Coproducts and the **cd**-index*

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Dedicated to Gian-Carlo Rota on his 2^6th birthday.

Abstract

The linear span of isomorphism classes of posets, $\mathcal{P}$, has a Newtonian coalgebra structure. We observe that the **ab**-index is a Newtonian coalgebra map from the vector space $\mathcal{P}$ to the algebra of polynomials in the non-commutative variables **a** and **b**. This enables us to obtain explicit formulas showing how the **cd**-index of the face lattice of a convex polytope changes when taking the pyramid and the prism of the polytope and the corresponding operations on posets. As a corollary, we have new recursion formulas for the **cd**-index of the Boolean algebra and the cubical lattice. Moreover, these operations also have interpretations for certain classes of permutations, including simsun and signed simsun permutations. We prove an identity for the shelling components of the simplex. Lastly, we show how to compute the **ab**-index of the Cartesian product of two posets given the **ab**-indexes of each poset.

1 Introduction

The **cd**-index is an efficient way to encode the flag f-vector (equivalently the flag h-vector) of an Eulerian poset. It gives an explicit basis for the generalized Dehn-Sommerville equations, also known as the Bayer-Billera relations [1]. An important example of an Eulerian poset is the face lattice of a convex polytope.

In this paper we study how the **cd**-index of the face lattice of a convex polytope changes after applying each of the following geometric operations to the convex polytope itself: taking the pyramid, taking the prism, truncating at a vertex, and pasting two polytopes together at a common facet. All four of these operations act on the face lattice of the polytope. The change in the **cd**-index from the pasting operation follows from a result of Stanley [20, Lemma 2.1]. Similarly the change from truncating at a vertex follows from the same result of Stanley and the pyramid and prism operations.

To understand how the **cd**-index changes under the prism and pyramid operations, we consider $\mathcal{P}$, the linear span of isomorphism classes of graded posets. This vector space is an algebra under the star product $*$ of posets, first described by Stanley [20]. More importantly, $\mathcal{P}$ has a coalgebra structure. The pair formed by the star product $*$ and the coproduct Δ do not form a bialgebra, but instead a

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Newtonian coalgebra, a concept introduced by Joni and Rota [13]. The main observation we make is that the **cd**-index is a Newtonian coalgebra map from the vector space $\mathcal{E}$ spanned by all isomorphism classes of Eulerian posets to the algebra $\mathcal{F}$ of polynomials in the non-commutative variables **c** and **d**. We thus obtain that the prism operation corresponds to a certain derivation D on **cd**-polynomials, and the pyramid operation corresponds to a second derivation G. Hence given the **cd**-index of a polytope, we may easily compute the **cd**-index of the prism and the pyramid of the polytope with the help of these two derivations. Using these two derivations, we obtain new explicit recursion formulas for the **cd**-index of the Boolean algebra B_n and the cubical lattice C_n.

There is a relation between the **cd**-index of the Boolean algebra B_n and certain classes of permutations. For instance, the **cd**-index of B_n is a refined enumeration of André permutations [17]. Similarly, it is also a refined enumeration of simsun permutations, first defined by Simion and Sundaram [22, 23]. Another known example of a poset–permutations pair is the cubical lattice and signed André permutations [8, 17]. This motivates us to ask the following question. Given an Eulerian poset P, is it possible to find a canonical class of permutations which correspond to the **cd**-index of the poset P? We show that given a poset–permutations pair (P, T), we can construct a class of permutations corresponding to the pyramid of P. A similar signed result holds for the prism of P. The simsun permutations may be built up by repeated use of this correspondence. Also, we define signed simsun permutations, which correspond to the cubical lattice C_n.

In [20] Stanley studies the shelling components of a simplex and their **cd**-indexes, given by a sum of $\check{\Phi}_i^n$'s. Using our techniques, we obtain a recursion formula for $\check{\Phi}_i^n$. As a corollary to this recursion we prove a version of Stanley's conjecture [20, Conjecture 3.1] concerning the correspondence between simsun permutations and the $\check{\Phi}_i^n$'s.

In Section 9 we consider the problem of computing the **ab**-index of the Cartesian product of two posets knowing the **ab**-indexes of each poset. This problem is solved by introducing mixing operators. The theorem of this section is motivated by the case where each poset has an R-labeling. As a corollary we obtain that for two convex polytopes U and V we can compute the **ab**-index of their Cartesian product in terms of their **ab**-indexes.

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2 Newtonian coalgebras

Let k be a field of characteristic 0. Let V be a vector space over the field k. A product on the vector space V is a linear map $\mu : V \otimes V \longrightarrow V$. The product μ is associative if $\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$. Similarly, a coproduct on the vector space V is a linear map $\Delta : V \longrightarrow V \otimes V$. The coproduct Δ is coassociative if $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$.

Definition 2.1 *Let V be a vector space with an associative product μ and a coassociative coproduct Δ. We call the triplet (V, μ, Δ) a* Newtonian coalgebra *if it satisfies the identity*

$$\Delta \circ \mu = (1 \otimes \mu) \circ (\Delta \otimes 1) + (\mu \otimes 1) \circ (1 \otimes \Delta).$$

The Sweedler notation of a coproduct Δ is to write $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$; see [24, pages 10-11]. Then the Newtonian condition may be written

$$\Delta(x \cdot y) = \sum_x x_{(1)} \otimes (x_{(2)} \cdot y) + \sum_y (x \cdot y_{(1)}) \otimes y_{(2)}.$$

Observe that this identity is a generalization of the product rule for a derivative. In fact, for any element $v \in V$, the linear map $x \longmapsto D_v(x) = \sum_x x_{(1)} \cdot v \cdot x_{(2)}$ is a derivative on the algebra (V, μ). That is, $D_v(x \cdot y) = D_v(x) \cdot y + x \cdot D_v(y)$, or $D_v \circ \mu = \mu \circ (D_v \otimes 1 + 1 \otimes D_v)$.

The definition of Newtonian coalgebra originated from Joni and Rota [13] under the name infinitesimal coalgebra. Our definition is from [7]. The first in–depth study of a Newtonian coalgebra was by Hirschhorn and Raphael [12], who studied the coalgebra on $\mathrm{k}[x]$ where the coproduct is given by $\Delta(x^n) = \sum_{i+j=n-1} x^i \otimes x^j$. In this section we will introduce two important examples of Newtonian coalgebras, which we denote by $\mathcal{A}$ and $\mathcal{P}$. These two examples appear in [7].

Let $\mathcal{A} = \mathrm{k}\langle \mathbf{a}, \mathbf{b} \rangle$ be the polynomial algebra in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$. Let the product on $\mathcal{A}$ be the ordinary multiplication. Define the coproduct Δ on a monomial $v_1 \cdot v_2 \cdots v_n$ by

$$\Delta(v_1 \cdot v_2 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n.$$

It is easy to see that this is a Newtonian coalgebra. The Newtonian coalgebra $\mathcal{A}$ is naturally graded, that is, we may write $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$, where $\mathcal{A}_n$ is spanned by monomials of degree n. Then $\dim(\mathcal{A}_n) = 2^n$ and we have $\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ and $\Delta(\mathcal{A}_n) \subseteq \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$.

Lemma 2.2 *Consider the coproduct Δ as a linear map $\Delta : \mathcal{A}_n \longrightarrow \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$. The kernel of Δ is one-dimensional and is spanned by the element $(\mathbf{a} - \mathbf{b})^n$.*

Proof: Let x be an element in $\mathcal{A}_n$ such that $\Delta(x) = 0$. Assume that $x = \sum_w \alpha_w \cdot w$, where w ranges over all monomials in $\mathcal{A}_n$.

Let $u \in \mathcal{A}_i$ and $v \in \mathcal{A}_j$, where $i + j = n - 1$. The only way to obtain the term $u \otimes v$ in a coproduct is by applying the coproduct to either $u \cdot \mathbf{a} \cdot v$ or $u \cdot \mathbf{b} \cdot v$. Hence consider the coefficient of $u \otimes v$ in $\Delta(x) = 0$. Then we obtain the identity $\alpha_{u \cdot \mathbf{a} \cdot v} + \alpha_{u \cdot \mathbf{b} \cdot v} = 0$. Since this identity holds for all u and v, we get

$$\alpha_w = (-1)^{\text{number of } \mathbf{b}\text{'s in } w} \cdot \alpha_{\mathbf{a}^n},$$

which completes the proof. $\square$

The linear map $\Delta : \mathcal{A}_n \longrightarrow \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$ is not surjective for $n \geq 2$, because $\dim(\Delta(\mathcal{A}_n)) = 2^n - 1$ and $\dim\left(\bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j\right) = n \cdot 2^{n-1} > 2^n - 1$.

We will now consider graded posets P whose minimal element differs from its maximal element. Hence the rank of such a poset is at least 1. (See [19] for terminology on posets.) If two posets are isomorphic we say that they have the same *type*. We denote the type of a poset P by $\overline{P}$. Let $\mathcal{P}$ be the vector space over the field k spanned by all types of posets.

We define a coproduct on the vector space $\mathcal{P}$ by

$$\Delta(\overline{P}) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \overline{[\hat{0}, x]} \otimes \overline{[x, \hat{1}]},$$

and extend this definition by linearity. Observe that this coproduct differs from the ordinary coproduct that is defined on the reduced incidence Hopf algebra of posets; see [5, 13, 18].

Let P and Q be two graded posets. We define their *star product*, $R = P * Q$, by letting R be the set $(P - \{\hat{1}\}) \cup (Q - \{\hat{0}\})$ and defining the order relation on R by $x \leq_R y$ if (*i*) $x, y \in P$ and $x \leq_P y$, (*ii*) $x, y \in Q$ and $x \leq_Q y$, or (*iii*) $x \in P$ and $y \in Q$. This product was first mentioned in [20]. Observe that the rank of the poset $P * Q$ is given by $\rho(P) + \rho(Q) - 1$. The product $*$ extends naturally to a product on $\mathcal{P}$.

Proposition 2.3 (Ehrenborg and Hetyei) *The triplet $(\mathcal{P}, *, \Delta)$ is a Newtonian coalgebra.*

This Newtonian algebra has a natural grading, $\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n$, where $\mathcal{P}_n$ is the linear span of types of graded posets of rank $n+1$. Then we have $\mathcal{P}_i * \mathcal{P}_j \subseteq \mathcal{P}_{i+j}$ and $\Delta(\mathcal{P}_n) \subseteq \bigoplus_{i+j=n-1} \mathcal{P}_i \otimes \mathcal{P}_j$.

There are two other products on posets that we consider. First, there is the *Cartesian product* of posets, which we denote by $P \times Q$, defined as $\{(x, y) \; : \; x \in P \text{ and } y \in Q\}$, with the order relation given by $(x, y) \leq_{P \times Q} (z, w)$ if and only if $x \leq_P z$ and $y \leq_Q w$. Secondly, define the *diamond product* by $P \diamond Q = (P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$. The diamond product corresponds to the Cartesian product of convex polytopes, that is, $\mathcal{L}(V \times W) = \mathcal{L}(V) \diamond \mathcal{L}(W)$, where V and W are two convex polytopes and $\mathcal{L}(V)$ denotes the face lattice of V. Both of these products on posets extend naturally to the linear space $\mathcal{P}$, and we have that $\mathcal{P}_i \times \mathcal{P}_j \subseteq \mathcal{P}_{i+j+1}$ and $\mathcal{P}_i \diamond \mathcal{P}_j \subseteq \mathcal{P}_{i+j}$.

3 The cd-index of Eulerian posets

To each graded poset P we will assign a non-commutative polynomial in the variables $\mathbf{a}$ and $\mathbf{b}$ called the $\mathbf{ab}$-index. Let P be a graded poset of rank $n+1$. To every chain $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ of the poset P we associate a *weight* $w_P(c) = w(c) = z_1 \cdots z_n$, where

$$z_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

Observe that the chain $\{\hat{0} < \hat{1}\}$ receives the weight $(\mathbf{a} - \mathbf{b})^n$ and a maximal chain has weight $\mathbf{b}^n$. Note also that the degree of the weight $w(c)$ is n. Define the $\mathbf{ab}$-*index* of the poset P to be the sum

$$\Psi(P) = \sum_c w(c),$$

where c ranges over all chains $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ in the poset P.

