CYCLOTOMIC FACTORS OF THE DESCENT SET POLYNOMIAL

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Abstract. We introduce the notion of the descent set polynomial as an alternative way of encoding the sizes of descent classes of permutations. Descent set polynomials exhibit interesting factorization patterns. We explore the question of when particular cyclotomic factors divide these polynomials. As an instance we deduce that the proportion of odd entries in the descent set statistics in the symmetric group $\mathfrak{S}_n$ only depends on the number on 1's in the binary expansion of n. We observe similar properties for the signed descent set statistics.

1. Introduction

The study of the behavior of the descent sets of permutations in the symmetric group $\mathfrak{S}_n$ on n elements usually involves such questions as maximizing the descent set or determining inequalities which hold among the entries [4, 8, 9, 13, 14, 15]. The usual way to encode the descent statistic information is via the *Eulerian polynomial* $E_n(t) = \sum_S \beta(S) \cdot t^{|S|}$, where S runs over all subsets of $[n-1] = \{1, \ldots, n-1\}$. We instead introduce the descent set polynomial where the statistic of interest appears in the exponent of the variable t, rather than as a coefficient. That is, the *nth descent set polynomial* is defined by

$$Q_n(t) = \sum_S t^{\beta(S)},$$

where S ranges over all subsets of $[n-1]$.

The degree of the descent set polynomial is given by the nth Euler number, which grows faster than an exponential. Despite this, these polynomials appear to have curious factorization properties, in particular, having factors which are *cyclotomic polynomials*. See Table 2. This paper explains the occurrence of certain cyclotomic factors. We have displayed these in boldface in the tables. Both combinatorial and number-theoretic properties (for example, the number of 1's in binary expansion of n and the prime factorization of n) are involved in our investigations.

The divisibility by cyclotomic factors is related to the remainders of sizes of descent classes modulo integers. As a simplest example, $Q_n(t)$ is divisible by the second cyclotomic polynomial Φ_2 if and only if the number of even descent classes

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is equal to the number of odd descent classes. In other words, the proportion of even and odd entries in the descent statistics is the same (in the notation below, $\rho(n) = 1/2$) if and only if -1 is a root of the descent polynomial. Somewhat surprisingly, whether or not n has this property depends only on the number of 1's in the binary expansion of n.

The paper proceeds as follows. In Section 2 we review quasisymmetric functions and their relation to posets. In Section 3 we look at the proportion of odd entries in the descent statistics. We consider similar properties for the signed descent statistics in Section 4. The natural setting for this question is to look at flag vectors of zonotopes. In Section 5 we explore patterns of descent statistics modulo $2p$ for prime p. Here we introduce the descent polynomial and consider divisibility by cyclotomic polynomials. In Section 7 we explore when the quadratic factors Φ_2^2, Φ_4^2 and Φ_{2p}^2 occur in the decomposition. In Section 8 we introduce type B quasisymmetric functions and the *signed descent polynomials*. We use the former to describe divisibility patterns of the latter. Finally, in the concluding remarks we make a number of observations on the data presented in Tables 2 and 3.

2. Quasisymmetric functions and posets

Consider the ring $\mathbb{Z}[[w_1, w_2, \ldots]]$ of power series with bounded degree. A function f in this ring is called *quasisymmetric* if for any positive integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ we have

$$\left[w_{i_1}^{\alpha_1} \cdots w_{i_k}^{\alpha_k}\right] f = \left[w_{j_1}^{\alpha_1} \cdots w_{j_k}^{\alpha_k}\right] f$$

whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$, and where $[w^\alpha]f$ denotes the coefficient of w^α in f. Denote by $\mathrm{QSym} \subseteq \mathbb{Z}[[w_1, w_2, \ldots]]$ the ring of quasisymmetric functions. With an appropriate choice of coproduct, QSym is a Hopf algebra. See for example [6].

For a positive integer n, a *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. For a composition α the *monomial quasisymmetric function* M_α is given by

$$M_\alpha = \sum_{i_1 < \cdots < i_k} w_{i_1}^{\alpha_k} \cdots w_{i_k}^{\alpha_k}.$$

Definition 2.1. *Let P be a graded poset with rank function ρ. Define the quasisymmetric function $F(P)$ of the poset P by*

$$F(P) = \sum_{c} M_{\rho(c)},$$

where the sum ranges over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in P, $\rho(c)$ denotes the composition $(\rho(x_0, x_1), \rho(x_1, x_2), \ldots, \rho(x_{k-1}, x_k))$, and where $\rho(x, y)$ denotes the rank difference $\rho(x, y) = \rho(y) - \rho(x)$.

The quasisymmetric function of a poset is multiplicative, that is, for two graded posets P and Q the quasisymmetric function of their Cartesian product is given by the product of the respective quasisymmetric functions

$$F(P \times Q) = F(P) \cdot F(Q).$$

See Proposition 4.4 in [6]. This result may be stated that F is an algebra map. In fact, a stronger result holds, namely, the function F is a Hopf algebra homomorphism from the Hopf algebra of graded posets to the Hopf algebra of quasisymmetric functions. However, this more general statement will not be needed here.

Let P be a graded poset of rank n. For a subset $S = \{s_1 < s_2 < \cdots < s_{k-1}\}$ of $[n-1]$, define the *flag f-vector* entry f_S to be the number of chains $\{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ such that $\rho(x_i) = s_i$ for $1 \leq i \leq k-1$. The *flag h-vector* is defined by the invertible relation

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

Define a bijection D between subsets of the set $[n-1]$ and compositions of n by sending the set $\{s_1 < s_2 < \cdots < s_{k-1}\}$ to the composition $(s_1, s_2 - s_1, s_3 - s_2, \ldots, n - s_{k-1})$. By abuse of notation we will write M_S instead of $M_{D(S)}$, where the degree of the quasisymmetric function is understood. Then Lemma 4.2 in [6] states that the quasisymmetric function of a poset encodes the flag f-vector, that is,

$$F(P) = \sum_{S \subseteq [n-1]} f_S \cdot M_S.$$

The *fundamental quasisymmetric function* L_T is given by

$$L_T = \sum_{T \subseteq S} M_S.$$

Note that one can write $F(P)$ in terms of fundamental quasisymmetric functions:

$$F(P) = \sum_{S \subseteq [n-1]} h_S \cdot L_S.$$

For $\pi = \pi_1 \cdots \pi_n$ a permutation in $\mathfrak{S}_n$, recall that the *descent set* of π is the subset of $[n-1]$ given by $\{i : \pi_i > \pi_{i+1}\}$, where $[m]$ denotes the set $\{1, \ldots, m\}$. For a subset S of $[n-1]$, let $\beta(S)$ denote the number of permutations in $\mathfrak{S}_n$ with descent set S. If the poset P is a Boolean algebra B_n then $h_S = \beta(S)$. This is straightforward to observe using the classical R-labeling of the Boolean algebra [16, Section 3.13] or by direct enumeration [16, Corollary 3.12.2].

Lemma 2.2. *For $m = 2^j$ we have $F(B_m) \equiv M_{(m)} \bmod 2$. Consequently, for $n = 2^{j_1} + \cdots + 2^{j_k}$ with $j_1 > \cdots > j_k \geq 0$ we have $F(B_n) \equiv \prod_{i=1}^{k} M_{(2^{j_i})} \bmod 2$.*

Proof. The first statement can be proved by induction using F as an algebra map and that $(a+b)^2 \equiv a^2 + b^2 \bmod 2$. The second statement follows from the facts that F is an algebra map and that the Boolean algebra can be realized as a Cartesian product. □

k	$n = 2^k - 1$	$\rho(n)$	$1/2 - \rho(n)$
1	1	1	$-1/2$
2	3	$1/2$	0
3	7	$1/2$	0
4	15	$29/2^6$	$3/2^6$
5	31	$3991/2^{13}$	$3 \cdot 5 \cdot 7/2^{13}$

TABLE 1. Values of the proportion $\rho(n)$ for at most five 1's in the binary expansion of n.

