Exponential Dowling Structures*

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Abstract

The notion of exponential Dowling structures is introduced, generalizing Stanley's original theory of exponential structures. Enumerative theory is developed to determine the Möbius function of exponential Dowling structures, including a restriction of these structures to elements whose types satisfy a semigroup condition. Stanley's study of permutations associated with exponential structures leads to a similar vein of study for exponential Dowling structures. In particular, for the extended r-divisible partition lattice we show the Möbius function is, up to a sign, the number of permutations in the symmetric group on $rn + k$ elements having descent set $\{r, 2r, \ldots, nr\}$. Using Wachs' original EL-labeling of the r-divisible partition lattice, the extended r-divisible partition lattice is shown to be EL-shellable.

1 Introduction

Stanley introduced the notion of exponential structures, that is, a family of posets that have the partition lattice Π_n as the archetype [19, 21]. His original motivation was to explain certain permutation phenomena. His theory ended up inspiring many mathematicians to study the partition lattice and other exponential structures from enumerative, representation theoretic and homological perspectives.

For example, Stanley studied the r-divisible partition lattice Π_n^r and computed its Möbius numbers [19]. Calderbank, Hanlon and Robinson [6] derived plethystic formulas in order to determine the character of the representation of the symmetric group on its top homology, while Wachs determined the homotopy type, gave explicit bases for the homology and cohomology and studied the $\mathfrak{S}_n$ action on the top homology [26]. For the poset of partitions with block sizes divisible by r and having cardinality at least rk, a similar array of questions have been considered by Björner and Wachs, Browdy, Linusson, Sundaram and Wachs [3, 5, 14, 22, 27]. Other related work can be found in [1, 4, 13, 23, 24, 28], as well as work of Sagan [17], who showed certain examples of exponential structures are CL-shellable.

In this paper we extend Stanley's notion of exponential structures to that of *exponential Dowling structures*. The prototypical example is the Dowling lattice [7]. It can most easily be viewed as the intersection lattice of the complex hyperplane arrangement in (2.1). See Section 2 for a review of the Dowling lattice.

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In Section 3 we introduce exponential Dowling structures. We derive the compositional formula for exponential Dowling structures analogous to Stanley's theorem on the compositional formula for exponential structures [19]. As an application, we give the generating function for the Möbius numbers of an exponential Dowling structure.

An important method to generate new exponential Dowling structures from old ones is given in Example 3.4. Loosely speaking, in this new structure an r-divisibility condition holds for the "non-zero blocks" and the cardinality of the "zero block" satisfies the more general condition of being greater than or equal to k and congruent to k modulo r. We will return to many important special cases of this example in later sections.

In Section 4 we consider restricted forms of both exponential and exponential Dowling structures. In the case the exponential Dowling structure is restricted to elements whose type satisfies a semigroup condition, the generating function for the Möbius function of this poset is particularly elegant. See Corollary 4.3 and Proposition 4.4. When the blocks have even size, the generating function is nicely expressed in terms of the hyperbolic functions. See Corollary 4.6.

In Section 5 we continue to develop the connection between permutations and structures first studied by Stanley in the case of exponential structures. In particular we consider the lattice $\Pi^{r,j}_m$, an extension of the r-divisible partition lattice Π^r_m. In Section 6 we verify that Wachs' EL-labeling of the r-divisible partition lattice Π^r_m naturally extends to the new lattice $\Pi^{r,j}_m$.

We end with remarks and open questions regarding further exponential Dowling structures and their connections with permutation statistics.

2 The Dowling lattice

Let G be a finite group of order s. The *Dowling lattice* $L_n(G) = L_n$ has the following combinatorial description. For the original formulation, see Dowling's paper [7]. Define an *enriched block* $\widetilde{B} = (B, f)$ to be a non-empty subset B of $\{1, \ldots, n\}$ and a function $f : B \longrightarrow G$. Two enriched blocks $\widetilde{B} = (B, f)$ and $\widetilde{C} = (C, g)$ are said to be equivalent if $B = C$ and the functions f and g differ only by a multiplicative scalar, that is, there exists $\alpha \in G$ such that $f(b) = g(b) \cdot \alpha$ for all b in B. Hence there are only $s^{|B|-1}$ possible ways to enrich a non-empty set B, up to equivalence. Let $\widetilde{B} = (B, f)$ and $\widetilde{C} = (C, g)$ be two disjoint enriched blocks and let α be an element in G. We can define a function h on the block $B \cup C$ by

$$h(b) = \begin{cases} f(b) & \text{if} \quad b \in B, \\ \alpha \cdot g(b) & \text{if} \quad b \in C. \end{cases}$$

Since the group element α can be chosen in s possible ways, there are s possible ways to merge two enriched blocks.