By linearity we may extend the map Ψ to a linear map $\Psi : \mathcal{P} \longrightarrow \mathcal{A}$.

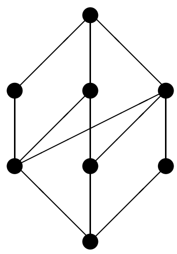

Figure 1: A non-Eulerian poset P with $\Psi(P) \in \mathcal{F}$.

Proposition 3.1 *The linear map $\Psi : \mathcal{P} \longrightarrow \mathcal{A}$ is a Newtonian coalgebra map. That is, $\Psi \circ \mu = \mu \circ (\Psi \otimes \Psi)$ and $\Delta \circ \Psi = (\Psi \otimes \Psi) \circ \Delta$.*

Proof: The first identity is equivalent to $\Psi(P * Q) = \Psi(P) \cdot \Psi(Q)$, for two posets P and Q. This is due to Stanley; see [20, Lemma 1.1]. In terms of posets, the second identity says that

$$\Delta(\Psi(P)) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \otimes \Psi([x, \hat{1}]). \tag{3.1}$$

Observe that the coproduct of the weight of a chain $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ is given by

$$\Delta(w(c)) = \sum_{i=1}^{k} w_{[\hat{0}, x_i]}(\{\hat{0} < x_1 < \cdots < x_i\}) \otimes w_{[x_i, \hat{1}]}(\{x_i < x_{i+1} < \cdots < x_k < \hat{1}\}).$$

Equation (3.1) follows now by summing over all chains c and regrouping the terms. $\square$

Recall that a poset P is *Eulerian* if the Möbius function μ on any interval $[x, y]$ in P is given by $\mu(x, y) = (-1)^{\rho(x,y)}$. Let $\mathcal{E}$ be the subspace of $\mathcal{P}$ spanned by all types of Eulerian posets. It is easy to see that $\mathcal{E}$ is closed under the product $*$ and the coproduct Δ. Hence $\mathcal{E}$ forms a Newtonian subalgebra of $\mathcal{P}$. The subspace $\mathcal{E}$ is also closed under the Cartesian product and the diamond product.

Fine observed that the **ab**-index of an Eulerian poset may be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$; see [2]. When the **ab**-index can be written as a polynomial in **c** and **d**, we call this polynomial the **cd**-*index*. See Stanley [20] for an elementary proof of this fact.

For a poset of P of rank $n + 1$, the coefficient of $\mathbf{a}^n$ in the **ab**-index $\Psi(P)$ is always equal to 1. Hence the coefficient of $\mathbf{c}^n$ in the **cd**-index of Eulerian poset is always equal to 1.

Let $\mathcal{F}$ be the subalgebra of $\mathcal{A}$ spanned by the elements **c** and **d**. $\mathcal{F}$ is closed under the coproduct Δ, since $\Delta(\mathbf{c}) = \Delta(\mathbf{a}+\mathbf{b}) = 1 \otimes 1 + 1 \otimes 1 = 2 \cdot 1 \otimes 1$ and $\Delta(\mathbf{d}) = \Delta(\mathbf{ab}+\mathbf{ba}) = \mathbf{a} \otimes 1 + 1 \otimes \mathbf{b} + \mathbf{b} \otimes 1 + 1 \otimes \mathbf{a} = \mathbf{c} \otimes 1 + 1 \otimes \mathbf{c}$. The Newtonian coalgebra $\mathcal{F}$ inherits the grading from $\mathcal{A}$. That is, $\mathcal{F} = \bigoplus_{n \geq 0} \mathcal{F}_n$, where $\mathcal{F}_n \subseteq \mathcal{A}_n$. Since $\dim(\mathcal{F}_0) = \dim(\mathcal{F}_1) = 1$ and $\mathcal{F}_n = \mathbf{c} \cdot \mathcal{F}_{n-1} + \mathbf{d} \cdot \mathcal{F}_{n-2}$ one has $\dim(\mathcal{F}_n) = f_{n+1}$, where f_n is the nth Fibonacci number. (Recall f_n is defined recursively by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$.)

The important observation to make here is that the linear map $\Psi : \mathcal{P} \longrightarrow \mathcal{A}$ restricts to a linear map from the Newtonian coalgebra $\mathcal{E}$ to the Newtonian coalgebra $\mathcal{F}$. Note that there exist posets which are not Eulerian, but whose **ab**-index may be expressed in terms of **c** and **d**. For example, the poset P in Figure 1 is not Eulerian, but $\Psi(P) = \mathbf{c}^2 + \mathbf{d} \in \mathcal{F}$.

The *dual* poset of a poset P is the poset P^* where the order relation is defined by $x \leq_{P^*} y$ if and only if $y \leq_P x$. This is an involution of posets which extends to an involution ω on the linear space $\mathcal{P}$. The map ω is also an involution on $\mathcal{E}$. Define an involution on $\mathcal{A}$, also denoted ω, by $\omega(v_1 \cdot v_2 \cdots v_n) = v_n \cdots v_2 \cdot v_1$, and extend it linearly to all of $\mathcal{A}$. This is also an involution on $\mathcal{F}$. It is easy see that ω commutes with the linear map Ψ, that is, $\omega \circ \Psi = \Psi \circ \omega$. Moreover, in our four Newtonian coalgebras $\mathcal{A}$, $\mathcal{F}$, $\mathcal{P}$, and $\mathcal{E}$, we have the two relations $\omega \circ \mu = \mu \circ \sigma \circ (\omega \otimes \omega)$ and $\Delta \circ \omega = (\omega \otimes \omega) \circ \sigma \circ \Delta$, where $\sigma(x \otimes y) = y \otimes x$.

Let V be a convex polytope. Then the face lattice of V, $\mathcal{L}(V)$, is an Eulerian poset. Hence we may compute the **cd**-index of $\mathcal{L}(V)$, that is, $\Psi(\mathcal{L}(V))$. For the remainder of this paper we will write $\Psi(V)$ instead of the more cumbersome $\Psi(\mathcal{L}(V))$. For a polytope V, the *polar* (or *dual*) polytope is denoted by V^Δ; see [26]. We have $\mathcal{L}(V^\Delta) = \mathcal{L}(V)^*$. Hence directly we obtain $\Psi(\mathcal{L}(V^\Delta)) = \Psi(\mathcal{L}(V))^*$, or in our shorthand, $\Psi(V^\Delta) = \Psi(V)^*$.

As two examples of **cd**-indexes of polytopes, the **cd**-index of a polygon V is given by

$$\Psi(V) = \mathbf{c}^2 + (f_0 - 2) \cdot \mathbf{d}, \tag{3.2}$$

and the **cd**-index of a three-dimensional polytope V is given by

$$\Psi(V) = \mathbf{c}^3 + (f_0 - 2) \cdot \mathbf{dc} + (f_2 - 2) \cdot \mathbf{cd}, \tag{3.3}$$

where f_0 denotes the number of vertices and f_2 the number of two-dimensional faces of the polytope.

Recall that $(\mathbf{a} - \mathbf{b})^2 = (\mathbf{c}^2 - 2 \cdot \mathbf{d})$. Hence when n is even, the element $(\mathbf{a} - \mathbf{b})^n$ belongs to $\mathcal{F}_n$. Given an element x in $\mathcal{F}_n$, expand it in terms of **a** and **b**. The coefficient of $\mathbf{a}^n$ and $\mathbf{b}^n$ in x both equal the coefficient of $\mathbf{c}^n$ in x. When n is odd, the coefficients of $\mathbf{a}^n$ and $\mathbf{b}^n$ in $(\mathbf{a} - \mathbf{b})^n$ differ, and hence $(\mathbf{a} - \mathbf{b})^n$ does not belong to $\mathcal{F}_n$. Thus we have the following result.

Corollary 3.2 *Consider the linear map $\Delta : \mathcal{F}_n \longrightarrow \bigoplus_{i+j=n-1} \mathcal{F}_i \otimes \mathcal{F}_j$. When n is odd, the linear map Δ is injective. When n is even, the kernel of the linear map Δ is one-dimensional and is spanned by the element $(\mathbf{c}^2 - 2 \cdot \mathbf{d})^{\frac{n}{2}}$.*

Given an Eulerian poset P, how do we compute its **cd**-index? If we instead consider the larger problem of determining the **cd**-index of all of the intervals of the poset P, the coproduct suggests an algorithm for this computation.

Assume that we know $\Delta(\Psi(P))$ for an Eulerian poset P. Recall the coefficient of $\mathbf{c}^{\rho(P)-1}$ in $\Psi(P)$ is equal to 1. Let $\mathcal{F}'_n$ be the linear span of all **cd**-monomials of degree n which contain at least one **d**, that is, we have excluded the monomial $\mathbf{c}^n$. So $\dim(\mathcal{F}'_n) = \dim(\mathcal{F}_n) - 1$. By Corollary 3.2, the kernel

set $w \leftarrow 0$
for $r = 0$ to $\lfloor n/2 \rfloor - 1$ do
  for $s = 0$ to $n - 2r - 1$ do
    set $v \leftarrow [\mathbf{c}^s \otimes]\ \delta_r(p)$
    if v is not divisible on the left by $\mathbf{c}$
      then terminate since p is not in $\Delta(\mathcal{F}'_n)$
      else $u \leftarrow \mathbf{c}^{-1} \cdot v$
    set $w \leftarrow w + \mathbf{c}^s\mathbf{d} \cdot u$
    set $p \leftarrow p - \Delta(\mathbf{c}^s\mathbf{d} \cdot u)$
  if $\delta_r(p) \neq 0$
    then terminate since p is not in $\Delta(\mathcal{F}'_n)$
return(w)

Figure 2: An algorithm for computing $\Delta^{-1}(p)$, where p belongs to $\Delta(\mathcal{F}'_n)$.

of Δ does not belong to the linear space $\mathcal{F}'_n$, hence the restricted map $\Delta : \mathcal{F}'_n \longrightarrow \oplus_{i+j=n-1}\mathcal{F}_i \otimes \mathcal{F}_j$ is injective.

In order to present a method to compute the inverse of Δ, we need to introduce some terminology and notation. We say that a $\mathbf{cd}$-polynomial v is *divisible on the left* by $\mathbf{c}$ if there exists a $\mathbf{cd}$-polynomial u such that $v = \mathbf{c} \cdot u$. If u exists, we will write $u = \mathbf{c}^{-1} \cdot v$. For z in $\mathcal{F}_n$, define $\delta_r(z)$ to be all of the terms of z that contain exactly r number of $\mathbf{d}$'s. An example is $\delta_1(\mathbf{c}^3 + 6 \cdot \mathbf{dc} + 4 \cdot \mathbf{cd}) = 6 \cdot \mathbf{dc} + 4 \cdot \mathbf{cd}$. Similarly, for p in $\oplus_{i+j=n-1}\mathcal{F}_i \otimes \mathcal{F}_j$ define $\delta_r(p)$ to be all of the terms of p that contain exactly r number of $\mathbf{d}$'s. In both cases the linear map δ_r is a projection. Finally, we write $[\mathbf{c}^s \otimes]\, p$ for the sum of all monomials v such that the term $\mathbf{c}^s \otimes v$ occurs in p. For example, for $p = 2 \cdot \mathbf{cd} \otimes \mathbf{c} + 3 \cdot \mathbf{c}^2 \otimes \mathbf{d} - \mathbf{d} \otimes \mathbf{d}$ we have $\delta_1(p) = 2 \cdot \mathbf{cd} \otimes \mathbf{c} + 3 \cdot \mathbf{c}^2 \otimes \mathbf{d}$ and $[\mathbf{c}^2 \otimes]\, p = 3 \cdot \mathbf{d}$.

Proposition 3.3 *The algorithm in Figure 2 computes the inverse of the linear map* $\Delta : \mathcal{F}'_n \longrightarrow \oplus_{i+j=n-1}\mathcal{F}_i \otimes \mathcal{F}_j$. *That is, given* p *in* $\Delta(\mathcal{F}'_n)$, *the algorithm computes* $\Delta^{-1}(p)$.