3. The proportion of odd entries

Let $\rho(n)$ denote the proportion of odd entries in the descent statistics in the symmetric group $\mathfrak{S}_n$, that is,

$$\rho(n) = \frac{|\{S \subseteq [n-1] \,:\, \beta(S) \equiv 1 \bmod 2\}|}{2^{n-1}}.$$

For instance, $\rho(3) = 1/2$ since in the data $\beta(\emptyset) = \beta(\{1,2\}) = 1$ and $\beta(\{1\}) = \beta(\{2\}) = 2$ exactly half of the entries are odd.

Our first result is that the proportion $\rho(n)$ depends on the number of 1's in the binary expansion of n.

Theorem 3.1. *The proportion of odd entries in the descent set statistics $\rho(n)$ only depend on the number of 1's in the binary expansion of the integer n.*

The first few values of the proportion $\rho(n)$ are shown in Table 1.

An *ordered partition* π of a set $[k]$ is a list of non-empty pairwise disjoint sets $(B_1, B_2, \ldots, B_j)$ such that their union is $[k]$. There is a natural ordering on ordered partitions with the cover relation $\pi \prec \sigma$, whenever σ is obtained from π by merging two adjacent blocks. This partial order coincides with that of the face poset of the dual of the boundary of the $(k-1)$-dimensional permutahedron.

Theorem 3.2. *For positive integers $m_1, m_2, \ldots, m_k$, the product of the quasi-symmetric functions $M_{(m_1)} \cdot M_{(m_2)} \cdots M_{(m_k)}$ is given by*

$$M_{(m_1)} \cdot M_{(m_2)} \cdots M_{(m_k)} = \sum_{\pi} M_{\left(\sum_{i \in B_1} m_i, \sum_{i \in B_2} m_i, \ldots, \sum_{i \in B_j} m_i\right)},$$

where $\pi = (B_1, B_2, \ldots, B_j)$ ranges over all ordered partitions of the set $[k]$.

Proof. The theorem can be proved by iterating Lemma 3.3 in [6]. □

Definition 3.3. *A* knapsack system *$m_1, m_2, \ldots, m_k$ is a list of positive integers such that the 2^k partial sums of these integers are all distinct. That is,*

$$\left| \left\{ \sum_{i \in S} m_i \;:\; S \subseteq [k] \right\} \right| = 2^k.$$

As an example, suppose that a, b, c is a knapsack system. Then the product $M_{(a)} \cdot M_{(b)} \cdot M_{(c)}$ has the expansion

$$\begin{aligned} & M_{(a)} \cdot M_{(b)} \cdot M_{(c)} \\ = \; & M_{(a,b,c)} + M_{(a,c,b)} + M_{(b,a,c)} + M_{(b,c,a)} + M_{(c,a,b)} + M_{(c,b,a)} \\ & + M_{(a+b,c)} + M_{(c,a+b)} + M_{(a+c,b)} + M_{(b,a+c)} + M_{(b+c,a)} + M_{(a,b+c)} \\ & + M_{(a+b+c)}. \end{aligned}$$

By changing the index notation from compositions to sets, we can write this as

$$\begin{aligned} & M_{(a)} \cdot M_{(b)} \cdot M_{(c)} \\ = \; & M_{\{a,a+b\}} + M_{\{a,a+c\}} + M_{\{b,a+b\}} + M_{\{b,b+c\}} + M_{\{c,a+c\}} + M_{\{c,b+c\}} \\ & + M_{\{a+b\}} + M_{\{c\}} + M_{\{a+c\}} + M_{\{b\}} + M_{\{b+c\}} + M_{\{a\}} \\ & + M_{\emptyset}. \end{aligned}$$

Observe that the only elements occurring in these sets are a, b, c, $a+b$, $a+c$ and $b+c$. Moreover, these six elements are distinct exactly when a, b, c is a knapsack system. The 13 sets appearing in the expansion encode the face poset of the boundary of a hexagon, that is, the hexagon is self-dual and is the two-dimensional permutahedron.

Theorem 3.4. *For a knapsack system $m_1, m_2, \ldots, m_k$ the proportion of odd coefficients in the quasisymmetric function $f = M_{(m_1)} \cdot M_{(m_2)} \cdots M_{(m_k)}$ when expressed in the L-basis only depends on k.*

Proof. We have to express the quasisymmetric function f in terms of the L-basis. That is,

$$\begin{aligned} f &= \sum_S M_S \\ &= \sum_S \sum_{S \subseteq T} (-1)^{|T-S|} \cdot L_T, \end{aligned}$$

where we restrict S to range over sets corresponding to ordered partitions; see Theorem 3.2. Thus the coefficient of L_T is given by the sum $\sum_{S \subseteq T} (-1)^{|T-S|}$.

Call the $2^k - 2$ elements of the set $[n-1]$ of the form $\sum_{i \in B} m_i$ for $\emptyset \subset B \subset [k]$ *essential*, and the other $n - 1 - (2^k - 2)$ elements *nonessential*. For T an index set not containing the nonessential element i, observe that the coefficients of L_T and $L_{T \cup \{i\}}$ are the same modulo 2. Hence the proportion of odd coefficients in the quasisymmetric function f only depends on sets T consisting solely of essential elements. Since the collection of sets S encodes the poset of ordered partitions of the set $[k]$, we conclude that the proportion depends on k and not on the elements of the knapsack system. □

Proof of Theorem 3.1: The proof follows by considering the knapsack system $2^{j_1}, 2^{j_2}, \ldots, 2^{j_k}$ where $j_1 > j_2 > \cdots > j_k \geq 0$ and $n = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_k}$. □

Theorem 3.1 implies that for $n = 2^j$ all the values in the descent set statistics are odd. Hence it is interesting to look at this data modulo 4.

Theorem 3.5. *For $n = 2^j \geq 4$ exactly half of the descent set statistics are congruent to* 1 *modulus* 4*, and the other half are congruent to* 3 *modulus* 4*.*

Proof. First we claim that $F(B_n) \equiv M_{(n)} + 2 \cdot M_{(n/2,n/2)} \bmod 4$. This identity follows from the observation $(a + 2 \cdot b)^2 \equiv a^2 \bmod 4$ and by induction on j, where the induction step is $F(B_{2n}) \equiv (M_{(n)} + 2 \cdot M_{(n/2,n/2)})^2 \equiv M_{(n)}^2 \equiv M_{(2n)} + 2 \cdot M_{(n,n)} \bmod 4$. We have the expansion

$$\begin{aligned} M_{(n)} + 2 \cdot M_{(n/2,n/2)} &= M_{\emptyset} + 2 \cdot M_{\{n/2\}} \\ &= \sum_{S} (-1)^{|S|} \cdot L_S - 2 \cdot \sum_{n/2 \in S} (-1)^{|S|} \cdot L_S \\ &= \sum_{n/2 \notin S} (-1)^{|S|} \cdot L_S - \sum_{n/2 \in S} (-1)^{|S|} \cdot L_S. \end{aligned}$$

Hence the descent set statistics modulo 4 are given by $\beta(S) \equiv (-1)^{|S - \{n/2\}|} \bmod 4$. Thus for $1 \notin S$ the values of $\beta(S)$ and $\beta(S \cup \{1\})$ have the opposite sign modulo 4, proving the result. $\square$

4. The signed descent set statistics

A *signed permutation* is of the form $\pi = \pi_1 \cdots \pi_n$ where each π_i belongs to the set $\{\pm 1, \ldots, \pm n\}$ and $|\pi_1| \cdots |\pi_n|$ is a permutation. Let $\mathfrak{S}_n^{\pm}$ be the set of signed permutations on n elements. For ease of notation let $\pi_0 = 0$. The descent set of a signed permutation π is a subset of $[n]$ defined as $\{i : \pi_{i-1} > \pi_i\}$. For $S \subseteq [n]$ let $\beta^{\pm}(S)$ denote the number of permutations in $\mathfrak{S}_n^{\pm}$ with descent set S.

