let the underlying label sets be Λ_1, respectively Λ_2. Define a new bounded labeled digraph H by*

$$V(H) = V(G_1) \cup V(G_2),$$
$$E(H) = E(G_1) \cup E(G_2) \cup \{h_1, h_2\}$$

where the new edges are $\mathrm{tail}(h_i) = \hat{1}_1$ *and* $\mathrm{head}(h_i) = \hat{0}_2$*. Let the new label set be $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \{\mu_1, \mu_2\}$ and the linear order be any shuffling of Λ_1 and Λ_2 with the condition that the new labels μ_1 and μ_2 are the minimal, respectively the maximal, element of the linear order Λ. Finally, let the labels of the* *new edges be $\lambda(h_i) = \mu_i$. Then the digraph H has a balanced labeling which is linear, and its* **cd***-index is given by*

$$\Psi(H) = \Psi(G_1) \cdot \mathbf{d} \cdot \Psi(G_2).$$

Proof. Every path p from $\hat{0}_{G_1} = \hat{0}_H$ to $\hat{1}_{G_2} = \hat{1}_H$ breaks into a path in G_1, a path in G_2 and one of the new edges h_1 or h_2. Observe that

$$u(p) = u(p|_{G_1}) \cdot v \cdot u(p|_{G_2}),$$

where $v = \mathbf{ba}$ if the new edge is h_1 and $v = \mathbf{ab}$ if the new edge is h_2. Hence summing over all paths we have

$$\Psi(H) = \Psi(G_1) \cdot (\mathbf{ba} + \mathbf{ab}) \cdot \Psi(G_2).$$

A similar argument shows that every interval of H has a **cd**-index and hence the labeling is balanced. □

Lemma 8.3. *Let G_1 and G_2 be two bounded digraphs with balanced linear edge labelings. Let H be the bounded digraph obtained by the disjoint union of G_1 and G_2 and identifying the minimal elements $\hat{0}_{G_1}$ and $\hat{0}_{G_2}$, and the maximal elements $\hat{1}_{G_1}$ and $\hat{1}_{G_2}$. Then H has a balanced linear edge labeling and its* **cd***-index is the sum*

$$\Psi(H) = \Psi(G_1) + \Psi(G_2).$$

Proof of Theorem 8.1. The strong Bruhat order on the dihedral group is the butterfly poset and hence its **cd**-index is $\mathbf{c}^n$. Hence by Lemma 8.2 for any **cd**-monomial v we can construct a bounded labeled acyclic digraph G with a balanced linear order such that $\Psi(G) = v$. By Lemma 8.3 this can be extended to any non-negative **cd**-polynomial. □

As a remark, Bayer and Hetyei's work on the cone spanned by Eulerian posets shows one can obtain a limiting poset which has **cd**-index $2^k \cdot w$, where w is a **cd** monomial with exactly k **d**'s; see [3, Proposition 2.9].

Theorem 8.1 motivates us to make the following conjecture.

Conjecture 8.4. *The* **cd***-index of a bounded labeled acyclic digraph G with a balanced linear edge labeling is non-negative.*

This conjecture implies the non-negativity of the **cd**-index of Bruhat graphs; see [7, Conjecture 6.1].

Conjecture 8.5 (Billera–Brenti)**.** *Let (W, S) be a Coxeter system with $u, v \in W$ and $u < v$. Then the* **cd***-index of the interval $[u, v]$ is non-negative.*

9. Concluding remarks

As was mentioned in the previous section, verifying Conjecture 8.4 would imply the non-negativity of the **cd**-index of Bruhat graphs [7]. In the case of Bruhat graphs of infinite Coxeter groups, recent work of the Ehrenborg, Hetyei and Readdy [25] on level Eulerian posets, that is, infinite Eulerian posets with a local regularity condition, may provide some insight. Blanco [16] has studied instances of non-negativity for the poset of shortest paths in the Bruhat order. Shellability arguments have been used by Stanley to prove the non-negativity of the **cd**-index of S-shellable spheres [48], in Karu's argument for the non-negativity of the toric g-vector of non-rational polytopes [39], as well as Karu's argument for the non-negativity of **cd**-index of Gorenstein* posets [40].