For E a subset of $\{1, \ldots, n\}$, an *enriched partition* $\widetilde{\pi} = \{\widetilde{B}_1, \ldots, \widetilde{B}_m\}$ on the set E is a partition $\pi = \{B_1, \ldots, B_m\}$ of E, where each block B_i is enriched with a function f_i. The elements of the Dowling lattice L_n are the collection

$$L_n = \Big\{(\widetilde{\pi}, Z) \;:\; Z \subseteq \{1, \ldots, n\} \text{ and } \widetilde{\pi} \text{ is an enriched partition of } \overline{Z} = \{1, \ldots, n\} - Z\Big\}.$$

The set Z is called the *zero block.* Define the cover relation on L_n by the following two relations:

$$\begin{array}{rcl} (\{\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_m\}, Z) & \prec & (\{\widetilde{B}_2, \ldots, \widetilde{B}_m\}, Z \cup B_1), \\ (\{\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_m\}, Z) & \prec & (\{\widetilde{B}_1 \cup \widetilde{B}_2, \ldots, \widetilde{B}_m\}, Z). \end{array}$$

The first relation says that a block is allowed to merge with the zero set. The second relation says that two blocks are allowed to be merged together. The minimal element $\hat{0}$ corresponds to the partition having all singleton blocks and empty zero block, while the maximal element $\hat{1}$ corresponds to the partition where all the elements lie in the zero block. Observe that the Dowling lattice L_n is graded of rank n.

When the group G is the cyclic group of order s, that is, $\mathbb{Z}_s$, the Dowling lattice has the following geometric description. Let ζ be a primitive sth root of unity. The *Dowling lattice* $L_n(\mathbb{Z}_s)$ is the intersection lattice of the complex hyperplane arrangement

$$\left\{ \begin{array}{rcll} z_i & = & \zeta^h \cdot z_j & \text{for } 1 \leq i < j \leq n \text{ and } 0 \leq h \leq s-1, \\ z_i & = & 0 & \text{for } 1 \leq i \leq n, \end{array} \right. \tag{2.1}$$

that is, the collection of all possible intersections of these hyperplanes ordered by reverse inclusion.

In the notation we will suppress the Dowling lattice's dependency on the group G. Only the order s of the group will matter in this paper. In Section 5 the order s will be specialized to the value 1.

For an element $x = (\widetilde{\pi}, Z)$ in the Dowling lattice L_n, define the *type* of x to be $(b; a_1, a_2, \ldots, a_n)$, where a_i is the number of blocks in $\widetilde{\pi}$ of size i in x and b is the size of the zero block Z. Observe that the interval $[x, \hat{1}]$ in the Dowling lattice is isomorphic to $L_{n-\rho(x)}$ where ρ denotes the rank function. Moreover, the interval $[\hat{0}, x]$ is isomorphic to $L_b \times \Pi_1^{a_1} \times \cdots \times \Pi_n^{a_n}$, where $(b; a_1, a_2, \ldots, a_n)$ is the type of x and $\Pi_j^{a_j}$ denotes the Cartesian product of a_j copies of the partition lattice on j elements.

Lemma 2.1 *In the Dowling lattice L_n there are*

$$\frac{s^n \cdot n!}{s^b \cdot b! \cdot (s \cdot 1!)^{a_1} \cdot a_1! \cdot (s \cdot 2!)^{a_2} \cdot a_2! \cdots (s \cdot n!)^{a_n} \cdot a_n!}$$

elements of type $(b; a_1, a_2, \ldots, a_n)$.

Proof: For an element of type $(b; a_1, a_2, \ldots, a_n)$ in the Dowling lattice L_n we can choose the b elements in the zero-set in $\binom{n}{b}$ ways. The underlying partition on the remaining $n-b$ elements can be chosen in

$$\binom{n-b}{\underbrace{1, \ldots, 1}_{a_1}, \underbrace{2, \ldots, 2}_{a_2}, \ldots, \underbrace{n, \ldots, n}_{a_n}} \cdot \frac{1}{a_1! \cdot a_2! \cdots a_n!}$$

ways. For a block of size k there are s^{k-1} signings, so the result follows. $\square$

3 Dowling exponential structures

Stanley introduced the notion of an exponential structure. See [19] and [21, Section 5.5].