An equivalent way to using quasisymmetric functions to encode the flag f-vector data of a poset is via the **ab**-index. Let **a** and **b** be two non-commutative variables. For $S \subseteq [n-1]$ let u_S be the monomial $u_1 u_2 \cdots u_{n-1}$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The **ab**-*index* of a poset P of rank n is defined as the sum

$$\Psi(P) = \sum_{S} h_S \cdot u_S.$$

When the poset P is Eulerian then its **ab**-index can be written in terms of $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This more compact form removes all linear redundancies among the flag vector entries [2]. The linear relations satisfied by the flag f-vectors of Eulerian posets are known as the generalized Dehn-Sommerville relations [1]. Similar to quasisymmetric functions, the **ab**-index and **cd**-index also have an underlying coalgebra structure. For more details, see [10].

The poset associated to signed permutations is the cubical lattice C_n, that is, the face lattice of an n-dimensional cube. Observe that C_n has rank $n + 1$. We have

$$\Psi(C_n) = \sum_{S} \beta^{\pm}(S) \cdot u_S,$$

where S ranges over all subsets of $[n]$.

A more general setting for the $\mathbf{cd}$-index of the cube is that of zonotopes. Recall that a *zonotope* is a Minkowski sum of line segments. Associated to every zonotope Z there is a central hyperplane arrangement $\mathcal{H}$. Let L be the intersection lattice of the arrangement $\mathcal{H}$. A result by Billera–Ehrenborg–Readdy shows how to compute the $\mathbf{cd}$-index of the zonotope from the $\mathbf{ab}$-index of the intersection lattice L. First, introduce the linear map ω from $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ to $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ defined on an $\mathbf{ab}$-monomial as follows. Replace each occurrence of $\mathbf{ab}$ by $2\mathbf{d}$ and then replace the remaining letters by $\mathbf{c}$. The main result in [3] states that the $\mathbf{cd}$-index of the zonotope Z is given by

$$\Psi(Z) = \omega(\mathbf{a} \cdot \Psi(L)). \tag{4.1}$$

Especially, for the cubical lattice we have that

$$\Psi(C_n) = \omega(\mathbf{a} \cdot \Psi(B_n)), \tag{4.2}$$

since the associated hyperplane arrangement is the coordinate arrangement and its intersection lattice is the Boolean algebra.

Considering Equation (4.1) modulo 2, we observe that $\Psi(Z) \equiv \mathbf{c}^n \bmod 2$ and hence we conclude the following result.

Lemma 4.1. *All the entries of the flag h-vector of a zonotope are odd. In particular, all the signed descent set statistics are odd.*

In order to understand the flag h-vector modulo 4, we need a few lemmas.

Lemma 4.2. *After expanding the* $\mathbf{cd}$*-polynomial*

$$\sum_{i=0}^{n-2} \mathbf{c}^i \cdot \mathbf{d} \cdot \mathbf{c}^{n-i-2}$$

into an $\mathbf{ab}$*-polynomial, exactly half of the coefficients are odd.*

Proof. Since $\mathbf{d} \equiv \mathbf{ab} - \mathbf{ba} \bmod 2$, it is sufficient to consider the identity

$$\sum_{i=0}^{n-2} \mathbf{c}^i \cdot (\mathbf{ab} - \mathbf{ba}) \cdot \mathbf{c}^{n-i-2} = \mathbf{a} \cdot (\mathbf{a} + \mathbf{b})^{n-2} \cdot \mathbf{b} - \mathbf{b} \cdot (\mathbf{a} + \mathbf{b})^{n-2} \cdot \mathbf{a}.$$

This identity holds since the coefficient of an $\mathbf{ab}$-polynomial in the sum is the number of occurrences of $\mathbf{ab}$ minus the number of occurrences of $\mathbf{ba}$ in the monomial. This difference only depends on the first and last letter in the monomial and the identity follows. Now the conclusion follows by observing that out of 2^n $\mathbf{ab}$-monomials of degree n, exactly 2^{n-1} appear in the right-hand side of the identity. $\square$

Lemma 4.3. *Let z and w be two homogeneous polynomials in $\mathbb{Z}\langle\mathbf{a},\mathbf{b}\rangle$ of degree m and n, respectively, each having exactly half of their coefficients odd. Then the two* $\mathbf{ab}$*-polynomials*

$$\mathbf{c}^i \cdot z \cdot \mathbf{c}^j \quad \textit{and} \quad z \cdot \mathbf{c}^n + \mathbf{c}^m \cdot w$$

also each have exactly half of their coefficients odd.

Proof. We only prove the second statement of the lemma. We omit the proof of the first, as it is similar and easier. Let u and v be two **ab**-monomials of degrees m and n, respectively. The coefficient of $u \cdot v$ in $z \cdot \mathbf{c}^n + \mathbf{c}^m \cdot w$ is given by the sum of the coefficients of u in z and of v in w. Hence the coefficient of $u \cdot v$ is even when the coefficients of u and v are both even ($2^{m-1} \cdot 2^{n-1}$ cases) or the coefficients of u and v are both odd ($2^{m-1} \cdot 2^{n-1}$ cases), completing this case. $\square$

Combining Lemmas 4.2 and 4.3, we have:

Proposition 4.4. *Let $\alpha_1, \ldots, \alpha_n$ be integers, not all of which are even. When expanding the* **cd***-polynomial*

$$\sum_{i=0}^{n-2} \alpha_i \cdot \mathbf{c}^i \cdot \mathbf{d} \cdot \mathbf{c}^{n-i-2}$$

into an **ab***-polynomial, exactly half of the coefficients are odd.*

Theorem 4.5. *For a zonotope Z either (i) exactly half of the flag h-vector entries are congruent to 1 modulo 4, and the other half are congruent to 3 modulo 4; or (ii) all the flag h-vector entries are congruent to 1 modulo 4.*

Proof. Considering the identity (4.1) modulo 4, we observe that the only terms appearing are $\mathbf{c}^n$ and those **cd**-monomials with even coefficients having exactly one $\mathbf{d}$, that is,

$$\Psi(Z) \equiv \mathbf{c}^n + 2 \cdot \left(\sum_{i=0}^{n-2} \alpha_i \cdot \mathbf{c}^i \cdot \mathbf{d} \cdot \mathbf{c}^{n-i-2} \right) \bmod 4.$$

If all the α_i's are even then the flag h-vector entries are congruent to 1 modulo 4. If at least one α_i is odd, then by Proposition 4.4 we know that exactly half of the flag h-vector entries are congruent to 1 modulo 4 and the other half are congruent to 3 modulo 4. $\square$

Now we consider the cubical lattice, that is, the signed descent set statistics modulo 4.

Theorem 4.6. *For an integer $n \geq 2$, exactly half of the signed descent set statistics are congruent to 1 modulo 4, and the other half are congruent to 3 modulo 4.*

Proof. Observe that there are 2^n atoms in the cubical lattice. Hence $h_{\{1\}} = 2^n - 1 \equiv 3 \bmod 4$. Thus the result follows from Theorem 4.5. $\square$

5. Descent set statistics modulo $2p$

For a set S of integers and a non-zero integer q, define the two notions $q \cdot S$ and S/q by

$$\begin{aligned} q \cdot S &= \{q \cdot s \ : \ s \in S\}, \\ S/q &= \{s/q \ : \ s \in S \text{ and } q \mid s\}. \end{aligned}$$

In what follows let $\beta_i(S)$ denote the descent set statistic $\beta(S)$ in the symmetric group $\mathfrak{S}_i$ and let $\alpha_i(S)$ denote the corresponding alpha statistic $\alpha(S)$ in $\mathfrak{S}_i$.

Proposition 5.1. *Let $q = p^t$, where p is a prime and t is a non-negative integer. Let $n = r \cdot q$, where r is a positive integer. Then the descent set statistics modulo p in the symmetric group $\mathfrak{S}_n$ are given by*

$$\beta_n(S) \equiv (-1)^{|S - q\cdot[r-1]|} \cdot \beta_r(S/q) \bmod p,$$

where $S \subseteq [n-1]$.