The proof of this proposition is an induction argument on the pair (r, s). The induction step from (r, s) to $(r, s + 1)$ is given by the lemma:

Lemma 3.4 *At step* (r, s) *in the algorithm in Figure 2, let* $z \in \mathcal{F}'_n$ *be the element such that* $\Delta(z) = p$. *Assume that the element* z *satisfies:*

(i) $\delta_i(z) = 0$ *for all* $0 \leq i < r$,

(ii) $[\mathbf{c}^j\mathbf{d} \cdot y]\, z = 0$ *for all* $0 \leq j < s$ *and for all* $\mathbf{cd}$*-monomials* y *such that* $\deg_{\mathbf{d}}(y) = r - 1$.

Then the element $z - \mathbf{c}^s\mathbf{d} \cdot u$ *satisfies* $[\mathbf{c}^s\mathbf{d} \cdot y]\, (z - \mathbf{c}^s\mathbf{d} \cdot u) = 0$ *for all* $\mathbf{cd}$*-monomials* y *such that* $\deg_{\mathbf{d}}(y) = r - 1$.

for all intervals $[x, y]$ of length 1
  set $\Psi([x, y]) \leftarrow 1$
for all intervals $[x, y]$ of length 2
  set $\Psi([x, y]) \leftarrow \mathbf{c}$
for $k = 3$ to $\rho(P)$ do
  for all intervals $[x, y]$ of length k
    set $\Psi([x, y]) \leftarrow \Delta^{-1} \left(\sum_{x<z<y} \Psi([x, z]) \otimes \Psi([z, y]) \right)$

Figure 3: An algorithm for computing the **cd**-index of all intervals of an Eulerian poset P.

First observe that when the term $x \otimes y$ occurs in $\Delta(z)$, where x and y are monomials, then one of the following three cases holds: the term $x \cdot \mathbf{c} \cdot y$ occurs in z, the term $x' \cdot \mathbf{d} \cdot y$ occurs in z with $x = x' \cdot \mathbf{c}$, or the term $x \cdot \mathbf{d} \cdot y'$ occurs in z with $y = \mathbf{c} \cdot y'$.

Consider the term $\mathbf{c}^s \otimes v$ in p. This term has exactly r $\mathbf{d}$'s and it is in the image of a monomial with either r or $r+1$ $\mathbf{d}$'s. The case with r $\mathbf{d}$'s is not possible by our first assumption on z. If v contains a monomial that begins with a $\mathbf{d}$, say $\mathbf{d} \cdot x$, then the term $\mathbf{c}^s \otimes \mathbf{d} \cdot x$ can only occur in $\Delta(\mathbf{c}^{s-1}\mathbf{dd}x)$, contradicting our second assumption on z. Hence we conclude that v is divisible on the left by $\mathbf{c}$. The term $\mathbf{c}^s \otimes v = \mathbf{c}^s \otimes \mathbf{c} \cdot u$ does occur in $\Delta(\mathbf{c}^s\mathbf{d} \cdot u)$. Now consider the expression $p - \Delta(\mathbf{c}^s\mathbf{d} \cdot u)$, which is equal to $\Delta(z - \mathbf{c}^s\mathbf{d} \cdot u)$. No monomial in $z - \mathbf{c}^s\mathbf{d} \cdot u$ is of the form $\mathbf{c}^s\mathbf{d} \cdot y$. This completes the proof of the lemma.

The suggested algorithm to compute the **cd**-index of a poset P and all of its intervals is presented in Figure 3.

4 The pyramid and the prism of a polytope

There are two well-known operations defined on convex polytopes: the pyramid and the prism. From a convex polytope V, we may construct the pyramid of V, $\mathrm{Pyr}(V)$, and the prism of V, $\mathrm{Prism}(V)$. Let B_n be the Boolean algebra of rank n, that is, the face lattice of the simplex of dimension $n-1$. Also let C_n be the cubical lattice of rank $n+1$, that is, the face lattice of an n-dimensional cube.

Proposition 4.1 *Let V be a convex polytope. Then the face lattice of the pyramid of V and the face lattice of the prism of V are given by*

$$\mathcal{L}(\mathrm{Pyr}(V)) = \mathcal{L}(V) \times B_1 \quad \textit{and} \quad \mathcal{L}(\mathrm{Prism}(V)) = \mathcal{L}(V) \diamond B_2.$$

These two identities follow from observations of Kalai [14, section 2].

We define the two operations pyramid and prism on a poset P by $\mathrm{Pyr}(P) = P \times B_1$ and $\mathrm{Prism}(P) = P \diamond B_2$. Two natural questions occur now. Given the **cd**-index $\Psi(V)$, are we able to compute $\Psi(\mathrm{Pyr}(V))$ and $\Psi(\mathrm{Prism}(V))$.

Proposition 4.2 *Let P be a graded poset. Then*

$$\begin{aligned}\Psi(\mathrm{Pyr}(P)) &= \frac{1}{2}\left[\Psi(P)\cdot\mathbf{c}+\mathbf{c}\cdot\Psi(P)+\sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}}\Psi([\hat{0},x])\cdot\mathbf{d}\cdot\Psi([x,\hat{1}])\right],\\ \Psi(\mathrm{Prism}(P)) &= \Psi(P)\cdot\mathbf{c}+\sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}}\Psi([\hat{0},x])\cdot\mathbf{d}\cdot\Psi([x,\hat{1}]).\end{aligned}$$

Proof: The proof of the second identity will follow as a special case of Proposition 4.3. The first identity follows by a careful chain argument. Consider a chain c in $P\times B_1$. We have $c=\{(\hat{0},\hat{0})=(x_0,y_0)<(x_1,y_1)<\cdots<(x_k,y_k)=(\hat{1},\hat{1})\}$. Let i be the smallest index such that $y_i=\hat{1}$. Let $x=x_i$. Hence we have $y_0=\cdots=y_{i-1}=\hat{0}$ and $y_i=\cdots=y_k=\hat{1}$. Moreover we have $x_{i-1}\le x_i$, and the two chains $c_1=\{\hat{0}=x_0<x_1<\cdots<x_{i-1}\le x\}$ in $[\hat{0},x]$ and $c_2=\{x<x_{i+1}<\cdots<x_k=\hat{1}\}$ in $[x,\hat{1}]$.

Three cases occur:

1. $\hat{0}<x<\hat{1}$. Then the element $(x,\hat{0})$ may or may not be in the chain c. Let c' denote the chain $c-\{(x,\hat{0})\}$, that is, the chain without the element $(x,\hat{0})$. Similarly, let c'' denote the chain $c\cup\{(x,\hat{0})\}$, that is, the chain with the element $(x,\hat{0})$. Observe that the element $(x,\hat{1})$ belongs to both the chains c' and c'', so the weight of these chains at rank $\rho(x)+1$ is $\mathbf{b}$. Hence we have

$$\begin{aligned}w(c') &= w_{[\hat{0},x]}(c_1)\cdot(\mathbf{a}-\mathbf{b})\cdot\mathbf{b}\cdot w_{[x,\hat{1}]}(c_2),\\ w(c'') &= w_{[\hat{0},x]}(c_1)\cdot\mathbf{b}\cdot\mathbf{b}\cdot w_{[x,\hat{1}]}(c_2),\\ w(c')+w(c'') &= w_{[\hat{0},x]}(c_1)\cdot\mathbf{a}\cdot\mathbf{b}\cdot w_{[x,\hat{1}]}(c_2).\end{aligned}$$

2. $x=\hat{1}$. Then the element $(\hat{1},\hat{0})$ may or may not be in the chain c. Let c' be the chain $c-\{(\hat{1},\hat{0})\}$ and let c'' be the chain $c\cup\{(\hat{1},\hat{0})\}$. Then

$$\begin{aligned}w(c') &= w_P(c_1)\cdot(\mathbf{a}-\mathbf{b}),\\ w(c'') &= w_P(c_1)\cdot\mathbf{b},\\ w(c')+w(c'') &= w_P(c_1)\cdot\mathbf{a}.\end{aligned}$$

3. $x=\hat{0}$. Then the element $(\hat{0},\hat{1})$ lies in the chain c, and the weight of the chain c is

$$w(c)=\mathbf{b}\cdot w_P(c_2).$$

Summing over all chains c in $P\times B_1$, we obtain

$$\Psi(P\times B_1)=\mathbf{b}\cdot\Psi(P)+\Psi(P)\cdot\mathbf{a}+\sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}}\Psi([\hat{0},x])\cdot\mathbf{a}\cdot\mathbf{b}\cdot\Psi([x,\hat{1}]). \tag{4.1}$$

Applying equation (4.1) to the poset P^* gives

$$\Psi(P^*\times B_1)=\mathbf{b}\cdot\Psi(P^*)+\Psi(P^*)\cdot\mathbf{a}+\sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}}\Psi([x,\hat{1}]^*)\cdot\mathbf{a}\cdot\mathbf{b}\cdot\Psi([\hat{0},x]^*).$$

Now applying the involution ω, we obtain

$$\Psi(P \times B_1) = \Psi(P) \cdot \mathbf{b} + \mathbf{a} \cdot \Psi(P) + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \cdot \mathbf{b} \cdot \mathbf{a} \cdot \Psi([x, \hat{1}]). \tag{4.2}$$

Adding equations (4.1) and (4.2) gives the desired result. $\square$

Since $C_{n+1} = \mathrm{Prism}(C_n)$, Proposition 4.2 gives a recursion formula for the **cd**-index of the cubical lattice C_n. This recursion was first developed by Purtill [17]. The second part of Proposition 4.2 may be generalized in the following manner. Let A_r be a graded poset of rank 2 which has r atoms (which are also coatoms). Note that $A_2 = B_2$. Let $\overline{\mathbf{c}}_r = \mathbf{a} + (r-1) \cdot \mathbf{b}$ and $\overline{\mathbf{d}}_r = \mathbf{ab} + (r-1) \cdot \mathbf{ba}$.

Proposition 4.3 *Let P be a graded poset. Then*

$$\Psi(P \diamond A_r) = \Psi(P) \cdot \overline{\mathbf{c}}_r + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \cdot \overline{\mathbf{d}}_r \cdot \Psi([x, \hat{1}]).$$

Proof: Denote the r atoms in A_r by $1, \ldots, r$. Consider a chain c in $P \diamond A_r$. We have $c = \{\hat{0} < (x_1, y_1) < \cdots < (x_k, y_k) = (\hat{1}, \hat{1})\}$. Let i be the smallest index such that $y_i = \hat{1}$. Two cases occur, each having two subcases.

1. Assume that y_{i-1} exists (that is, $i > 1$) such that $1 \leq y_{i-1} \leq r-1$. Let $x = x_{i-1} > \hat{0}$. The element $(x, \hat{1})$ may or may not be in the chain. Let c' be the chain $c - \{(x, \hat{1})\}$ and let c'' be the chain $c \cup \{(x, \hat{1})\}$. Moreover, let c_1 be the chain $\{\hat{0} < x_1 < \cdots < x_{i-2} < x\}$ and c_2 the chain $\{x < x_{i+1} < \cdots < x_{k-1} < \hat{1}\}$. The first subcase is when $x < \hat{1}$. Then the sum of the weights of the chains c' and c'' is given by

   $$w(c') + w(c'') = w_{[\hat{0},x]}(c_1) \cdot \mathbf{b} \cdot \mathbf{a} \cdot w_{[x,\hat{1}]}(c_2).$$

   The second subcase is $x = \hat{1}$. Then the weight of the chain is

   $$w(c) = w_P(c_1) \cdot \mathbf{b}.$$

   Observe in both subcases that there are $r-1$ choices for y_{i-1}.

2. Assume that either $i = 1$ (so y_{i-1} does not exist) or $y_{i-1} = r$. Let $x = x_i > \hat{0}$. Let c' be the chain $c - \{(x, r)\}$ and let c'' be the chain $c \cup \{(x, r)\}$. Moreover, let c_1 be the chain $\{\hat{0} < x_1 < \cdots < x_{i-1} < x\}$ and c_2 the chain $\{x < x_{i+1} < \cdots < x_{k-1} < \hat{1}\}$. In the first subcase when $x < \hat{1}$, we obtain that the sum of the weights of the chains c' and c'' is given by

   $$w(c') + w(c'') = w_{[\hat{0},x]}(c_1) \cdot \mathbf{a} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2).$$

   For the second subcase when $x = \hat{1}$, we similarly get that the sum of the weights of the chains c' and c'' is

   $$w(c') + w(c'') = w_P(c_1) \cdot \mathbf{a}.$$

Now summing over all chains c in $P \diamond A_r$, we obtain

$$\Psi(P \diamond A_r) = \Psi(P) \cdot (\mathbf{a} + (r-1) \cdot \mathbf{b}) + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \cdot (\mathbf{ab} + (r-1) \cdot \mathbf{ba}) \cdot \Psi([x, \hat{1}]). \qquad \square$$

Define a linear operator $D : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D(w) = \sum_{w} w_{(1)} \cdot \mathbf{d} \cdot w_{(2)}.$$

Recall that D is a derivation. We could have defined D directly as a derivation on $\mathcal{A}$ such that $D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab} + \mathbf{ba} = \mathbf{d}$. Note that D is also a derivation on $\mathcal{F}$ since $D(\mathbf{c}) = 2 \cdot \mathbf{d}$ and $D(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$.