Definition 3.1 *An* exponential structure $\mathbf{Q} = (Q_1, Q_2, \ldots)$ *is a sequence of posets such that*

(E1) The poset Q_n has a unique maximal element $\hat{1}$ and every maximal chain in Q_n contains n elements.

(E2) For an element x in Q_n of rank k, the interval $[x, \hat{1}]$ is isomorphic to the partition lattice on $n-k$ elements, Π_{n-k}.

(E3) The lower order ideal generated by $x \in Q_n$ is isomorphic to $Q_1^{a_1} \times \cdots \times Q_n^{a_n}$. We call $(a_1, \ldots, a_n)$ the type *of x.*

(E4) The poset Q_n has $M(n)$ minimal elements. The sequence $(M(1), M(2), \ldots)$ is called the denominator sequence.

Analogous to the definition of an exponential structure, we introduce the notion of an exponential Dowling structure.

Definition 3.2 *An* exponential Dowling structure $\mathbf{R} = (R_0, R_1, \ldots)$ *associated to an exponential structure* $\mathbf{Q} = (Q_1, Q_2, \ldots)$ *is a sequence of posets such that*

(D1) The poset R_n has a unique maximal element $\hat{1}$ and every maximal chain in R_n contains $n+1$ elements.

(D2) For an element $x \in R_n$, $[x, \hat{1}] \cong L_{n-\rho(x)}$.

(D3) Each element x in R_n has a type $(b; a_1, \ldots, a_n)$ *assigned such that the lower order ideal generated by x in R_n is isomorphic to $R_b \times Q_1^{a_1} \times \cdots \times Q_n^{a_n}$.*

(D4) The poset R_n has $N(n)$ minimal elements. The sequence $(N(0), N(1), \ldots)$ is called the denominator sequence.

Observe that R_0 is the one element poset and thus $N(0) = 1$. Also note if x has type $(b; a_1, \ldots, a_n)$ then $a_{n-b+1} = \cdots = a_n = 0$.

Condition *(D3)* has a different formulation than condition *(E3)*. The reason is that there could be cases where the lower order ideal generated by an element does not factor uniquely into the form $R_b \times Q_1^{a_1} \times \cdots \times Q_n^{a_n}$. However, in the examples we consider the type of an element will be clear.

Proposition 3.3 *Let $\mathbf{R} = (R_0, R_1, \ldots)$ be an exponential Dowling structure with associated exponential structure $\mathbf{Q} = (Q_1, Q_2, \ldots)$. The number of elements in R_n of type $(b; a_1, \ldots, a_n)$ is given by*

$$\frac{N(n) \cdot s^n \cdot n!}{N(b) \cdot s^b \cdot b! \cdot (M(1) \cdot s \cdot 1!)^{a_1} \cdot a_1! \cdots (M(n) \cdot s \cdot n!)^{a_n} \cdot a_n!} \tag{3.1}$$

Proof: Consider pairs of elements (x, y) satisfying $y \leq x$, where the element x has type $(b; a_1, \ldots, a_n)$ and y is a minimal element of R_n. We count such pairs in two ways. The number of minimal elements $y \in R_n$ is given by $N(n)$. Given such a minimal element y, the number of x's is given in Lemma 2.1. Alternatively, we wish to count the number of x's. The number of y's given an element x equals the number of minimal elements occurring in the lower order ideal generated by x. This equals the number of minimal elements in $R_b \times Q_1^{a_1} \times \cdots \times Q_n^{a_n}$, that is, $N(b) \cdot M(1)^{a_1} \cdots M(n)^{a_n}$. Thus the answer is as in (3.1). $\square$

Let $\mathbf{Q}$ be an exponential structure and r a positive integer. Stanley defines the exponential structure $\mathbf{Q}^{(r)}$ by letting $Q_n^{(r)}$ be the subposet Q_{rn} of all elements x of type $(a_1, a_2, \ldots)$ where $a_i = 0$ unless r divides i. The denominator sequence of $\mathbf{Q}^{(r)}$ is given by