Proof. Observe that

$$M_{(1)}^r = F(B_r) = \sum_{S\subseteq[r-1]} \alpha_r(S) \cdot M_S.$$

In this quasisymmetric function identity make the plethystic substitution $w_i \longmapsto w_i^q$. We then obtain

$$\begin{aligned}
M_{(q)}^r &= \sum_{S\subseteq[r-1]} \alpha_r(S) \cdot M_{q\cdot S} \\
&= \sum_{S\subseteq[r-1]} \sum_{\substack{T\subseteq[n-1] \\ q\cdot S\subseteq T}} \alpha_r(S) \cdot (-1)^{|T - q\cdot S|} \cdot L_T \\
&= \sum_{T\subseteq[n-1]} \sum_{q\cdot S\subseteq T} \alpha_r(S) \cdot (-1)^{|T - q\cdot S|} \cdot L_T \\
&= \sum_{T\subseteq[n-1]} (-1)^{|T - q\cdot[r-1]|} \cdot \left(\sum_{S\subseteq T/q} \alpha_r(S) \cdot (-1)^{|T/q - S|} \right) \cdot L_T \\
&= \sum_{T\subseteq[n-1]} (-1)^{|T - q\cdot[r-1]|} \cdot \beta_r(T/q) \cdot L_T.
\end{aligned}$$

Since $M_1^q \equiv M_{(q)} \bmod p$, we have $F(B_n) = (M_1^q)^r \equiv M_{(q)}^r \bmod p$. Now by reading off the coefficients of L_T, the result follows. $\square$

Corollary 5.2. *Let $q = p^t$, where p is a prime and t is a non-negative integer. Then the descent set statistics for the symmetric group $\mathfrak{S}_q$ satisfy*

$$\beta(S) \equiv (-1)^{|S|} \bmod p.$$

Proof: The claim can be deduced from Proposition 5.1 by setting $r = 1$. A direct argument proceeds as follows. Since $(a + b)^q \equiv a^q + b^q \bmod p$, we have

$$F(B_q) = (w_1 + w_2 + \cdots)^q \equiv w_1^q + w_2^q + \cdots = M_{(q)} = \sum_S (-1)^{|S|} \cdot L_S \bmod p. \quad \square$$

Corollary 5.3. *Let $q = p^t$, where p is a prime and t is a non-negative integer. Then the descent set statistics for the symmetric group $\mathfrak{S}_n$ of permutations on $n = 2q$ elements satisfy*

$$\beta(S) \equiv (-1)^{|S-\{q\}|} \bmod p.$$

Proof. Follows from Proposition 5.1 by setting $r = 2$ and noting that $\beta_2(\emptyset) = \beta_2(\{1\}) = 1$. $\square$

For n having the binary expansion $n = 2^{j_1} + 2^{j_2} + \cdots + 2^{j_k}$, where $j_1 > j_2 > \cdots > j_k \geq 0$, recall an element $j \in [n-1]$ is nonessential if j is not a sum of a subset of $\{2^{j_1}, 2^{j_2}, \ldots, 2^{j_k}\}$.

Theorem 5.4. *Let $q = p^t$, for p an odd prime and t a non-negative integer, and let $n = r \cdot q$, where r is a positive integer. Assume that there is a nonessential element $j \in [n-1]$ that is not divisible by q. Finally, assume that the descent set statistics β in the symmetric group $\mathfrak{S}_n$ take on exactly one of two values modulo p, say a and b, where a and b have the same parity. Then*

$$\begin{aligned} |\{S \subseteq [n-1] \,:\, \beta(S) \equiv a\}| &= |\{S \subseteq [n-1] \,:\, \beta(S) \equiv b\}|, \\ |\{S \subseteq [n-1] \,:\, \beta(S) \equiv a+p\}| &= |\{S \subseteq [n-1] \,:\, \beta(S) \equiv b+p\}|, \end{aligned}$$

where all the congruences are modulo $2p$. In the case when the proportion $\rho(n)$ is $1/2$, the four cardinalities above are all equal to 2^{n-3}.

Proof. Consider the collection of sets $S \subseteq [n-1]$ such that $\beta(S) \equiv a \equiv b \bmod 2$. For S in this collection such that $j \notin S$, we have $\beta(S) \equiv \beta(S \cup \{j\}) \bmod 2$. However, since q does not divide j we have $\beta(S) \equiv -\beta(S \cup \{j\}) \bmod p$. Hence this collection splits into two classes of equal size when divided according to the value of $\beta(S)$ modulo $2p$. The same argument holds for the sets S satisfying $\beta(S) \equiv a + p \equiv b + p \bmod 2$. $\square$

By the Chinese remainder theorem, we have $\beta(S) \equiv \pm 1, p \pm 1 \bmod 2p$. For non-Mersenne primes we can say more.

Theorem 5.5. *Let $q = p^t$ be an odd prime power which has k 1's in its binary expansion. Assume that $q > 2^k - 1$, that is, q is not a Mersenne prime. Then*

$$\begin{aligned} |\{S \subseteq [q-1] \,:\, \beta(S) \equiv 1\}| &= |\{S \subseteq [q-1] \,:\, \beta(S) \equiv -1\}|, \\ |\{S \subseteq [q-1] \,:\, \beta(S) \equiv p-1\}| &= |\{S \subseteq [q-1] \,:\, \beta(S) \equiv p+1\}|, \end{aligned}$$

where all the congruences are modulo $2p$ and β denotes descent set statistic in $\mathfrak{S}_q$. In the case the proportion $\rho(q)$ is $1/2$, the four cardinalities above are equal to 2^{q-3}.

Proof. Since $q > 2^k - 1$, as in the proof of Theorem 3.4 there exists a nonessential element $j \in [q-1]$. Now Theorem 5.4 applies. $\square$

When $q = p$ is a prime and $k = 2$, Theorem 5.5 applies only to the Fermat primes which are greater than 3, that is, 5, 17, 257 and 65537. We also know that the proportion is $1/2$ for the case $k = 3$, that is, primes whose binary expansion has three 1's. The first few such primes are $7, 11, 13, 19, 37, 41, 67, 73, 97$; see sequence A081091 in The On-Line Encyclopedia of Integer Sequences.

For prime powers of the form $q = p^t$ with $t \geq 2$, the only case with $k = 2$ we know is $q = 3^2$. Similarly, with $k = 3$ we know six cases: 5^2, 7^2, 3^4, 17^2, 23^2, 257^2 and 65537^2. It is not surprising that the squares of the Fermat's primes and the square of 3^2 appear in this list. The two sporadic cases are 7^2 and 23^2.

The next theorem concerns permutations of length twice a prime power.

Theorem 5.6. *Let $q = p^t$ be an odd prime power which has k 1's in its binary expansion. Then*

$$\begin{aligned} |\{S \subseteq [2q-1] \,:\, \beta(S) \equiv 1\}| &= |\{S \subseteq [2q-1] \,:\, \beta(S) \equiv -1\}|, \\ |\{S \subseteq [2q-1] \,:\, \beta(S) \equiv p-1\}| &= |\{S \subseteq [2q-1] \,:\, \beta(S) \equiv p+1\}|, \end{aligned}$$

where all the congruences are modulo $2p$ and β denotes descent set statistic in $\mathfrak{S}_{2q}$. In the case the proportion $\rho(q)$ is $1/2$, the four cardinalities above are equal to 2^{2q-3}.

Proof. By Corollary 5.3 we know that $\beta(S) \equiv (-1)^{|S-\{q\}|} \bmod p$. Furthermore, since $2q$ is even, the element 1 is nonessential, and Theorem 5.4 applies. □

6. The descent set polynomial

A different approach to view the results from the previous sections is in terms of the *descent set polynomial*

$$Q_n(t) = \sum_{S \subseteq [n-1]} t^{\beta(S)}.$$

The degree of this polynomial is the nth Euler number E_n. For $n \geq 2$ the polynomial is divisible by $2t$. Theorems 3.1, 3.5, 5.5 and 5.6 can be reformulated as follows.

Theorem 6.1. (i) *For a positive integer n we have $Q_n(-1) = 2^n \cdot (1/2 - \rho(n))$. In particular, when n has two or three 1's in its binary expansion, then -1 is a root of $Q_n(t)$.*

(ii) *For $n = 2^j \geq 4$ the imaginary unit i is a root of $Q_n(t)$.*

(iii) *Let $q = p^t$ be a prime power, where p is an odd prime. Assume that q has k 1's in its binary expansion and satisfies $q > 2^k - 1$. Let ζ be a primitive $2p$-th root of unity. Then*

$$Q_q(\zeta) = 2^q \cdot \Re(\zeta) \cdot \left(\rho(q) - \frac{1}{2}\right),$$

where $\Re(\zeta)$ denotes the real part of ζ.