Combining Proposition 4.2 with the fact that Ψ is a Newtonian coalgebra map, we obtain:

Theorem 4.4 *Let P be a graded poset. Then*

$$\begin{aligned} \Psi(\mathrm{Pyr}(P)) &= \frac{1}{2} [\Psi(P) \cdot \mathbf{c} + \mathbf{c} \cdot \Psi(P) + D(\Psi(P))], \\ \Psi(\mathrm{Prism}(P)) &= \Psi(P) \cdot \mathbf{c} + D(\Psi(P)). \end{aligned}$$

Similarly, let V be a convex polytope. Then

$$\begin{aligned} \Psi(\mathrm{Pyr}(V)) &= \frac{1}{2} [\Psi(V) \cdot \mathbf{c} + \mathbf{c} \cdot \Psi(V) + D(\Psi(V))], \\ \Psi(\mathrm{Prism}(V)) &= \Psi(V) \cdot \mathbf{c} + D(\Psi(V)). \end{aligned}$$

In Theorem 5.2 we will improve the formula for the pyramid.

Theorem 4.4 gives a new recursion formula for the **cd**-index of the cubical lattice C_n. Directly we have

$$\Psi(C_{n+1}) = \Psi(C_n) \cdot \mathbf{c} + D(\Psi(C_n)).$$

This is a different recursion formula than Purtill obtained in [17].

Similar to Theorem 4.4, define a derivation D_r on $\mathcal{A}$ by $D_r(\mathbf{a}) = D_r(\mathbf{b}) = \overline{\mathbf{d}}_r$. Then by Proposition 4.3 we obtain that

Corollary 4.5 *Let P be a graded poset. Then*

$$\Psi(P \diamond A_r) = \Psi(P) \cdot \overline{\mathbf{c}}_r + D_r(\Psi(P)).$$

Proposition 4.3 and Corollary 4.5 generalize the recursion for the $\mathbf{r}$-$\mathbf{cd}$-index given in [8].

Example 4.6 *Let the convex polytope V be a 3-cube with a vertex cut off. The polytope V has 10 vertices and 7 facets. By equation (3.3), the* $\mathbf{cd}$*-index of V is* $\Psi(V) = \mathbf{c}^3 + (10-2)\mathbf{dc} + (7-2)\mathbf{cd} = \mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}$. *We have*

$$\begin{aligned}\Delta(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) &= 7 \cdot \mathbf{c}^2 \otimes 1 + 15 \cdot \mathbf{c} \otimes \mathbf{c} + 10 \cdot 1 \otimes \mathbf{c}^2 + 16 \cdot \mathbf{d} \otimes 1 + 10 \cdot 1 \otimes \mathbf{d}. \\ D(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) &= 7 \cdot \mathbf{c}^2\mathbf{d} + 15 \cdot \mathbf{cdc} + 10 \cdot \mathbf{dc}^2 + 26 \cdot \mathbf{d}^2.\end{aligned}$$

Hence the $\mathbf{cd}$*-index of the prism of V is equal to*

$$\Psi(\mathrm{Prism}(V)) = \mathbf{c}^4 + 7 \cdot \mathbf{c}^2\mathbf{d} + 20 \cdot \mathbf{cdc} + 18 \cdot \mathbf{dc}^2 + 26 \cdot \mathbf{d}^2.$$

There is another operation on polytopes, namely the *bipyramid*, $\mathrm{Bipyr}(V)$. It is well-known that $\mathrm{Bipyr}(V) = \mathrm{Prism}(V^\Delta)^\Delta$. Since the involution $w \longmapsto w^*$ commutes with the derivation D, that is, $D(w^*) = D(w)^*$, we obtain:

Corollary 4.7 *For a polytope V*

$$\Psi(\mathrm{Bipyr}(V)) = \mathbf{c} \cdot \Psi(V) + D(\Psi(V)).$$

5 The derivation G

On the algebra $\mathcal{A}$ define two derivations G and G' by letting

$$\begin{aligned} G(\mathbf{a}) &= \mathbf{ba}, & G'(\mathbf{a}) &= \mathbf{ab}, \\ G(\mathbf{b}) &= \mathbf{ab}, & G'(\mathbf{b}) &= \mathbf{ba}, \end{aligned}$$

and extending G and G' to all $\mathbf{ab}$-polynomials by linearity and the product rule of derivations. Since $D(\mathbf{a}) = G(\mathbf{a}) + G'(\mathbf{a})$ and $D(\mathbf{b}) = G(\mathbf{b}) + G'(\mathbf{b})$, we obtain that $D(w) = G(w) + G'(w)$ for all $\mathbf{ab}$-polynomials w. That is, $D = G + G'$.

Observe that $G(\mathbf{c}) = G(\mathbf{a}+\mathbf{b}) = \mathbf{ba}+\mathbf{ab} = \mathbf{d}$ and $G(\mathbf{d}) = G(\mathbf{a})\cdot\mathbf{b}+\mathbf{a}\cdot G(\mathbf{b})+G(\mathbf{b})\cdot\mathbf{a}+\mathbf{b}\cdot G(\mathbf{a}) = \mathbf{bab}+\mathbf{aab}+\mathbf{aba}+\mathbf{bba} = \mathbf{cd}$. A similar computation gives $G'(\mathbf{c}) = \mathbf{d}$ and $G'(\mathbf{d}) = \mathbf{dc}$. Hence G and G' restrict to be derivations on $\mathcal{F}$.

Lemma 5.1 *For all* $\mathbf{ab}$*-monomials w, the identity $w \cdot \mathbf{c} + G(w) = \mathbf{c} \cdot w + G'(w)$ holds.*

Proof: The proof is by induction on the length of w. The base case is the three cases $w = 1$, $w = \mathbf{a}$, and $w = \mathbf{b}$. When $w = 1$, both sides are equal to $\mathbf{c}$. When $w = \mathbf{a}$, we have that $\mathbf{a} \cdot \mathbf{c} + G(\mathbf{a}) = \mathbf{a} \cdot (\mathbf{a}+\mathbf{b}) + \mathbf{ba} = (\mathbf{a}+\mathbf{b}) \cdot \mathbf{a} + \mathbf{ab} = \mathbf{c} \cdot \mathbf{a} + G'(\mathbf{a})$. A similar computation holds when $w = \mathbf{b}$.

For the induction step, consider w where w has length at least 2. We can write $w = u \cdot v$, where $u \neq w$ and $v \neq w$. By the induction hypothesis, we obtain

$$u \cdot \mathbf{c} + G(u) = \mathbf{c} \cdot u + G'(u) \quad \text{and} \quad v \cdot \mathbf{c} + G(v) = \mathbf{c} \cdot v + G'(v).$$

Multiplying the first equality on the right with v, the second equality on the left with u and then adding the two equations gives

$$u \cdot \mathbf{c} \cdot v + G(u) \cdot v + u \cdot v \cdot \mathbf{c} + u \cdot G(v) = \mathbf{c} \cdot u \cdot v + G'(u) \cdot v + u \cdot \mathbf{c} \cdot v + u \cdot G'(v).$$

By cancelling the term $u \cdot \mathbf{c} \cdot v$ and rewriting the equation using the product rule, we obtain

$$u \cdot v \cdot \mathbf{c} + G(u \cdot v) = \mathbf{c} \cdot u \cdot v + G'(u \cdot v),$$

which is the desired equality. $\square$

Theorem 5.2 *Let P be a graded poset. Then*

$$\Psi(\mathrm{Pyr}(P)) = \Psi(P) \cdot \mathbf{c} + G(\Psi(P)).$$

Similarly, let V be a convex polytope. Then

$$\Psi(\mathrm{Pyr}(V)) = \Psi(V) \cdot \mathbf{c} + G(\Psi(V)).$$

Proof: Since $\Psi(P) \in \mathcal{F}$, we know $G(\Psi(P))$ and $G'(\Psi(P))$ are well-defined. Thus by Theorem 4.4 we have

$$\begin{aligned} 2 \cdot \Psi(P \times B_1) &= \Psi(P) \cdot \mathbf{c} + \mathbf{c} \cdot \Psi(P) + D(\Psi(P)) \\ &= (\Psi(P) \cdot \mathbf{c} + G(\Psi(P))) + (\mathbf{c} \cdot \Psi(P) + G'(\Psi(P))) . \end{aligned}$$

But by Lemma 5.1 the two terms are equal. Thus we have $\Psi(P \times B_1) = \Psi(P) \cdot \mathbf{c} + G(\Psi(P)) = \mathbf{c} \cdot \Psi(P) + G'(\Psi(P))$. $\square$

This theorem gives us a new recursion formula for the $\mathbf{cd}$-index of the Boolean algebra B_n different from the one Purtill obtained in [17]. It is

$$\Psi(B_{n+1}) = \Psi(B_n) \cdot \mathbf{c} + G(\Psi(B_n)).$$

Webster [25] has found similar recursion formulas for the Boolean algebra and the cubical lattice.

Example 5.3 *Let V be the polytope in Example 4.6, with $\mathbf{cd}$-index $\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}$. We have*

$$\begin{aligned} \Gamma(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) &= 6 \cdot \mathbf{c}^2 \otimes 1 + 9 \cdot \mathbf{c} \otimes \mathbf{c} + 1 \otimes \mathbf{c}^2 + 8 \cdot \mathbf{d} \otimes 1 + 5 \cdot 1 \otimes \mathbf{d}. \\ G(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) &= 6 \cdot \mathbf{c}^2\mathbf{d} + 9 \cdot \mathbf{cdc} + \mathbf{dc}^2 + 13 \cdot \mathbf{d}^2. \end{aligned}$$

Hence the $\mathbf{cd}$-index of the pyramid of V is given by

$$\Psi(\mathrm{Pyr}(V)) = \mathbf{c}^4 + 6 \cdot \mathbf{c}^2\mathbf{d} + 14 \cdot \mathbf{cdc} + 9 \cdot \mathbf{dc}^2 + 13 \cdot \mathbf{d}^2.$$

6 Other operations on polytopes

Let W be an n-dimensional convex polytope with vertex v. Let $\mathbf{u}$ be a vector such that $W \cap \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{u} \cdot \mathbf{x} \geq c\} = \{v\}$. The vertex figure V of W at the vertex v is defined as the polytope $V = W \cap \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{u} \cdot \mathbf{x} = c - \epsilon\}$, for small enough $\epsilon > 0$. We define the truncated polytope $\widehat{W}$ as the polytope $W \cap \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{u} \cdot \mathbf{x} \leq c - \epsilon\}$. The combinatorial structure of V and $\widehat{W}$ only depends on W and v, and not on $\mathbf{u}$, c, or ϵ.

Proposition 6.1 *Let W be a convex polytope and let v be a vertex of W. Assume that the vertex figure at v is the polytope V. Let $\widehat{W}$ be the polytope W with the vertex v cut off. Then the difference in the* **cd***-index of $\widehat{W}$ and W is given by*

$$\Psi(\widehat{W}) - \Psi(W) = D(w) - G(w) = G'(w),$$

where $w = \Psi(V)$.