$$M^{(r)}(n) = \frac{M(rn) \cdot (rn)!}{M(r)^n \cdot n! \cdot r!^n}.$$

Example 3.4 Let $\mathbf{R}$ be an exponential Dowling structure associated with the exponential structure $\mathbf{Q}$. Let r be a positive integer and k a non-negative integer. Let $R_n^{(r,k)}$ be the subposet of Q_{rn+k} consisting of all elements x of type $(b; a_1, a_2, \ldots)$ such that $b \geq k$, $b \equiv k \bmod r$ and $a_i = 0$ unless r divides i. Then $\mathbf{R}^{(r,k)} = (R_0^{(r,k)}, R_1^{(r,k)}, \ldots)$ is an exponential Dowling structure associated with the exponential structure $\mathbf{Q}^{(r)}$. The minimal elements of $R_n^{(r,k)}$ are the elements of R_{rn+k} having types given by $b = k$, $a_r = n$ and $a_i = 0$ for $i \neq n$. The denominator sequence of $\mathbf{R}^{(r,k)}$ is given by

$$N^{(r,k)}(n) = \frac{N(rn+k) \cdot (rn+k)! \cdot s^{(r-1) \cdot n}}{N(k) \cdot k! \cdot M(r)^n \cdot r!^n \cdot n!}.$$

Stanley [19] proved the following structure theorem.

Theorem 3.5 *(The Compositional Formula for Exponential Structures) Let $\mathbf{Q} = (Q_1, Q_1, \ldots)$ be an exponential structure with denominator sequence $(M(1), M(2), \ldots)$. Let $f : \mathbb{P} \to \mathbb{C}$ and $g : \mathbb{N} \to \mathbb{C}$ be given functions such that $g(0) = 1$. Define the function $h : \mathbb{N} \to \mathbb{C}$ by*

$$h(n) = \sum_{x \in Q_n} f(1)^{a_1} \cdot f(2)^{a_2} \cdots f(n)^{a_n} \cdot g(a_1 + \cdots + a_n), \tag{3.2}$$

for $n \geq 1$, where $\text{type}(x) = (a_1, \ldots, a_n)$, and $h(0) = 1$. Define the formal power series $F, G, K \in \mathbb{C}[[x]]$ by

$$\begin{aligned} F(x) &= \sum_{n \geq 1} f(n) \cdot \frac{x^n}{M(n) \cdot n!} \\ G(x) &= \sum_{n \geq 0} g(n) \cdot \frac{x^n}{n!} \\ H(x) &= \sum_{n \geq 0} h(n) \cdot \frac{x^n}{M(n) \cdot n!}. \end{aligned}$$

Then $H(x) = G(F(x))$.

For Dowling structures we have an analogous theorem.

Theorem 3.6 *(The Compositional Formula for Exponential Dowling Structures) Let* $\mathbf{R} = (R_0, R_1, \ldots)$ *be an exponential Dowling structure with denominator sequence* $(N(0), N(1), \ldots)$ *and associated exponential structure* $\mathbf{Q} = (Q_1, Q_2, \ldots)$ *with denominator sequence* $(M(1), M(2), \ldots)$*. Let* $f : \mathbb{P} \rightarrow \mathbb{C}$*,* $g : \mathbb{N} \rightarrow \mathbb{C}$ *and* $k : \mathbb{N} \rightarrow \mathbb{C}$ *be given functions. Define the function* $h : \mathbb{N} \rightarrow \mathbb{C}$ *by*

$$h(n) = \sum_{x \in R_n} k(b) \cdot f(1)^{a_1} \cdot f(2)^{a_2} \cdots f(n)^{a_n} \cdot g(a_1 + \cdots + a_n), \tag{3.3}$$

for $n \geq 0$*, where* $\mathrm{type}(x) = (b; a_1, \ldots, a_n)$*. Define the formal power series* $F, G, K, H \in \mathbb{C}[[x]]$ *by*

$$\begin{aligned} F(x) &= \sum_{n \geq 1} f(n) \cdot \frac{x^n}{M(n) \cdot n!} \\ G(x) &= \sum_{n \geq 0} g(n) \cdot \frac{x^n}{n!} \\ K(x) &= \sum_{n \geq 0} k(n) \cdot \frac{x^n}{N(n) \cdot n!} \\ H(x) &= \sum_{n \geq 0} h(n) \cdot \frac{x^n}{N(n) \cdot n!} \end{aligned}$$