(iv) *Let $q = p^t$ be a prime power, where p is an odd prime. Assume that q has k 1's in its binary expansion. Let ζ be a primitive $2p$-th root of unity. Then*

$$Q_{2q}(\zeta) = 2^{2q} \cdot \Re(\zeta) \cdot \left(\rho(q) - \frac{1}{2}\right).$$

It is curious to observe that the polynomial $Q_n(t)$ quite often has zeroes occurring at roots of unity. An equivalent formulation is that $Q_n(t)$ often has cyclotomic polynomials $\Phi_k(t)$ as factors. (Recall that the *cyclotomic polynomial* $\Phi_k(t)$ is defined as the product $\prod_\zeta (t-\zeta)$, where ζ ranges over all primitive kth roots of unity.) See Table 2 for the cyclotomic factors of $Q_n(t)$ for $n \leq 23$.

Lemma 6.2. *Let q be an odd prime power. Then the cyclotomic polynomial Φ_q does not divide the descent set polynomial $Q_n(t)$.*

Proof. The cyclotomic polynomial Φ_q evaluated at 1 is given by the odd prime p where $q = p^t$. Since $Q_n(1) = 2^{n-1}$ has no odd factors, the lemma follows. $\square$

Lemma 6.3. *Let q be the odd prime power p^t. Then*

(i) *If $n = 2^j$ then the cyclotomic polynomial Φ_{2q} does not divide $Q_n(t)$.*
(ii) *If n has four 1's in its binary expansion and $p \geq 5$ then the cyclotomic polynomial Φ_{2q} does not divide $Q_n(t)$.*
(iii) *If n has five 1's in its binary expansion and $p \geq 11$ then the cyclotomic polynomial Φ_{2q} does not divide $Q_n(t)$.*

Proof. The cyclotomic polynomial Φ_{2q} evaluated at -1 equals the prime p. Since $Q_n(-1) = 2^n \cdot (1/2 - \rho(n))$ the result follows by consulting Table 1. $\square$

7. Quadratic factors in the descent set polynomial

In order to study the double root behavior of the descent set polynomial $Q_n(t)$, equivalently, quadratic factors in $Q_n(t)$, we need to prove a few identities for the descent set statistics. We begin by introducing the multivariate **ab**- and **cd**-indexes. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots$ be non-commutative variables. For $S \subseteq [n-1]$ let u_S be the monomial $u_1 u_2 \cdots u_{n-1}$ where $u_i = \mathbf{a}_i$ if $i \notin S$ and $u_i = \mathbf{b}_i$ if $i \in S$. The *multivariate* **ab**-*index* of a poset P of rank n is defined as the sum

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where S ranges over all subsets of $[n-1]$.

Lemma 7.1. *For an Eulerian poset P the multivariate* **ab**-*index can be written in terms of the non-commutative variables $\mathbf{c}_i = \mathbf{a}_i + \mathbf{b}_i$ and $\mathbf{d}_{i,i+1} = \mathbf{a}_i \mathbf{b}_{i+1} + \mathbf{b}_i \mathbf{a}_{i+1}$.*

In this case, we call the resulting polynomial the *multivariate* **cd**-*index*. Observe that for a rank n Eulerian poset each of the indices 1 through n appears in each monomial of the multivariate **cd**-index.

Proposition 7.2. *Let h_S be the flag h-vector of an Eulerian poset P of rank n, or more generally, belong to the generalized Dehn-Sommerville subspace. Let $T \subseteq [n-1]$ such that T contains an interval $[s,t] = \{s, s+1, \ldots, t\}$ of odd cardinality with $s-1, t+1 \notin T$. Then*

$$\sum_S (-1)^{|S \cap T|} \cdot h_S = 0,$$

where S ranges over all subsets of $[n-1]$.

Proof. The sum is obtained from the multivariate **ab**-index of the poset P by setting $\mathbf{a}_i = 1$ and

$$\mathbf{b}_i = \begin{cases} -1 & \text{if } i \in T, \\ 1 & \text{otherwise.} \end{cases}$$

Notice that $\mathbf{c}_i = 0$ for $i \in [s,t]$ and that $\mathbf{d}_{s-1,s} = \mathbf{d}_{t,t+1} = 0$. If $s = 1$ we set $\mathbf{d}_{0,1} = 0$ and if $t = n-1$ set $\mathbf{d}_{n-1,n} = 0$. Since P is Eulerian, the multivariate **ab**-index can be written in terms of multivariate **cd**-monomials. A multivariate **cd**-monomial that contains $\mathbf{d}_{s-1,s}$ or $\mathbf{d}_{t,t+1}$ evaluates to zero. Since the interval $[s,t]$ has odd size, a multivariate **cd**-monomial not containing $\mathbf{d}_{s-1,s}$ and $\mathbf{d}_{t,t+1}$ must contain at least one variable $\mathbf{c}_i$ with $i \in [s,t]$. Hence this monomial also evaluates to zero. □

Observe that the identity in Proposition 7.2 is a part of the generalized Dehn-Sommerville relations; see [1].

Theorem 7.3. *If the binary expansion of n has two 1's and $n > 3$, then Φ_2^2 divides $Q_n(t)$.*

Proof. Assume that $n = m_1 + m_2$, where $m_1 = 2^{j_1}$, $m_2 = 2^{j_2}$, and $j_1 > j_2$. From the proof of Theorem 3.4 we have

$$\beta(S) \equiv \begin{cases} 1 \mod 2 & \text{if } |S \cap \{m_1, m_2\}| = 0, 2, \\ 0 \mod 2 & \text{if } |S \cap \{m_1, m_2\}| = 1. \end{cases}$$

Hence

$$\begin{aligned} Q_n'(-1) &= \sum_S \beta(S) \cdot (-1)^{\beta(S)-1} \\ &= \sum_S (-1)^{|S \cap \{m_1, m_2\}|} \cdot \beta(S) \end{aligned}$$

which is zero by Proposition 7.2. □

Theorem 7.4. *If $n = 2^j \geq 4$ then Φ_4^2 divides $Q_n(t)$.*

Proof. Let $m = n/2$. The proof of Theorem 3.5 states that $\beta(S) \equiv (-1)^{|S - \{m\}|} \bmod 4$. Let i be the imaginary unit, so that $\mathrm{i}^2 = -1$. Observe $\mathrm{i}^{(-1)^k - 1} = (-1)^k$. Evaluating $Q_n'(t)$ at the imaginary unit i, we obtain

$$\begin{aligned} Q_n'(\mathrm{i}) &= \sum_S \beta(S) \cdot \mathrm{i}^{\beta(S)-1} \\ &= \sum_S (-1)^{|S - \{m\}|} \cdot \beta(S). \end{aligned}$$

By Proposition 7.2, $Q_n'(\mathrm{i}) = 0$, since $S - \{m\} = S \cap \{1, \ldots, m-1, m+1, \ldots, n-1\}$ and $m - 1$ is odd. □

The next result applies to prime powers that have two 1's in their binary expansion. The only cases known so far are the five known Fermat primes $3, 5, 17, 257, 65537$ and the prime power 3^2.