Sketch of proof: Stanley showed in [20, Lemma 2.1] that when we make local changes in a polytope the difference in the **cd**-indexes only depends on what happens locally. Thus it is enough to consider the case when $W = \mathrm{Pyr}(V)$ with vertex v. The vertex figure at v is V, and $\widehat{W}$ is the prism of V, $\mathrm{Prism}(V)$. Hence the difference in the **cd**-indexes is $(w \cdot \mathbf{c} + D(w)) - (w \cdot \mathbf{c} + G(w))$, and the result follows. $\Box$

Example 6.2 *Let W be a four-dimensional convex polytope such that at the vertex v it has the vertex figure V, where V is the three-dimensional polytope mentioned in Examples 4.6 and 5.3. Hence*

$$\begin{aligned} \Psi(\widehat{W}) - \Psi(W) &= D(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) - G(\mathbf{c}^3 + 8\mathbf{dc} + 5\mathbf{cd}) \\ &= \mathbf{c}^2\mathbf{d} + 6 \cdot \mathbf{cdc} + 9 \cdot \mathbf{dc}^2 + 13 \cdot \mathbf{d}^2. \end{aligned}$$

Another operation on polytopes is pasting two polytopes along a common facet. We may still speak about the face lattice and the **cd**-index of the union, even though the union may not be a convex polytope.

Lemma 6.3 (Stanley) *Let V and W be two polytopes which intersect in a facet F, that is, $V \cap W = F$. Then the* **cd***-index of the union $V \cup W$ is given by*

$$\Psi(V \cup W) = \Psi(V) + \Psi(W) - \Psi(F) \cdot \mathbf{c}.$$

Proof: Let P be the face lattice of the facet F. We rewrite the identity as $\Psi(V \cup W) + \Psi(P * B_2) = \Psi(V) + \Psi(W)$. Label the facet F in the face lattice of V by F_V and similarly label F in the face lattice of W by F_W. Moreover, label the two coatoms in $P * B_2$ by F_V and F_W. The lemma follows by noting that each chain that occurs in these four posets contributes either one term or two terms to both sides of the identity. $\square$

As a corollary we obtain:

Corollary 6.4 *Let V be a polytope and let F be a facet of V. Let $V \cup \mathrm{Pyr}(F)$ denote the polytope which is formed by making a pyramid over the facet F. Then*

$$\Psi(V \cup \mathrm{Pyr}(F)) = \Psi(V) + G(\Psi(F)).$$

The *Minkowski sum* of two subsets X and Y of $\mathbb{R}^n$ is defined as

$$X + Y = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^n \ : \ \mathbf{x} \in X, \ \mathbf{y} \in Y\}.$$

Notably, the Minkowski sum of two convex polytopes is another convex polytope. For a vector $\mathbf{x}$ we denote the set $\{\lambda \cdot \mathbf{x} \ : \ 0 \leq \lambda \leq 1\}$ by $[\mathbf{0}, \mathbf{x}]$. We say that the non-zero vector $\mathbf{x}$ lies in *general position* with respect to the convex polytope V if for each $\mathbf{u} \in \mathbb{R}^n$ the line $\{\lambda \cdot \mathbf{x} + \mathbf{u} \in \mathbb{R}^n \ : \ \lambda \in \mathbb{R}\}$ intersects the boundary of the polytope V in at most two points.

Proposition 6.5 *Let V be an n-dimensional convex polytope and $\mathbf{x}$ a non-zero vector that lies in general position with respect to the polytope V. Let H be a hyperplane orthogonal to the vector $\mathbf{x}$ and let $\mathrm{Proj}(V)$ be the orthogonal projection of V onto the hyperplane H. Observe that $\mathrm{Proj}(V)$ is an $(n-1)$-dimensional convex polytope. Then the* **cd***-index of the Minkowski sum of V and $[\mathbf{0}, \mathbf{x}]$ is given by*

$$\Psi(V + [\mathbf{0}, \mathbf{x}]) = \Psi(V) + D(\Psi(\mathrm{Proj}(V))).$$

In order to prove this proposition, we need to consider a larger class of geometric objects than convex polytopes, namely regular cell complexes. See [19, section 3.8] for more information about regular cell complexes. Note that the face lattice of a regular cell complex is an Eulerian poset.

Proof of Proposition 6.5: For each facet F_i of the polytope V choose a normal vector $\mathbf{u}_i$. Since the vector $\mathbf{x}$ lies in general position, we have that $\mathbf{x} \cdot \mathbf{u}_i \neq 0$ for all indexes i. Let S^+ be the union of all facets F_i such that $\mathbf{x} \cdot \mathbf{u}_i > 0$, and let S^- be the union of all facets F_i such that $\mathbf{x} \cdot \mathbf{u}_i < 0$. Both S^+ and S^- are homeomorphic to an $(n-1)$-dimensional ball and their boundaries agree, that is, $\partial(S^+) = \partial(S^-)$.

Let $\sigma \subseteq V$ be a closed $(n-1)$-dimensional cell such that $\partial(\sigma) = \partial(S^+)$ and for each $\mathbf{u} \in \mathbb{R}^n$ the line $\{\lambda \cdot \mathbf{x} + \mathbf{u} \in \mathbb{R}^n \ : \ \lambda \in \mathbb{R}\}$ intersects σ in at most two points. Observe that σ forms a regular cell complex and its face lattice is isomorphic to the face lattice of $\mathrm{Proj}(V)$. Moreover, $\sigma + [\mathbf{0}, \mathbf{x}]$ is also a regular cell complex whose face lattice is isomorphic to the face lattice of $\mathrm{Prism}(\mathrm{Proj}(V))$.

We may now divide the polytope V into two pieces V^+ and V^- such that $V = V^+ \cup V^-$, $\sigma = V^+ \cap V^-$, $\partial(V^+) = S^+ \cup \sigma$, and $\partial(V^-) = S^- \cup \sigma$.

We can now decompose the Minkowski sum $V + [\mathbf{0}, \mathbf{x}]$ into three pieces. Namely,

$$V + [\mathbf{0}, \mathbf{x}] = V^- \cup (\sigma + [\mathbf{0}, \mathbf{x}]) \cup (V^+ + \mathbf{x}).$$

Moreover, we have that $V^- \cap (\sigma + [\mathbf{0}, \mathbf{x}]) = \sigma$ and $(\sigma + [\mathbf{0}, \mathbf{x}]) \cap (V^+ + \mathbf{x}) = \sigma + \mathbf{x}$. We can now compute the desired **cd**-index by rearranging these pieces. We obtain

$$\begin{aligned}
\Psi(V + [\mathbf{0}, \mathbf{x}]) &= \Psi(V^-) + \Psi(\sigma + [\mathbf{0}, \mathbf{x}]) + \Psi(V^+ + \mathbf{x}) - \Psi(\sigma) \cdot \mathbf{c} - \Psi(\sigma + \mathbf{x}) \cdot \mathbf{c} \\
&= \Psi(V^-) + \Psi(V^+) - \Psi(\sigma) \cdot \mathbf{c} + \Psi(\mathrm{Prism}(\mathrm{Proj}(V))) - \Psi(\mathrm{Proj}(V)) \cdot \mathbf{c} \\
&= \Psi(V) + D(\Psi(\mathrm{Proj}(V))). \qquad \square
\end{aligned}$$

7 On Simsun permutations

Let S be a set such that $S \cup \{0\}$ is a linearly ordered set.

Definition 7.1 *An* augmented permutation π *of length* n *on* S *is a list* $\pi = (0 = s_0, s_1, \ldots, s_n)$, *where* $s_1, \ldots, s_n$ *are* n *distinct elements from the set* S.

The descent set of the augmented permutation π is the set $D(\pi) = \{i \ : \ s_{i-1} > s_i\}$. Observe the descent set of π is a subset of $[n] = \{1, \ldots, n\}$. We say that π has no double descents if there is no index i such that $s_i > s_{i+1} > s_{i+2}$. The *variation* of a permutation π is given by $U(\pi) = u_{D(\pi)}$, where u_S is the **ab**-monomial $u_1 \cdots u_n$ such that $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$.

Let $R_n(S)$ be the set of augmented permutations on the set S of length n so that any such permutation begins with an ascent and has no double descents. We let $R_0(S)$ be the singleton set containing the permutation (0). For an augmented permutation π in $R_n(S)$, we define the *reduced variation* of π, which we denote by $V(\pi)$, by replacing each **ab** in $U(\pi)$ with **d** and then replacing each remaining **a** by a **c**. For a subset T of $R_n(S)$ we define $V(T) = \sum_{\pi \in T} V(\pi)$.

We now ask the following question. Given an Eulerian poset P of rank $n+1$, is it possible to find in a canonical manner a linearly ordered set S and a subset T of $R_n(S)$ such that $\Psi(P) = V(T)$? Examples of such posets and permutation sets are the Boolean algebra and André permutations, and the cubical lattice and signed André permutations. See [8, 17]. For more refined identities using such a poset–permutation set correspondence, see [6, 11, 20].

We will now define three operations on permutations. These will give us a partial answer to our question.

For a permutation $\pi = (0, s_1, \ldots, s_n)$ and an element x, we define the *concatenation* $\pi \cdot x = (0, s_1, \ldots, s_n, x)$. We extend this notion for a class T of permutations by $T \cdot x = \{\pi \cdot x \ : \ \pi \in T\}$. Let M be an element larger than all the elements in the linear order $S \cup \{0\}$. For T a subset of $R_n(S)$ we have that $T \cdot M \subseteq R_{n+1}(S \cup \{M\})$. Moreover, we have that $V(T \cdot M) = V(T) \cdot \mathbf{c}$.

We will now define the insert operation. Let M be as just defined and let m be an element smaller than all the elements in $S \cup \{0\}$. For $T \subseteq R_n(S)$ and $x \in \{m, M\}$, we define $\mathrm{Insert}(T, x)$ to be set of all augmented permutations $(0, s_1, \ldots, s_i, x, s_{i+1}, \ldots, s_n)$ such that

1. $(0, s_1, \ldots, s_n) \in T$,
2. $(0, s_1, \ldots, s_i, x, s_{i+1}, \ldots, s_n) \in R_{n+1}(S \cup \{x\})$,
3. if x is the maximal element M then $i \neq n$, and
4. if x is the minimal element m then $i \neq 0$.

That is, we insert x into the permutation $(0, s_1, \ldots, s_n) \in T$ such that no double descents occur and we do not allow the maximal element at the end nor the minimal element at the beginning of the permutation. Observe that we have $\mathrm{Insert}(T, M) \cup \mathrm{Insert}(T, m) \subseteq R_{n+1}(S \cup \{M, m\})$.

Lemma 7.2 *For $T \subseteq R_n(S)$ the two identities $V(\mathrm{Insert}(T, M)) = G(V(T))$ and $V(\mathrm{Insert}(T, m)) = G'(V(T))$ hold.*

Theorem 7.3 *Let P be an Eulerian poset of rank $n+1$. Let $S \cup \{0\}$ be a linearly ordered set, and let T be a subset of $R_n(S)$ such that $\Psi(P) = V(T)$. Introduce a new maximal element M and a minimal element m to the set $S \cup \{0\}$. Then the following identities hold:*

$$\begin{aligned} \Psi(\mathrm{Pyr}(P)) &= V\,(\,\mathrm{Insert}(T, M)\,\cup\,T \cdot M\,)\,, \\ \Psi(\mathrm{Prism}(P)) &= V\,(\,\mathrm{Insert}(T, M)\,\cup\,\mathrm{Insert}(T, m)\,\cup\,T \cdot M\,)\,. \end{aligned}$$

Simion and Sundaram defined a class of permutations called simsun permutations; see [22, page 267] and [23]. We will now see how simsun permutations are closely related with the operations $\mathrm{Insert}(T, n)$ and $T \cdot n$ on permutations.

A *simsun permutation* π of length n is a augmented permutation $\pi = (0, s_1, \ldots, s_n)$ on the set $\{1, \ldots, n\}$ of length n such that for all $0 \leq k \leq n$ if we remove the k entries $n, n-1, \ldots, n-k+1$ from the permutation π, the resulting permutation does not have any double descents. Let $\mathcal{S}_n$ denote the set of all simsun permutations of length n. We have that $\mathcal{S}_n \subseteq R_n(\{1, \ldots, n\})$.

Similarly, we may define a *signed simsun permutation* π of length n as an augmented permutation of length n on the set $\{-n, \ldots, -1, 1, \ldots, n\}$ such that exactly one of the elements $+i$ and $-i$ occurs in the permutation and for all $0 \leq k \leq n$ if we remove the k entries $\pm n, \pm(n-1), \ldots, \pm(n-k+1)$ from the permutation π, the resulting permutation belongs to $R_{n-k}(\{-(n-k), \ldots, -1, 1, \ldots, n-k\})$. This implies that the resulting permutation have no double descents. Let $\mathcal{S}_n^{\pm}$ denote the set of all signed simsun permutations of length n.