Recent work of Fan and He [36] have applied Karu's flip condition [41] to show the coefficient of $\mathbf{d}c^i\mathbf{d}\mathbf{c}^j$ is non-negative in the **cd**-index of any Bruhat graph. See also [35]. Perhaps an analogous result can be established for digraphs having a balanced linear order.

Another research direction is to develop Kazhdan–Lusztig polynomials for directed graphs. In order to do this, one must require the graph to be bipartite, just as the bipartite condition holds for Bruhat graphs. Brenti, Caselli and Marietti's theory of special matchings of a Hasse diagram of a poset parallels the notion of perfect matchings in the Bruhat graph [19]. Can this be extended to balanced graphs? Morel [45] has given a geometric interpretation of Brenti's [17] non-recursive lattice path formulation of the Kazhdan–Lusztig polynomials in the case of Weyl groups. This may give some insight into Kazhdan–Lusztig polynomials of directed graphs.

In Billera and Brenti's original paper, restricting the **cd**-index of Bruhat graphs to the highest degree terms yields the **cd**-index of the Eulerian poset $[u, v]$. Understanding the degree restricted **cd**-index may suggest a natural decomposition of the paths in the Bruhat graph. For dihedral Coxeter systems Blanco showed the **cd**-index is given by the Fibonacci polynomials [15]. This gives evidence for non-negativity of the **cd**-index for Bruhat graphs.

Billera and Brenti's [7] expression for the Kazhdan–Lusztig polynomial $P_{u,v}(q)$ in terms of the **cd**-index $\Psi([u, v])$ was based upon showing the generalized Dehn–Sommerville relations hold for coefficients of polynomials arising in Kazhdan–Lusztig polynomials [17, Theorem 8.4], quasisymmetric functions, and the peak algebra. Their expression is

$$P_{u,v}(q) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_i \cdot q^i \cdot \mathcal{B}_{n-2i}(-q), \tag{9.1}$$

where $\mathcal{B}_k(q)$ is the kth ballot polynomial $\mathcal{B}_k(q) = 1/(k+1) \cdot \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{k+1}{i} \cdot (k+1-2i) \cdot q^i$ and the coefficients a_i arise out of a nontrivial computation from the coefficients of $\Psi([u, v])$. Although a remarkable identity, it does not reveal why $P_{u,v}(q)$ is non-negative for Coxeter groups.

As was noted in [7], restricting the identity (9.1) for the coefficients a_i to the highest degree **cd**-monomials yields the Bayer–Ehrenborg [2] expression for the g-polynomial of $\Psi([u, v])^*$, implying the difference $P_{u,v}(q) - g([u, v]^*, q)$ is a function of lower degree **cd**-coefficients; see [7, Remark 4.2]. Brenti and Caselli [18] have a new identity for the Kazhdan–Lusztig polynomials in terms of signed polynomials arising from an index set of what they call “slalom paths” associated to a sparse subset $T \subseteq \{1, \ldots, n-1\}$. This may be related to the Bayer–Ehrenborg S-diagram approach; see [2, Section 7]. Other recent papers describing the **cd**-index as weighted sum of lattice paths include that of Slone [47] for the **cd**-index of the mixing operator, and B. Fox [37] for the the **cd**-index of the diamond product of two Eulerian posets. On the polytope level, these poset operations correspond to taking the join of polytopes and the Cartesian product of polytopes, respectively. We expect these ideas to be fruitful for developing the Kazhdan–Lusztig polynomial of certain balanced graphs.

Reading [46] provided a recursive method to compute **cd**-index of any interval in the Bruhat order. Can his methods be extended to any interval in the Bruhat graph? Do they generalize to balanced graphs?

Returning our attention to graded posets and especially Eulerian posets, when do these posets possess a labeling? There are Eulerian posets which do not have an R-labeling; see [31]. What more can be said about Eulerian posets that have an R-labeling? Conjecture 8.4 implies that Eulerian posets with an R-labeling have a non-negative **cd**-index.

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