Theorem 7.5. *Let $q = p^t$ be a prime power, where p is an odd prime and assume that q has two 1's in its binary expansion. Then the cyclotomic polynomial Φ_{2p}^2 divides $Q_{2q}(t)$.*

Proof. In this case $n = 2 \cdot q = m + 2$, where $m = 2^j$. From the proof of Theorem 3.4 we have

$$\beta(S) \equiv \begin{cases} 1 \mod 2 & \text{if } |S \cap \{2, m\}| = 0, 2, \\ 0 \mod 2 & \text{if } |S \cap \{2, m\}| = 1. \end{cases}$$

Hence combining it with the proof of Corollary 5.3, we have

$$\beta(S) \equiv \begin{cases} (-1)^{|S - \{q\}|} \mod 2p & \text{if } |S \cap \{2, m\}| = 0, 2, \\ p + (-1)^{|S - \{q\}|} \mod 2p & \text{if } |S \cap \{2, m\}| = 1. \end{cases}$$

Thus for $\zeta = \Re(\zeta) + \Im(\zeta) \cdot i$ a $2p$-th primitive root of unity, we have that

$$\begin{aligned} \zeta^{\beta(S)} &= (-1)^{|S \cap \{2,m\}|} \cdot \zeta^{(-1)^{|S - \{q\}|}} \\ &= (-1)^{|S \cap \{2,m\}|} \cdot \Big(\Re(\zeta) + (-1)^{|S - \{q\}|} \cdot \Im(\zeta) \cdot i \Big) . \end{aligned}$$

Evaluating the sum and using the fact that $|S \cap \{2, m\}| + |S - \{q\}| \equiv |S \cap \{2, q, m\}| \mod 2$ we have

$$\begin{aligned} \zeta \cdot Q'_n(\zeta) &= \sum_S \beta(S) \cdot \zeta^{\beta(S)} \\ &= \Re(\zeta) \cdot \sum_S (-1)^{|S \cap \{2,m\}|} \cdot \beta(S) + \Im(\zeta) \cdot i \cdot \sum_S (-1)^{|S \cap \{2,q,m\}|} \cdot \beta(S), \end{aligned}$$

where both sums vanish by Proposition 7.2. □

8. The signed descent set polynomial

Similar to the descent set polynomial we can define the *signed descent set polynomial:*

$$Q_n^{\pm}(t) = \sum_S t^{\beta^{\pm}(S)},$$

where S ranges over all subsets of $[n]$. The degree of this polynomial is the nth signed Euler number $E_n^{\pm}$. Yet again, for $n \geq 1$ this polynomial is divisible by $2t$. Theorem 4.6 can now be stated as follows.

Theorem 8.1. *For $n \geq 2$ the signed descent set polynomial $Q_n^{\pm}(t)$ has the cyclotomic factor Φ_4.*

The space of quasisymmetric function of type B are defined as $\mathrm{BQSym} = \mathbb{Z}[s] \otimes \mathrm{QSym}$. Quasisymmetric functions of type B were first defined by Chow [5]. We will view them to be functions in the variables $s, w_1, w_2, \ldots$, that are quasisymmetric in $w_1, w_2, \ldots$ for a composition $(\alpha_0, \ldots, \alpha_k)$. Define the monomial quasisymmetric function of type B to be

$$M^B_{(\alpha_0, \alpha_1, \ldots, \alpha_k)} = s^{\alpha_0 - 1} \cdot M_{(\alpha_1, \ldots, \alpha_k)}.$$

A third method to encode the flag vector data of a poset P of rank at least 1 is the *quasisymmetric function of type B:*

$$F_B(P) = \sum_c M^B_{\rho(c)}, \tag{8.1}$$

where the sum ranges over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the poset P. See [11]. A different way to write equation (8.1) is

$$F_B(P) = \sum_{\hat{0} < x \leq \hat{1}} s^{\rho(x)-1} \cdot F([x, \hat{1}]). \tag{8.2}$$

Recall that the diamond product of two posets P and Q is $P \diamond Q = (P - \{\hat{0}\}) \times (Q - \{\hat{0}\}) \cup \{\hat{0}\}$. Using identity (8.2) one can show that the type B quasisymmetric function of a poset is multiplicative with respect to the diamond product of posets, that is, $F_B(P \diamond Q) = F_B(P) \cdot F_B(Q)$. Applying the bijection D between compositions and subsets, we have

$$F_B(P) = \sum_{S \subseteq [n-1]} f_S \cdot M^B_S,$$

where we write M^B_S instead of $M^B_{D(S)}$, and the poset P has rank n. The *fundamental quasisymmetric function of type B*, denoted by L^B_T, is given by

$$L^B_T = \sum_{T \subseteq S} M^B_S.$$

Then the flag h-vector appears as the coefficients in the type B quasisymmetric function

$$F_B(P) = \sum_{S \subseteq [n-1]} h_S \cdot L^B_S,$$

where the poset P has rank n.

Finally, recall that the cubical lattice C_n has rank $n+1$ and is obtained as a diamond power of the Boolean algebra B_2, that is, $C_n = B_2^{\diamond n}$. Therefore we have the following result.

Lemma 8.2. *The type B quasisymmetric function of the cubical lattice is given by*

$$F_B(C_n) = (s + 2 \cdot M_{(1)})^n.$$

Theorem 8.3. *For p an odd prime the cyclotomic polynomial Φ_{4p} divides the signed descent set polynomial $Q^{\pm}_p(t)$.*

Proof. Observe that modulo 4 we have

$$\begin{aligned}
F_B(C_p) &\equiv (s + 2 \cdot M_{(1)})^p \\
&\equiv s^p + 2 \cdot p \cdot s^{p-1} \cdot M_{(1)} \\
&\equiv M_{(p+1)} + 2 \cdot p \cdot M_{(p,1)} \\
&\equiv M_\emptyset + 2 \cdot M_{\{p\}} \\
&\equiv \sum_S (-1)^{|S|} \cdot L^B_S + 2 \cdot \sum_{p \in S} (-1)^{|S|-1} \cdot L^B_S \\
&\equiv \sum_{p \notin S} (-1)^{|S|} \cdot L^B_S + \sum_{p \in S} (-1)^{|S|-1} \cdot L^B_S \bmod 4.
\end{aligned}$$

Hence the signed descent statistics satisfy $\beta^{\pm}(S) \equiv (-1)^{|S-\{p\}|} \bmod 4$ for $S \subseteq [p]$. Now modulo p we have

$$\begin{aligned} F_B(C_p) &\equiv (s + 2 \cdot M_{(1)})^p \\ &\equiv s^p + 2 \cdot M_{(p)} \\ &\equiv M^B_{(p+1)} + 2 \cdot M^B_{(1,p)} \\ &\equiv M_{\emptyset} + 2 \cdot M_{\{1\}} \bmod p. \end{aligned}$$

This directly implies that $\beta^{\pm}(S) \equiv (-1)^{|S-\{1\}|} \bmod p$. Combining these two statements we obtain

$$\beta^{\pm}(S) \equiv \begin{cases} (-1)^{|S|} & \bmod 4p \quad \text{if } 1, p \notin S, \\ (-1)^{|S|-1} & \bmod 4p \quad \text{if } 1, p \in S, \\ 2 \cdot p + (-1)^{|S|} & \bmod 4p \quad \text{if } 1 \notin S, p \in S, \\ 2 \cdot p + (-1)^{|S|-1} & \bmod 4p \quad \text{if } 1 \in S, p \notin S. \end{cases} \tag{8.3}$$

Observe that for $p \notin S$ we have $\beta^{\pm}(S) \equiv \beta^{\pm}(S \cup \{p\}) + 2 \cdot p \bmod 4p$, implying that $\zeta^{\beta^{\pm}(S)} = -\zeta^{\beta^{\pm}(S \cup \{p\})}$ for ζ a $4p$-th primitive root of unity. Now sum over all subsets of $[p]$, and the result follows. □

Theorem 8.4. *For p an odd prime, Φ_{4p}^2 does not divide the signed descent set polynomial $Q_p^{\pm}(t)$. In fact, evaluating the derivative of the signed descent set polynomial $Q_p^{\pm}(t)$ at ζ, where ζ is a $4p$-th primitive root of unity, gives*

$$\zeta \cdot Q_p^{\pm\prime}(\zeta) = \Im(\zeta) \cdot \mathrm{i} \cdot (-1)^{(p-1)/2} \cdot 2^p \cdot p \cdot E_{p-1}.$$