Recall that $\mathcal{S}_{n-1} \cdot n$ denotes the set of all permutations π in $\mathcal{S}_n$ so that $\pi(n) = n$. We make the similar convention for $\mathcal{S}_{n-1}^{\pm} \cdot n$.

Corollary 7.4 *The sets of all simsun permutations and all signed simsun permutations satisfy the following recursions:*

$$\begin{aligned} \mathcal{S}_n &= \text{Insert}(\mathcal{S}_{n-1}, n) \cup \mathcal{S}_{n-1} \cdot n, \\ \mathcal{S}_n^{\pm} &= \text{Insert}(\mathcal{S}_{n-1}^{\pm}, n) \cup \text{Insert}(\mathcal{S}_{n-1}^{\pm}, -n) \cup \mathcal{S}_{n-1}^{\pm} \cdot n. \end{aligned}$$

Thus the **cd***-indexes of the Boolean algebra and the cubical lattice is given by*

$$\begin{aligned} \Psi(B_{n+1}) &= V(\mathcal{S}_n), \\ \Psi(C_n) &= V(\mathcal{S}_n^{\pm}). \end{aligned}$$

8 The shelling components of the simplex

Stanley [20] studies the shelling components of the simplex in order to obtain a formula for the **cd**-index of a simplicial Eulerian poset. Namely, if P is a simplicial Eulerian poset of rank $n+1$ with h-vector $(h_0, \ldots, h_n)$ then the **cd**-index of P is given by $\Psi(P) = \sum_{i=0}^{n-1} h_i \cdot \check{\Phi}_i^n$. By using the techniques we have developed, we now study the **cd**-polynomials $\check{\Phi}_i^n$.

Recall that B_n is the Boolean algebra, that is, all the subsets of $\{1, \ldots, n\}$ ordered by inclusion. Let c_i be the coatom $\{1, \ldots, n\} - \{n+1-i\}$. Similarly, for $i \neq j$ let $c_{i,j}$ be the element $\{1, \ldots, n\} - \{n+1-i, n+1-j\}$, that is, $c_{i,j}$ is the intersection of the two sets c_i and c_j. Define the poset $B'_{n,i}$ for $1 \leq i \leq n-1$ by

$$B'_{n,i} = \bigcup_{j=1}^{i} [\emptyset, c_j] \cup \{\{1, \ldots, n\}\}.$$

That is, $B'_{n,i}$ consists of the maximal element $\{1, \ldots, n\}$ and all the elements below the coatoms $c_1, \ldots, c_i$. Since the elements $c_{j,k}$, where $1 \leq j \leq i$ and $i+1 \leq k \leq n$, are only covered by one element in $B'_{n,i}$, we know that $B'_{n,i}$ is not an Eulerian poset. However we can obtain an Eulerian poset by adding an element γ in the following manner. Let $B_{n,i}$ be the poset $B'_{n,i} \cup \{\gamma\}$, where the coatom γ covers all elements $c_{j,k}$ with $1 \leq j \leq i$ and $i+1 \leq k \leq n$. The poset $B_{n,i}$ is Eulerian. Observe that $B_{n,1} = B_{n-1} * B_2$ and $B_{n,n-1} = B_n$. Stanley defines $\check{\Phi}_i^n$ by the relation

$$\Psi(B_{n,i}) = \check{\Phi}_0^{n-1} + \cdots + \check{\Phi}_{i-1}^{n-1}.$$

That is, $\check{\Phi}_0^n = \Psi(B_{n+1,1}) = \Psi(B_n) \cdot \mathbf{c}$, and for $1 \leq i \leq n-1$, $\check{\Phi}_i^n = \Psi(B_{n+1,i+1}) - \Psi(B_{n+1,i})$.

We now state the main result of this section.

Theorem 8.1 *The following recursion holds for $\check{\Phi}_i^n$: $G(\check{\Phi}_i^n) = \check{\Phi}_{i+1}^{n+1}$.*

Proof: We claim that the following identity is true:

$$\Psi(B_{n,i} \times B_1) + \Psi(B_n * B_2) = \Psi(B_{n+1,i+1}) + \Psi(B_{n,i} * B_2). \tag{8.1}$$

We may view $B'_{n,i} \times B_1$ as a subposet of B_{n+1} by viewing B_1 as the poset on $\{\emptyset, \{n+1\}\}$. Hence $B_{n,i} \times B_1$ is the poset $B'_{n,i} \times B_1$ with two extra elements of ranks $n-1$ and n. Label both of these elements by γ. Moreover, label the two coatoms in the two posets $B_n * B_2$ and $B_{n,i} * B_2$ by $\{1, \ldots, n\}$ and γ. It is now straightforward to prove equation (8.1) since there is a rank-preserving bijection between the chains on the right-hand side and the left-hand side. Except for the chains labeled by $(\emptyset, \{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n+1\})$, the bijection is given by reading off the labels of a chain. For the case when a chain is labeled with $(\emptyset, \{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n+1\})$, observe that each of the four posets has one such chain. This is the only chain that contributes two terms to each side of equation (8.1).

Recall that $\Psi(B_{n,i}) = \sum_{j=0}^{i-1} \check{\Phi}_j^{n-1}$. Thus by Theorem 5.2 we have $\Psi(B_{n,i} \times B_1) = \left(\sum_{j=0}^{i-1} \check{\Phi}_j^{n-1}\right) \cdot \mathbf{c} + G\left(\sum_{j=0}^{i-1} \check{\Phi}_j^{n-1}\right)$. Also $\Psi(B_n * B_2) = \Psi(B_n) \cdot \Psi(B_2) = \Psi(B_n) \cdot \mathbf{c} = \check{\Phi}_0^n$. Similarly, $\Psi(B_{n+1,i+1}) = \sum_{j=0}^{i} \check{\Phi}_j^n = \check{\Phi}_0^n + \sum_{j=1}^{i} \check{\Phi}_j^n$ and $\Psi(B_{n,i} * B_2) = \Psi(B_{n,i}) \cdot \mathbf{c} = \left(\sum_{j=0}^{i-1} \check{\Phi}_j^{n-1}\right) \cdot \mathbf{c}$. Hence when we expand the identity (8.1), we have

$$\left(\sum_{j=0}^{i-1} \check{\Phi}_j^{n-1}\right) \cdot \mathbf{c} + G\left(\sum_{j=0}^{i-1} \check{\Phi}_j^{n-1}\right) + \check{\Phi}_0^n = \check{\Phi}_0^n + \sum_{j=1}^{i} \check{\Phi}_j^n + \left(\sum_{j=0}^{i-1} \check{\Phi}_j^{n-1}\right) \cdot \mathbf{c}.$$

By cancelling terms we have $G(\check{\Phi}_0^{n-1} + \cdots + \check{\Phi}_{i-1}^{n-1}) = \check{\Phi}_1^n + \cdots + \check{\Phi}_i^n$, which is equivalent to the conclusion of the theorem. $\square$

Stanley conjectured [20, Conjecture 3.1] that the reduced variation of certain classes of permutations is equal to $\check{\Phi}_i^n$. This conjecture was proved by Hetyei in [11]. We now present a slightly modified result of this kind. It follows easily by Theorem 8.1 and the techniques of Section 7. Let $\mathcal{S}_{n,k}$ be the set of simsun permutations of length n ending with the element k.

Corollary 8.2 *The reduced variation of the set $\mathcal{S}_{n,k}$ is given by $V(\mathcal{S}_{n,k}) = \check{\Phi}_{n-k}^n$.*

This result follows from induction on n and noting that $\mathcal{S}_{n,k} = \mathrm{Insert}(\mathcal{S}_{n-1,k}, n)$ and $\mathcal{S}_{n,n} = \mathcal{S}_{n-1} \cdot n$.

9 The mixing operators

Given two posets P and Q, assume that we know $\Psi(P)$ and $\Psi(Q)$. Are we then able to compute $\Psi(P \times Q)$ from $\Psi(P)$ and $\Psi(Q)$? The answer is yes, and it may seen by using the quasi-symmetric function of a poset P; see [5]. It is shown in [5] that knowing $\Psi(P)$ is equivalent to knowing the quasi-symmetric function $F(P)$. Also the identity $F(P \times Q) = F(P) \cdot F(Q)$ holds. In fact, it is proved that F is a Hopf algebra homomorphism. Hence we may compute $\Psi(P \times Q)$ from $\Psi(P)$ and $\Psi(Q)$.

In terms of the **ab**-index, this quasi-symmetric function method is not explicit. We will now devise an explicit method. We begin to define the mixing operator. Let I be the set

$$I = \{(r, s, n) \;\;:\;\; r, s \in \{1, 2\}, n \geq 2, n \equiv r + s + 1 \bmod 2\}.$$

This set will be the index set of the mixing operator. For a coassociative coproduct $\Delta : V \longrightarrow V \otimes V$, we define the map $\Delta^k : V \longrightarrow V^{\otimes k}$ by Δ^1 is the identity map 1 and $\Delta^{k+1} = (\Delta^k \otimes 1) \circ \Delta$. Observe that $\Delta^2 = \Delta$. The Sweedler notation [24, pages 10-11] for the linear map Δ^k is

$$\Delta^k(x) = \sum_x x_{(1)} \otimes \cdots \otimes x_{(k)}.$$

We call each factor $x_{(i)}$ an x-piece. We will use this notation on the two coalgebras $\mathcal{A}$ and $\mathcal{P}$.

Definition 9.1 *Let u and v be in $\mathcal{A}$ and $(r, s, n) \in I$. The* mixing operator $M_{r,s}(u, v, n)$, *is defined by the following recursion:*

$$\begin{aligned} M_{1,2}(u, v, 2) &= u \cdot \mathbf{a} \cdot v, \\ M_{2,1}(u, v, 2) &= v \cdot \mathbf{b} \cdot u, \\ M_{1,s}(u, v, n+1) &= \sum_u u_{(1)} \cdot \mathbf{a} \cdot M_{2,s}(u_{(2)}, v, n), \\ M_{2,s}(u, v, n+1) &= \sum_v v_{(1)} \cdot \mathbf{b} \cdot M_{1,s}(u, v_{(2)}, n). \end{aligned}$$

As an example, by the coassociativity of the coproduct we have

$$M_{1,1}(u, v, 5) = \sum_u \sum_v u_{(1)} \cdot \mathbf{a} \cdot v_{(1)} \cdot \mathbf{b} \cdot u_{(2)} \cdot \mathbf{a} \cdot v_{(2)} \cdot \mathbf{b} \cdot u_{(3)}.$$

Observe that n is the number of pieces in each summand of $M_{r,s}(u, v, n)$. When $r = 1$ each summand begins with a u-piece, while when $r = 2$ the summands begin with a v-piece. Similarly, $s = 1$ says that each summand ends with a u-piece.

In general to compute $M_{1,1}(u, v, n)$ we apply Δ^k to u, where $k = \frac{n+1}{2}$. Observe that n is odd in this case. Similarly apply Δ^{k-1} to v. We obtain

$$\Delta^k(u) = \sum_u u_{(1)} \otimes \cdots \otimes u_{(k)} \quad \text{and} \quad \Delta^{k-1}(v) = \sum_v v_{(1)} \otimes \cdots \otimes v_{(k-1)}.$$

We then combine the u- and v-pieces alternatingly such that there is an $\mathbf{a}$ between an adjacent u-piece and v-piece (reading left to right), otherwise there is a $\mathbf{b}$ between. Lastly, we sum over all possible ways to split u and v by the coproduct. There are similar rules for $M_{1,2}(u, v, n)$, $M_{2,1}(u, v, n)$, and $M_{2,2}(u, v, n)$.

Theorem 9.2 *Let P and Q be two posets. Then*

$$\Psi(P \times Q) = \sum_{(r,s,n) \in I} M_{r,s}(\Psi(P), \Psi(Q), n).$$

Equations (4.1) and (4.2) in the proof of Proposition 4.2 are special cases of this theorem.

If $w \in \mathcal{F}_k$ then $\Delta^{k+2}(w) = 0$. Hence $M_{r,s}(u, v, n) = 0$ if $\frac{n+3-r-s}{2} \geq \deg(u) + 2$ or $\frac{n-3+r+s}{2} \geq \deg(v) + 2$. Hence the sum in Theorem 9.2 has a finite number of non-zero terms, so it is well-defined.