Then $H(x) = K(x) \cdot G(1/s \cdot F(s \cdot x))$*.*

Proof: By applying the compositional formula of generating functions to the (exponential) generating functions $1/s \cdot F(sx) = \sum_{n \geq 1} f(n)/(M(n) \cdot s) \cdot (sx)^n/n!$ and $G(x)$, we obtain

$$\begin{aligned} G(1/s \cdot F(sx)) &= \sum_{n \geq 0} \sum_{\pi \in \Pi_n} \prod_{B \in \pi} \frac{f(|B|)}{M(|B|) \cdot s} \cdot g(|\pi|) \cdot \frac{(sx)^n}{n!} \\ &= \sum_{n \geq 0} \sum_{1 \cdot a_1 + \cdots + n \cdot a_n = n} \frac{s^n \cdot n!}{(M(1) \cdot s \cdot 1!)^{a_1} \cdot a_1! \cdots (M(n) \cdot s \cdot n!)^{a_n} \cdot a_n!} \\ &\qquad \cdot f(1)^{a_1} \cdots f(n)^{a_n} \cdot g(a_1 + \cdots + a_n) \cdot \frac{x^n}{n!}. \end{aligned}$$

Multiply this identity with the (exponential) generating function $K(x) = \sum_{n \geq 0} k(n)/N(n) \cdot x^n/n!$ to obtain

$$\begin{aligned} & K(x) \cdot G(1/s \cdot F(sx)) \\ = & \sum_{n \geq 0} \sum_{b=0}^{n} \sum_{1 \cdot a_1 + \cdots + (n-b) \cdot a_{n-b} = n-b} \binom{n}{b} \cdot \frac{k(b)}{N(b)} \\ & \cdot \frac{s^{n-b} \cdot (n-b)!}{(M(1) \cdot s \cdot 1!)^{a_1} \cdot a_1! \cdots (M(n-b) \cdot s \cdot (n-b)!)^{a_{n-b}} \cdot a_{n-b}!} \\ & \cdot f(1)^{a_1} \cdots f(n)^{a_n} \cdot g(a_1 + \cdots + a_{n-b}) \cdot \frac{x^n}{n!} \\ = & \sum_{n \geq 0} \sum_{b=0}^{n} \sum_{1 \cdot a_1 + \cdots + n \cdot a_n = n-b} \frac{s^n \cdot n!}{N(b) \cdot s^b \cdot b! \cdot (M(1) \cdot s \cdot 1!)^{a_1} \cdot a_1! \cdots (M(n) \cdot s \cdot n!)^{a_n} \cdot a_n!} \end{aligned}$$

$$\cdot k(b) \cdot f(1)^{a_1} \cdots f(n)^{a_n} \cdot g(a_1 + \cdots + a_n) \cdot \frac{x^n}{n!}$$
$$= \sum_{n \geq 0} \sum_{x \in R_n} k(b) \cdot f(1)^{a_1} \cdots f(n)^{a_n} \cdot g(a_1 + \cdots + a_n) \cdot \frac{x^n}{N(n) \cdot n!} = H(x). \qquad \square$$

Example 3.7 Let $\mathbf{R} = (R_0, R_1, \ldots)$ be an exponential Dowling structure with denominator sequence $(N(0), N(1), \ldots)$ and associated exponential structure $\mathbf{Q} = (Q_1, Q_2, \ldots)$ with denominator sequence $(M(1), M(2), \ldots)$. Let $V_n(t)$ be the polynomial

$$V_n(t) = \sum_{x \in Q_n} t^{\rho(x, \hat{1})}.$$

In Example 5.5.6 in [21] Stanley obtains the generating function

$$\sum_{n \geq 0} V_n(t) \cdot \frac{x^n}{M(n) \cdot n!} = \exp\left(\sum_{n \geq 1} \frac{x^n}{M(n) \cdot n!}\right)^t,$$

by setting $f(n) = 1$ and $g(n) = t^n$ in Theorem 3.5. Similarly, defining $W_n(t)$ by

$$W_n(t) = \sum_{x \in R_n} t^{\rho(x, \hat{1})},$$

we obtain

$$\sum_{n \geq 0} W_n(t) \cdot \frac{x^n}{N(n) \cdot n!} = \left(\sum_{n \geq 0} \frac{x^n}{N(n) \cdot n!}\right) \cdot \exp\left(\sum_{n \geq 1} \frac{(s \cdot x)^n}{M(n) \cdot n!}\right)^{\frac{t}{s}},$$

by setting $f(n) = 1$, $g(n) = t^n$ and $k(n) = 1$ in Theorem 3.6.