Proof. From (8.3) we have:

$$\begin{aligned} \zeta \cdot Q_p^{\pm\prime}(\zeta) &= \sum_S \beta^{\pm}(S) \cdot \zeta^{\beta^{\pm}(S)} \\ &= \sum_S \beta^{\pm}(S) \cdot (-1)^{|S \cap \{1,p\}|} \cdot \zeta^{(-1)^{|S-\{1\}|}} \\ &= \Re(\zeta) \cdot \sum_S \beta^{\pm}(S) \cdot (-1)^{|S \cap \{1,p\}|} \\ &\quad + \Im(\zeta) \cdot \mathrm{i} \cdot \sum_S \beta^{\pm}(S) \cdot (-1)^{|S \cap \{1,p\}|} \cdot (-1)^{|S-\{1\}|}. \end{aligned}$$

The first sum is zero by Proposition 7.2. The second sum simplifies to

$$\Im(\zeta) \cdot \mathrm{i} \cdot \sum_S \beta^{\pm}(S) \cdot (-1)^{|S \cap [1,p-1]|}.$$

This sum can evaluated by setting $\mathbf{a}_j = 1$, $\mathbf{b}_1 = \cdots = \mathbf{b}_{p-1} = -1$ and $\mathbf{b}_p = 1$ in the the multivariate **ab**-index of the cubical lattice C_p. Observe that $\mathbf{c}_1 = \cdots = \mathbf{c}_{p-1} = 0$ and $\mathbf{d}_{p-1,p} = 0$. Hence the only surviving **cd**-monomial is $\mathbf{d}_{1,2} \cdots \mathbf{d}_{p-2,p-1} \mathbf{c}_p$. The

n	degree	cyclotomic factors of $Q_n(t)$
3	2	$\mathbf{\Phi_2}$
4	5	$\mathbf{\Phi_4^2}$
5	16	$\mathbf{\Phi_2^2} \cdot \mathbf{\Phi_{10}}$
6	61	$\mathbf{\Phi_2^2} \cdot \mathbf{\Phi_6^2} \cdot \Phi_{10}$
7	272	$\mathbf{\Phi_2}$
8	1385	$\mathbf{\Phi_4^2} \cdot \Phi_{28}$
9	7936	$\mathbf{\Phi_2^2} \cdot \mathbf{\Phi_6} \cdot \Phi_{18}$
10	50521	$\mathbf{\Phi_2^2} \cdot \Phi_6 \cdot \mathbf{\Phi_{10}^2} \cdot \Phi_{18} \cdot \Phi_{30}$
11	353792	$\mathbf{\Phi_2} \cdot \Phi_6 \cdot \mathbf{\Phi_{22}}$
12	2702765	$\mathbf{\Phi_2^2} \cdot \Phi_6 \cdot \Phi_{10} \cdot \Phi_{18} \cdot \Phi_{22} \cdot \Phi_{22} \cdot \Phi_{66} \cdot \Phi_{110} \cdot \Phi_{198}$
13	22368256	$\mathbf{\Phi_2} \cdot \mathbf{\Phi_{26}}$
14	$1.993 \cdot 10^8$	$\mathbf{\Phi_2} \cdot \Phi_2 \cdot \Phi_4 \cdot \mathbf{\Phi_{14}} \cdot \Phi_{14} \cdot \Phi_{26} \cdot \Phi_{28} \cdot \Phi_{182}$
15	$1.904 \cdot 10^9$	–
16	$1.939 \cdot 10^{10}$	$\mathbf{\Phi_4^2} \cdot \Phi_{12} \cdot \Phi_{20} \cdot \Phi_{44} \cdot \Phi_{52} \cdot \Phi_{60} \cdot \Phi_{156} \cdot \Phi_{220} \cdot \Phi_{260} \cdot \Phi_{572}$
17	$2.099 \cdot 10^{11}$	$\mathbf{\Phi_2^2} \cdot \mathbf{\Phi_{34}}$
18	$2.405 \cdot 10^{12}$	$\mathbf{\Phi_2^2} \cdot \mathbf{\Phi_6^2} \cdot \Phi_{18} \cdot \Phi_{34} \cdot \Phi_{102} \cdot \Phi_{306}$
19	$2.909 \cdot 10^{13}$	$\mathbf{\Phi_2} \cdot \mathbf{\Phi_{38}}$
20	$3.704 \cdot 10^{14}$	$\mathbf{\Phi_2^2} \cdot \Phi_6 \cdot \Phi_{10} \cdot \Phi_{30} \cdot \Phi_{34} \cdot \Phi_{38}^2 \cdot \Phi_{102} \cdot \Phi_{114} \cdot \Phi_{170}$ $\cdot \Phi_{190} \cdot \Phi_{510} \cdot \Phi_{570} \cdot \Phi_{646} \cdot \Phi_{1938} \cdot \Phi_{3230} \cdot \Phi_{9690}$
21	$4.951 \cdot 10^{15}$	$\mathbf{\Phi_2} \cdot \Phi_6 \cdot \Phi_{14} \cdot \Phi_{42}$
22	$6.935 \cdot 10^{16}$	$\mathbf{\Phi_2} \cdot \Phi_2 \cdot \Phi_{14} \cdot \mathbf{\Phi_{22}} \cdot \Phi_{22} \cdot \Phi_{154}$
23	$1.015 \cdot 10^{18}$	–

TABLE 2. Cyclotomic factors of $Q_n(t)$.

coefficient of this monomial is computed as follows.

$$\begin{aligned}
\left[\mathbf{d}^{(p-1)/2}\mathbf{c}\right] \Psi(C_p) &= 2^{(p-1)/2} \cdot \left[(2\mathbf{d})^{(p-1)/2}\mathbf{c}\right] \Psi(C_p) \\
&= 2^{(p-1)/2} \cdot \Big(\left[(\mathbf{ab})^{(p-1)/2}\mathbf{a}\right] \mathbf{a} \cdot \Psi(B_p) \\
&\qquad + \left[(\mathbf{ab})^{(p-1)/2}\mathbf{b}\right] \mathbf{a} \cdot \Psi(B_p) \Big) \\
&= 2^{(p-1)/2} \cdot p \cdot \left[\mathbf{b}(\mathbf{ab})^{(p-3)/2}\right] \Psi(B_{p-1}) \\
&= 2^{(p-1)/2} \cdot p \cdot E_{p-1}.
\end{aligned}$$

The third step is MacMahon's "Multiplication Theorem" (see [12, Article 159]). It can be stated in terms of the **ab**-indices as follows.

$$[u\mathbf{a}v]\Psi(B_{m+n}) + [u\mathbf{b}v]\Psi(B_{m+n}) = \binom{m+n}{m} \cdot [u]\Psi(B_m) + [v]\Psi(B_n),$$

where u and v have degrees $m-1$ and $n-1$, respectively. The monomial itself evaluates to $(-2)^{(p-1)/2} \cdot 2$, since $\mathbf{d}_{1,2} = \cdots = \mathbf{d}_{p-2,p-1} = -2$ and $\mathbf{c}_p = 2$. Combining all the factors, the evaluation at ζ follows. $\square$