In order to motivate Theorem 9.2, we will first prove it in the case when the posets P and Q have R-labelings. We will assume that the reader is familiar with R-labelings. Otherwise, see [8] or [19, section 3.13]

An *edge-labeling* λ of a finite poset P is a map which assigns to each edge in the Hasse diagram of P an element from a total linear order. If y covers x in P then we denote the label on this edge by $\lambda(x, y)$. If $c = \{\hat{0} = x_0 \prec x_1 \prec \ldots \prec x_{\rho(P)} = \hat{1}\}$ is a chain then we write the labeling of the chain c to be the list $\lambda(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \cdots, \lambda(x_{\rho(P)-1}, x_{\rho(P)}))$. An *R-labeling* λ is an edge-labeling such that in every interval there is a unique maximal chain where the labels are weakly increasing.

For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec \ldots \prec x_{\rho(P)} = \hat{1}\}$ we define the *descent monomial* of the chain c to be $u(c) = u_1 \cdots u_{\rho(P)-1}$, where $u_i = \mathbf{a}$ if $\lambda(x_{i-1}, x_i) \leq \lambda(x_i, x_{i+1})$, and $u_i = \mathbf{b}$ otherwise. The following lemma follows directly from a result of Björner and Stanley [4, Theorem 2.7].

Lemma 9.3 *Let P be a graded poset of rank $n+1$. If λ is an R-labeling of P then the* **ab***-index of P is equal to*

$$\Psi(P) = \sum_{c} u(c),$$

where the sum is over all maximal chains c.

Assume that the posets P and Q have R-labelings. Then the poset $P \times Q$ has an R-labeling given as follows. Each edge in the poset $P \times Q$ either comes from an edge in P or an edge in Q. Hence label the edge between (x, z) and (y, z) by the label $\lambda_P(x, y)$ and label the edge between (x, z) and (x, w) by the label $\lambda_Q(z, w)$. Moreover let all the labels of the poset P be smaller than the labels of the poset Q.

A maximal chain c in the poset $P \times Q$ corresponds to two maximal chains, one in P and one in Q. Hence the labels of the chain c are the labels of the corresponding chains in P and Q. If a label is from the poset P, we call it a P-label. Similarly, a label from Q is called a Q-label. If c begins with a P-label then we classify this as $r = 1$, otherwise as $r = 2$. Similarly, if c ends with a P-label then we classify this as $s = 1$, otherwise as $s = 2$. Moreover, when reading the labels of the chain c in order, group the P-labels and the Q-labels into runs. Let n be the number of such runs. We say that the maximal chain c has the type (r, s, n). Observe that $(r, s, n) \in I$.

Let us consider the case when $(r, s, n) = (1, 1, 5)$. This means that the labeling of the chain c looks like

$$\lambda(c) = (\lambda_1, \ldots, \lambda_i, \nu_1, \ldots, \nu_k, \lambda_{i+1}, \ldots, \lambda_j, \nu_{k+1}, \ldots, \nu_{\rho(Q)}, \lambda_{j+1}, \ldots, \lambda_{\rho(P)}),$$

where $(\lambda_1, \ldots, \lambda_{\rho(P)})$ is the labeling of a maximal chain c_P in P and $(\nu_1, \ldots, \nu_{\rho(Q)})$ is the labeling of a maximal chain c_Q in Q. Observe also that $1 \leq i < j \leq \rho(P) - 1$ and $1 \leq k \leq \rho(Q) - 1$.

Assume that $u(c_P) = u = u_1 \cdots u_{\rho(P)-1}$ and $u(c_P) = v = v_1 \cdots v_{\rho(Q)-1}$. Then the weight of the maximal chain c is given by

$$u(c) = u_1 \cdots u_{i-1} \cdot \mathbf{a} \cdot v_1 \cdots v_{k-1} \cdot \mathbf{b} \cdot u_{i+1} \cdots u_{j-1} \cdot \mathbf{a} \cdot v_{k+1} \cdots v_{\rho(Q)-1} \cdot \mathbf{b} \cdot u_{j+1} \cdots u_{\rho(P)-1}.$$

Observe that the variables u_i, u_j, and v_k are not in the expression for $u(c)$. This because when we compute $u(c)$, we do not need to compare the label λ_i with λ_{i+1}, the label λ_j with λ_{j+1}, and the label

ν_k with ν_{k+1}. Moreover the $\mathbf{a}$'s and the $\mathbf{b}$'s occur in the expression for $u(c)$ since at those places we compare a label from the poset P with a label from Q. There will be an $\mathbf{a}$ if the P-label occurs before the Q-label in the list since the P-labels are smaller than the Q-labels. Similarly, there will be a $\mathbf{b}$ if the Q-label occurs before the P-label.

Recall that we have

$$\begin{aligned}
\Delta^3(u) &= \sum_{1 \leq i < j \leq \rho(P)-1} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_{j-1} \otimes u_{j+1} \cdots u_{\rho(P)-1}, \\
\Delta(v) &= \sum_{1 \leq k \leq \rho(Q)-1} v_1 \cdots v_{k-1} \otimes v_{k+1} \cdots v_{\rho(Q)-1}.
\end{aligned}$$

Hence if we sum $u(c)$ over all maximal chains c such that c restricted to P is c_P, c restricted to Q is c_Q, and c has the type $(r, s, n) = (1, 1, 5)$, we obtain

$$\begin{aligned}
\sum_c u(c) &= \sum_{1 \leq i < j \leq \rho(P)-1} \sum_{1 \leq k \leq \rho(Q)-1} u_1 \cdots u_{i-1} \cdot \mathbf{a} \cdot v_1 \cdots v_{k-1} \cdot \mathbf{b} \cdot u_{i+1} \cdots u_{j-1} \\
&\qquad\qquad \cdot \mathbf{a} \cdot v_{k+1} \cdots v_{\rho(Q)-1} \cdot \mathbf{b} \cdot u_{j+1} \cdots u_{\rho(P)-1} \\
&= \sum_u \sum_v u_{(1)} \cdot \mathbf{a} \cdot v_{(1)} \cdot \mathbf{b} \cdot u_{(2)} \cdot \mathbf{a} \cdot v_{(2)} \cdot \mathbf{b} \cdot u_{(3)} \\
&= M_{1,1}(u, v, 5).
\end{aligned}$$

Since the mixing operator is linear in u and v, and using Lemma 9.3, we obtain:

$$\sum_c u(c) = M_{1,1}(\Psi(P), \Psi(Q), 5),$$

where the sum ranges over all maximal chains c which have the type $(r, s, n) = (1, 1, 5)$. Proceeding along these lines one can generalize this argument to prove Theorem 9.2 in the case when the two posets have R-labelings.

10 Proof of Theorem 9.2

We will now prove Theorem 9.2 in the general case, that is, we will not assume P and Q have R-labelings. We begin by defining a map K from the set of chains of the poset $P \times Q$ to the set of quadruples (d_P, d_Q, r, s) such that d_P is a chain in P, d_Q is a chain in Q, and $r, s \in \{1, 2\}$. We will do this so that $l(d_Q) - l(d_P) = r + s - 3$, where $l(\cdot)$ denotes the length of the chain. This condition implies that lengths of the chains d_P and d_Q differ by at most one.

We now describe the map K. Let $c = \{(\hat{0}, \hat{0}) = (x_0, y_0) < (x_1, y_1) < \cdots < (x_k, y_k) = (\hat{1}, \hat{1})\}$ be a chain in the poset $P \times Q$. Observe that the chains $\{\hat{0} = x_0 \leq x_1 \leq \cdots \leq x_k = \hat{1}\}$ and $\{\hat{0} = y_0 \leq y_1 \leq \cdots \leq y_k = \hat{1}\}$ are weakly increasing.

We will now find two other chains in the posets P and Q. Let $z_0 = \hat{0}_P$ and $w_0 = \hat{0}_Q$. Recursively define

$$\begin{aligned}
z_i &= \min\{x_j \ : \ y_j > w_{i-1}\}, \\
w_i &= \max\{y_j \ : \ x_j = z_i\}.
\end{aligned}$$

This recursion ends when $w_k = \hat{1}_Q$ since then we cannot find z_{k+1}. Observe that $\hat{0} = z_0 \leq z_1 < z_2 < \cdots < z_{k-1} < z_k \leq \hat{1}$ and $\hat{0} = w_0 < w_1 < w_2 < \cdots < w_k = \hat{1}$.

Let d_Q be the chain $\{\hat{0} = w_0 < w_1 < w_2 < \cdots < w_k = \hat{1}\}$. Let d_P be the chain $\{\hat{0} = z_0 \leq z_1 < z_2 < \cdots < z_{k-1} < z_k \leq \hat{1}\}$. Observe that we consider d_P as a set, not as a multiset. If $z_0 = z_1$ let $r = 2$, otherwise $r = 1$. If $z_k = \hat{1}$ let $s = 2$, otherwise $s = 1$. Now let $K(c) = (d_P, d_Q, r, s)$.

For a set S of chains of the poset $P \times Q$, define $W(S) = W_{P\times Q}(S)$ to be the sum $\sum_{c\in S} w_{P\times Q}(c)$, where $w_{P\times Q}$ is the weight function defined in Section 3.

Lemma 10.1 *Let d_P be the chain $\{\hat{0} = p_0 < p_1 < \cdots < p_k = \hat{1}\}$ and let d_Q be the chain $\{\hat{0} = q_0 < q_1 < \cdots < q_{k'} = \hat{1}\}$, where $k' = k + r + s - 3$. For the four cases $(r, s) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ we have:*

$$\begin{aligned}
W(K^{-1}(d_P, d_Q, 1, 1)) &= \Psi([p_0, p_1]) \cdot \mathbf{a} \cdot \Psi([q_0, q_1]) \cdot \mathbf{b} \cdots \mathbf{a} \cdot \Psi([q_{k-2}, q_{k-1}]) \cdot \mathbf{b} \cdot \Psi([p_{k-1}, p_k]), \\
W(K^{-1}(d_P, d_Q, 1, 2)) &= \Psi([p_0, p_1]) \cdot \mathbf{a} \cdot \Psi([q_0, q_1]) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \Psi([p_{k-1}, p_k]) \cdot \mathbf{a} \cdot \Psi([q_{k-1}, q_k]), \\
W(K^{-1}(d_P, d_Q, 2, 1)) &= \Psi([q_0, q_1]) \cdot \mathbf{b} \cdot \Psi([p_0, p_1]) \cdot \mathbf{a} \cdots \mathbf{a} \cdot \Psi([q_{k-1}, q_k]) \cdot \mathbf{b} \cdot \Psi([p_{k-1}, p_k]), \\
W(K^{-1}(d_P, d_Q, 2, 2)) &= \Psi([q_0, q_1]) \cdot \mathbf{b} \cdot \Psi([p_0, p_1]) \cdot \mathbf{a} \cdots \mathbf{b} \cdot \Psi([p_{k-1}, p_k]) \cdot \mathbf{a} \cdot \Psi([q_k, q_{k+1}]).
\end{aligned}$$

Here the inserted **a***'s and* **b***'s alternate.*

Proof: We will prove the case when $(r, s) = (1, 2)$. The other three cases are proved by a similar argument.

Let c be a chain such that $K(c) = (d_P, d_Q, 1, 2)$. Observe that the pair (p_i, q_i) belongs to the chain c for all $i = 0, \ldots, k$. Similarly, the pair (p_i, q_{i-1}) may or may not belong to c for $i = 1, \ldots, k$. Any other element of the chain c belongs to either an interval of the form $[(p_{i-1}, q_{i-1}), (p_i, q_{i-1})]$ or of the form $[(p_i, q_{i-1}), (p_i, q_i)]$.

More formally, the chain c contains the pairs (p_i, q_i). That is,

$$\{(p_0, q_0), (p_1, q_1), \ldots, (p_k, q_k)\} \subseteq c.$$

Also, the chain c is a subset of the union of these intervals. That is,

$$c \subseteq \bigcup_{i=1}^{k} [(p_{i-1}, q_{i-1}), (p_i, q_{i-1})] \cup [(p_i, q_{i-1}), (p_i, q_i)].$$

It is easy to see that any chain c' of $P \times Q$ that fulfills these two conditions satisfies $K(c') = (d_P, d_Q, 1, 2)$.

Hence when computing the weight of the chain c, the interval $[(p_{i-1}, q_{i-1}), (p_i, q_{i-1})]$ contributes $\Psi([p_{i-1}, p_i])$, the pair (p_i, q_{i-1}) contributes $\mathbf{a}$, the interval $[(p_i, q_{i-1}), (p_i, q_i)]$ contributes $\Psi([q_{i-1}, q_i])$, and finally the pair (p_i, q_i) contributes $\mathbf{b}$. This completes the proof of the lemma in the case when $(r, s) = (1, 2)$. $\square$

Proof of Theorem 9.2: For (r, s, n) in the index set I, define $P(r, s, n)$ be the set of chains in the poset $P \times Q$ such that

$$P(r, s, n) = \{c \ : \ K(c) = (d_P, d_Q, r, s), \text{ where } l(d_P) + l(d_Q) = n\}.$$

We would like to compute $W(P(1, 2, 2 \cdot k))$. To do this we must consider all possible ways to have a chain in P of length k and a chain in Q of length k. We may compute all such possibilities by considering the two expressions $\Delta^k(P)$ and $\Delta^k(Q)$. By Lemma 10.1, we have that

$$\begin{aligned} W(P(1, 2, 2 \cdot k)) &= \sum_P \sum_Q \Psi(P_{(1)}) \cdot \mathbf{a} \cdot \Psi(Q_{(1)}) \cdot \mathbf{b} \cdots \mathbf{b} \cdot \Psi(P_{(k)}) \cdot \mathbf{a} \cdot \Psi(Q_{(k)}) \\ &= \sum_u \sum_v u_{(1)} \cdot \mathbf{a} \cdot v_{(1)} \cdot \mathbf{b} \cdots \mathbf{b} \cdot u_{(k)} \cdot \mathbf{a} \cdot v_{(k)} \\ &= M_{1,2}(u, v, 2 \cdot k), \end{aligned}$$

where $u = \Psi(P)$ and $v = \Psi(Q)$. The second equality holds since Ψ is a coalgebra map. The last equality is the expression for the mixing operator.

One may generalize this argument to obtain that $W(P(r, s, n)) = M_{r,s}(u, v, n)$ for (r, s, n) belonging to the index set I. The theorem follows by summing over all triplets in the index set I. $\square$

Define an algebra map κ from $\mathcal{F}$ to itself by $\kappa(\mathbf{a}) = \mathbf{a} - \mathbf{b}$ and $\kappa(\mathbf{b}) = 0$. Since the monomial $\mathbf{a}^{\rho(P)-1}$ has coefficient 1 in the expansion of $\Psi(P)$, we have that $\kappa(\Psi(P)) = (\mathbf{a} - \mathbf{b})^{\rho(P)-1}$.

Recall that in the definition of $\Psi(P)$ we sum over all chains in the poset P. If we condition on the smallest non-zero element in the chain, we obtain the following expression:

$$\begin{aligned} \Psi(P) &= (\mathbf{a} - \mathbf{b})^{\rho(P)-1} + \sum_{\hat{0} < x < \hat{1}} (\mathbf{a} - \mathbf{b})^{\rho(x)-1} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]) \\ &= \kappa(\Psi(P)) + \sum_{\hat{0} < x < \hat{1}} \kappa(\Psi([\hat{0}, x])) \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]) \\ &= \kappa(\Psi(P)) + \sum_P \kappa(\Psi(P_{(1)})) \cdot \mathbf{b} \cdot \Psi(P_{(2)}). \end{aligned}$$

We use this identity to find a formula for $\Psi(P \diamond Q)$ in terms of $\Psi(P)$ and $\Psi(Q)$. Note that a non-zero element in $P \diamond Q$ is of the form (x, y), where $\hat{0}_P < x \leq \hat{1}_P$ and $\hat{0}_Q < y \leq \hat{1}_Q$. Moreover, the rank of the element (x, y) is $\rho(x) + \rho(y) - 1$. Hence, $\kappa(\Psi([\hat{0}, (x, y)])) = (\mathbf{a} - \mathbf{b})^{\rho((x,y))-1} = (\mathbf{a} - \mathbf{b})^{\rho(x)+\rho(y)-2} = \kappa(\Psi([\hat{0}, x])) \cdot \kappa(\Psi([\hat{0}, y]))$.

We now obtain

$$\begin{aligned} \Psi(P \diamond Q) &= \kappa(\Psi(P \diamond Q)) + \sum_{\hat{0} < (x,y) < (\hat{1}, \hat{1})} \kappa(\Psi([\hat{0}, (x, y)])) \cdot \mathbf{b} \cdot \Psi([(x, y), (\hat{1}, \hat{1})]) \\ &= \kappa(\Psi(P)) \cdot \kappa(\Psi(Q)) + \sum_{\hat{0} < (x,y) < (\hat{1}, \hat{1})} \kappa(\Psi([\hat{0}, x])) \cdot \kappa(\Psi([\hat{0}, y])) \cdot \mathbf{b} \cdot \Psi([x, \hat{1}] \times [y, \hat{1}]) \\ &= \kappa(\Psi(P)) \cdot \kappa(\Psi(Q)) + \sum_{\hat{0} < x < \hat{1}} \kappa(\Psi([\hat{0}, x])) \cdot \kappa(\Psi(Q)) \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]) \end{aligned}$$

$$+\sum_{\hat{0}<y<\hat{1}} \kappa(\Psi(P)) \cdot \kappa(\Psi([\hat{0}, y])) \cdot \mathbf{b} \cdot \Psi([y, \hat{1}])$$
$$+\sum_{\substack{\hat{0}<x<\hat{1} \\ \hat{0}<y<\hat{1}}} \kappa(\Psi([\hat{0}, x])) \cdot \kappa(\Psi([\hat{0}, y])) \cdot \mathbf{b} \cdot \Psi([x, \hat{1}] \times [y, \hat{1}]).$$

Here the second term comes from the case when $y = \hat{1}$, the third from the case $x = \hat{1}$, and the last from the remaining case. This equation can be expressed in Sweedler notation as

$$\begin{aligned}
\Psi(P \diamond Q) &= \kappa(\Psi(P)) \cdot \kappa(\Psi(Q)) + \sum_{P} \kappa(\Psi(P_{(1)})) \cdot \kappa(\Psi(Q)) \cdot \mathbf{b} \cdot \Psi(P_{(2)}) \\
&\quad + \sum_{Q} \kappa(\Psi(P)) \cdot \kappa(\Psi(Q_{(1)})) \cdot \mathbf{b} \cdot \Psi(Q_{(2)}) \\
&\quad + \sum_{P} \sum_{Q} \kappa(\Psi(P_{(1)})) \cdot \kappa(\Psi(Q_{(1)})) \cdot \mathbf{b} \cdot \Psi(P_{(2)} \times Q_{(2)}).
\end{aligned}$$

Letting $u = \Psi(P)$ and $v = \Psi(Q)$, and using the fact that the **ab**-index is a coalgebra homomorphism we have

$$\begin{aligned}
\Psi(P \diamond Q) &= \kappa(u) \cdot \kappa(v) + \sum_{u} \kappa(u_{(1)}) \cdot \kappa(v) \cdot \mathbf{b} \cdot u_{(2)} \\
&\quad + \sum_{v} \kappa(u) \cdot \kappa(v_{(1)}) \cdot \mathbf{b} \cdot v_{(2)} + \sum_{u} \sum_{v} \kappa(u_{(1)}) \cdot \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \mathcal{M}(u_{(2)}, v_{(2)}).
\end{aligned}$$

Here $\mathcal{M}(u, v)$ denotes the expression in Theorem 9.2. Hence, we conclude that we can compute $\Psi(P \diamond Q)$ in terms of $\Psi(P)$ and $\Psi(Q)$. Since $\mathcal{L}(U \times V) = \mathcal{L}(U) \diamond \mathcal{L}(V)$ for two convex polytopes U and V, we obtain the following proposition.

Proposition 10.2 *Let U and V be two convex polytopes. Then the* **ab***-index of their Cartesian product $U \times V$ is given by*

$$\begin{aligned}
\Psi(U \times V) &= \kappa(u) \cdot \kappa(v) + \sum_{u} \kappa(u_{(1)}) \cdot \kappa(v) \cdot \mathbf{b} \cdot u_{(2)} \\
&\quad + \sum_{v} \kappa(u) \cdot \kappa(v_{(1)}) \cdot \mathbf{b} \cdot v_{(2)} + \sum_{u} \sum_{v} \kappa(u_{(1)}) \cdot \kappa(v_{(1)}) \cdot \mathbf{b} \cdot \mathcal{M}(u_{(2)}, v_{(2)}),
\end{aligned}$$

where $u = \Psi(U)$ and $v = \Psi(V)$.

11 Concluding Remarks

There are a number of questions that appear at this point in the research. We put forward a few of them.

In Section 8 we found new properties that hold for the **cd**-index of the shelling components of the simplex. In [6] the **cd**-index of shelling components of the cube have been studied. Are there any identities between the **cd**-indexes of the shelling components of the cube involving coproducts?

Stanley conjectured that among all Gorenstein* lattices of rank n, the Boolean algebra B_n minimizes all the coefficients of the **cd**-index [21, Conjecture 2.7]. We present the following generalization:

Conjecture 11.1 *Let F be a polytope of dimension $d-1$. Then among all d-dimensional polytopes having F as a facet, the pyramid of F minimizes all the coefficients of the* **cd***-index.*

Let L be a linear functional on the Newtonian coalgebra V. Then the linear map D^L defined on V by

$$D^L(x) = \sum_x x_{(1)} \cdot L(x_{(2)}) \cdot x_{(3)}$$

is a coderivation on V. That is, D^L satisfies the relation $\Delta \circ D^L = (D^L \otimes 1 + 1 \otimes D^L) \circ \Delta$. In the Newtonian coalgebras $\mathcal{P}$, $\mathcal{E}$, $\mathcal{A}$, and $\mathcal{F}$ are there any coderivations which have a combinatorial interpretation?

In Section 9 when computing $\Psi(P \diamond Q)$ in terms of $\Psi(P)$ and $\Psi(Q)$, many terms with negative signs occur. Could one find a more bijective formula for $\Psi(P \diamond Q)$, such as the formula for $\Psi(P \times Q)$ in Theorem 9.2? More importantly, consider the case when P and Q are Eulerian, and we have $\Psi(P)$ and $\Psi(Q)$ as **cd**-indexes. Do there exist formulas for $\Psi(P \times Q)$ and $\Psi(P \diamond Q)$ where the computation is completely inside the algebra $\mathcal{F}$, that is, where all terms are **cd**-monomials. For instance, Hetyei has asked if there are any explicit formulas for the **cd**-index of products of simplices.

There is one more operation on graded posets which preserves the Eulerian property. Let P and Q be two posets of the same rank $n+1$. Define $P \circ Q$ to be the poset $(P - \{\hat{0}, \hat{1}\}) + (Q - \{\hat{0}, \hat{1}\}) \cup \{\hat{0}, \hat{1}\}$. That is, $P \circ Q$ is formed by pairwise identifying the extremal elements of P and Q. We have $\Psi(P \circ Q) = \Psi(P) + \Psi(Q) - (\mathbf{a} - \mathbf{b})^n$. When P and Q are Eulerian of the same odd rank $2k+1$, we have that $P \circ Q$ is Eulerian and $\Psi(P \circ Q) = \Psi(P) + \Psi(Q) - (\mathbf{c}^2 - 2 \cdot \mathbf{d})^k$.

In order to obtain a better understanding of the **cd**-index of an Eulerian poset P, one would need to compute more examples. It would be interesting to implement the algorithm in Section 3 in either Maple or Mathematica.

Let V and W be two convex polytopes in $\mathbb{R}^n$. The Minkowski sum $V+W$ is also a convex polytope. Assume that we know the **cd**-index of the two polytopes V and W. This does not give us enough information to compute the **cd**-index of the Minkowski sum $V+W$. What additional information do we need of V and W in order to compute $\Psi(V+W)$? Recall that in Proposition 6.5 this was solved when one of the polytopes is a line segment in general direction. Recently, the authors together with Louis Billera have found an answer in the case when one of the polytopes is a line segment not in general direction.

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