n	degree	cyclotomic factors of $Q_n^{\pm}(t)$
2	3	$\boldsymbol{\Phi_4}$
3	11	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \boldsymbol{\Phi_{12}}$
4	57	$\boldsymbol{\Phi_4} \cdot \Phi_{16} \cdot \Phi_{32}$
5	361	$\boldsymbol{\Phi_4} \cdot \Phi_{16} \cdot \boldsymbol{\Phi_{20}} \cdot \Phi_{32} \cdot \Phi_{80}$
6	2763	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \Phi_{24} \cdot \Phi_{32} \cdot \Phi_{40} \cdot \Phi_{96} \cdot \Phi_{120} \cdot \Phi_{160}$
7	24611	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \Phi_{24} \cdot \boldsymbol{\Phi_{28}} \cdot \Phi_{32} \cdot \Phi_{56} \cdot \Phi_{168} \cdot \Phi_{224}$
8	250737	$\boldsymbol{\Phi_4} \cdot \Phi_{32} \cdot \Phi_{64} \cdot \Phi_{224} \cdot \Phi_{448} \cdot \Phi_{512}$
9	2873041	$\boldsymbol{\Phi_4} \cdot \Phi_{12} \cdot \Phi_{32} \cdot \Phi_{36} \cdot \Phi_{64} \cdot \Phi_{96} \cdot \Phi_{192} \cdot \Phi_{288} \cdot \Phi_{448}$
		$\cdot \Phi_{512}^2 \cdot \Phi_{576} \cdot \Phi_{1344} \cdot \Phi_{1536} \cdot \Phi_{4032} \cdot \Phi_{4608}$
10	36581523	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \Phi_{24} \cdot \Phi_{40} \cdot \Phi_{64} \cdot \Phi_{72} \cdot \Phi_{120} \cdot \Phi_{192} \cdot \Phi_{320} \cdot \Phi_{360}$
		$\cdot \Phi_{448} \cdot \Phi_{512} \cdot \Phi_{960} \cdot \Phi_{1344} \cdot \Phi_{1536} \cdot \Phi_{2240} \cdot \Phi_{2560} \cdot \Phi_{6720} \cdot \Phi_{7680}$
11	$5.123 \cdot 10^8$	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \Phi_{40} \cdot \boldsymbol{\Phi_{44}} \cdot \Phi_{64} \cdot \Phi_{88} \cdot \Phi_{192} \cdot \Phi_{320} \cdot \Phi_{440}$
		$\cdot \Phi_{512} \cdot \Phi_{704} \cdot \Phi_{960} \cdot \Phi_{2112} \cdot \Phi_{2560} \cdot \Phi_{3520} \cdot \Phi_{5632}$
12	$7.828 \cdot 10^9$	$\boldsymbol{\Phi_4} \cdot \Phi_{16} \cdot \Phi_{32} \cdot \Phi_{48} \cdot \Phi_{96} \cdot \Phi_{160} \cdot \Phi_{176} \cdot \Phi_{288} \cdot \Phi_{352}$
		$\cdot \Phi_{480} \cdot \Phi_{512} \cdot \Phi_{528} \cdot \Phi_{1056} \cdot \Phi_{1440} \cdot \Phi_{1536} \cdot \Phi_{1760} \cdot \Phi_{2560}$
		$\cdot \Phi_{3168} \cdot \Phi_{4608} \cdot \Phi_{5280} \cdot \Phi_{5632}$
13	$1296 \cdot 10^{11}$	$\boldsymbol{\Phi_4} \cdot \Phi_{16} \cdot \Phi_{32} \cdot \Phi_{48} \cdot \boldsymbol{\Phi_{52}} \cdot \Phi_{160} \cdot \Phi_{208} \cdot \Phi_{352} \cdot \Phi_{416}$
		$\cdot \Phi_{512}^2 \cdot \Phi_{624} \cdot \Phi_{1536} \cdot \Phi_{1760} \cdot \Phi_{2080} \cdot \Phi_{4576} \cdot \Phi_{5632} \cdot \Phi_{6656}$
14	$2.310 \cdot 10^{12}$	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \Phi_{32} \cdot \Phi_{56} \cdot \Phi_{104} \cdot \Phi_{224} \cdot \Phi_{352} \cdot \Phi_{416} \cdot \Phi_{512}$
		$\cdot \Phi_{728} \cdot \Phi_{1536} \cdot \Phi_{2464} \cdot \Phi_{2912} \cdot \Phi_{3584} \cdot \Phi_{4576} \cdot \Phi_{5632} \cdot \Phi_{6656}$
15	$4.411 \cdot 10^{13}$	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \Phi_{12} \cdot \Phi_{20} \cdot \Phi_{24} \cdot \Phi_{32} \cdot \Phi_{40} \cdot \Phi_{56} \cdot \Phi_{60}$
		$\cdot \Phi_{96} \cdot \Phi_{120} \cdot \Phi_{160} \cdot \Phi_{168} \cdot \Phi_{224} \cdot \Phi_{280} \cdot \Phi_{416} \cdot \Phi_{480} \cdot \Phi_{512}$
		$\cdot \Phi_{672} \cdot \Phi_{840} \cdot \Phi_{1120} \cdot \Phi_{1248} \cdot \Phi_{1536} \cdot \Phi_{2080} \cdot \Phi_{2560} \cdot \Phi_{2912} \cdot \Phi_{3360}$
		$\cdot \Phi_{5632} \cdot \Phi_{6240} \cdot \Phi_{6656} \cdot \Phi_{7680} \cdot \Phi_{8736}$
16	$8.986 \cdot 10^{14}$	$\boldsymbol{\Phi_4} \cdot \Phi_{64} \cdot \Phi_{128} \cdot \Phi_{192} \cdot \Phi_{320} \cdot \Phi_{640} \cdot \Phi_{896} \cdot \Phi_{960}$
		$\cdot \Phi_{1024} \cdot \Phi_{1664} \cdot \Phi_{3072} \cdot \Phi_{4480} \cdot \Phi_{5120} \cdot \Phi_{8320}$
17	$1.945 \cdot 10^{16}$	$\boldsymbol{\Phi_4} \cdot \Phi_{64} \cdot \boldsymbol{\Phi_{68}} \cdot \Phi_{128} \cdot \Phi_{640} \cdot \Phi_{896}$
		$\cdot \Phi_{1024}^2 \cdot \Phi_{1088} \cdot \Phi_{2176} \cdot \Phi_{4480} \cdot \Phi_{5120}$
18	$4.458 \cdot 10^{17}$	$\boldsymbol{\Phi_4} \cdot \Phi_8 \cdot \Phi_{24} \cdot \Phi_{72} \cdot \Phi_{128} \cdot \Phi_{136} \cdot \Phi_{384} \cdot \Phi_{408} \cdot \Phi_{640}$
		$\cdot \Phi_{1024} \cdot \Phi_{1152} \cdot \Phi_{1224} \cdot \Phi_{1920} \cdot \Phi_{2176} \cdot \Phi_{3072} \cdot \Phi_{5760} \cdot \Phi_{6528} \cdot \Phi_{9216}$

TABLE 3. Cyclotomic factors of $Q_n^{\pm}(t)$.

9. Concluding remarks

Is there a reason why $\rho(n) - 1/2$ factors so nicely?

The two main results for unsigned permutations in Section 3, Theorems 3.1 and 3.4, can also be proved using the **ab**-index and the mixing operator; see [7]. We have omitted this approach since the quasisymmetric viewpoint is more succinct in this case.

Here are several observations about the data in Table 2:

(i) All the indices k of the cyclotomic polynomials are even.

(ii) Any prime factor p that occurs in an index k of a cyclotomic polynomial Φ_k factor of $Q_n(t)$ is less than or equal to n.
(iii) If Φ_{k_1} and Φ_{k_2} are factors of $Q_n(t)$, so is $\Phi_{\gcd(k_1,k_2)}$. That is, the set of indices is closed under the meet operation in the divisor lattice.
(iv) If k_1 divides k_2, k_2 divides k_3 and Φ_{k_1} and Φ_{k_3} occur as factors in $Q_n(t)$, then so does Φ_{k_2}. This is convexity in the divisor lattice.
(v) If both Φ_{k_1} and Φ_{k_2} divide $Q_n(t)$, where k_1 divides k_2, then the multiplicity of Φ_{k_1} is greater than or equal to the multiplicity of Φ_{k_2}.
(vi) If p is not a Mersenne prime then the largest cyclotomic factor occurring in $Q_p(t)$ is Φ_{2p}.
(vii) When $\rho(n) \neq 1/2$ then there are no cyclotomic factors in the descent set polynomial $Q_n(t)$.
(viii) For all primes p we conjecture Φ_{2p}^2 divides Q_{2p}.

Moreover for the signed descent set polynomial we observe that:

(ix) For $n \geq 3$ the the cyclotomic polynomial Φ_{4n} divides the signed descent set polynomial $Q_n^{\pm}(t)$.
(x) For $n \geq 5$ the the cyclotomic polynomial $\Phi_{4n(n-1)}$ divides the signed descent set polynomial $Q_n^{\pm}(t)$.

Can these phenomena be explained?

For what pairs of an integer n and a prime number p does the descent set statistics $\beta_n(S)$ only take two values modulo p?

Finally, we end with two number theoretic questions. Are there infinitely many primes whose binary expansion has three 1's? The only reference for these primes we found is The On-Line Encyclopedia of Integer Sequences, sequence A081091. Are there any more prime powers with two or three ones in its binary expansion?

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