Corollary 3.8 *Let $\mathbf{R} = (R_0, R_1, \ldots)$ be an exponential Dowling structure with denominator sequence $(N(0), N(1), \ldots)$ and associated exponential structure $\mathbf{Q} = (Q_1, Q_2, \ldots)$ with denominator sequence $(M(1), M(2), \ldots)$. Then the Möbius function of the posets $Q_n \cup \{\hat{0}\}$, respectively $R_n \cup \{\hat{0}\}$, has the generating function:*

$$\sum_{n \geq 1} \mu(Q_n \cup \{\hat{0}\}) \cdot \frac{x^n}{M(n) \cdot n!} = -\ln\left(\sum_{n \geq 0} \frac{x^n}{M(n) \cdot n!}\right), \tag{3.4}$$

$$\sum_{n \geq 0} \mu(R_n \cup \{\hat{0}\}) \cdot \frac{x^n}{N(n) \cdot n!} = -\left(\sum_{n \geq 0} \frac{x^n}{N(n) \cdot n!}\right) \cdot \left(\sum_{n \geq 0} \frac{(s \cdot x)^n}{M(n) \cdot n!}\right)^{-1/s}. \tag{3.5}$$

Proof: Setting $f(n) = 1$ and $g(n) = (-1)^{n-1} \cdot (n-1)!$ and using that

$$\mu(Q_n \cup \{\hat{0}\}) = -\sum_{x \in Q_n} \mu(x, \hat{1}) = -\sum_{x \in Q_n} g(a_1 + \cdots + a_n),$$

equation (3.4) follows by Theorem 3.5. Similarly, to prove the second identity (3.5), redefine $g(n)$ to be the Möbius function of the Dowling lattice L_n of rank n, that is,

$$g(n) = (-1)^n \cdot 1 \cdot (s+1) \cdot (2 \cdot s + 1) \cdots ((n-1) \cdot s + 1).$$

By the binomial theorem we have $\sum_{n \geq 0} g(n) \frac{x^n}{n!} = (1 + s \cdot x)^{-1/s}$. Moreover, let $k(n) = 1$. Using the recurrence

$$\mu(R_n \cup \{\hat{0}\}) = - \sum_{x \in R_n} \mu(x, \hat{1}) = \sum_{x \in R_n} g(a_1 + \cdots + a_n),$$

and Theorem 3.6, the result follows. □

4 The Möbius function of restricted structures

Let I be a subset of the positive integers $\mathbb{P}$. For an exponential structure $\mathbf{Q} = (Q_1, Q_2, \ldots)$ define the *restricted poset* Q_n^I to be all elements x in Q_n whose type $(a_1, \ldots, a_n)$ satisfies $a_i > 0$ implies $i \in I$. For $n \in I$ let $\mu_I(n)$ denote the Möbius function of the poset Q_n^I with a $\hat{0}$ adjoined, that is, the poset $Q_n^I \cup \{\hat{0}\}$. For $n \notin I$ let $\mu_I(n) = 0$.

For any positive integer n define

$$m_n = \sum_{x \in Q_n^I \cup \{\hat{0}\}} \mu_I(\hat{0}, x) = 1 + \sum_{x \in Q_n^I} \mu_I(\hat{0}, x).$$

Observe that for $n \in I$ we have that Q_n^I has a maximal element and hence $m_n = 0$. Especially for $n \notin I$ we have the expansion

$$m_n = 1 - \sum_{\sum_{i \in I} i \cdot a_i = n} (-1)^{\sum_{i \in I} a_i} \frac{M(n) \cdot n!}{(M(1) \cdot 1!)^{a_1} \cdot a_1! \cdots (M(n) \cdot n!)^{a_n} \cdot a_n!} \cdot \mu_I(1)^{a_1} \cdots \mu_I(n)^{a_n}.$$

The following theorem was inspired by work of Linusson [14].

Theorem 4.1

$$\sum_{i \in I} \mu_I(i) \frac{x^i}{M(i) \cdot i!} = - \ln \left(\sum_{n \geq 0} \frac{x^n}{M(n) \cdot n!} - \sum_{n \notin I} m_n \cdot \frac{x^n}{M(n) \cdot n!} \right).$$

Proof: Expand the product

$$-1 + \prod_{i \in I} \exp \left(-\mu_I(i) \frac{x^i}{M(i) \cdot i!} \right) \quad = \quad -1 + \prod_{i \in I} \left(1 - \mu_I(i) \frac{x^i}{M(i) \cdot i!} + \frac{1}{2} \cdot \left(\mu_I(i) \frac{x^i}{M(i) \cdot i!} \right)^2 - \cdots \right)$$

$$
\begin{aligned}
&= \sum_{n\geq 1} \sum_{\sum_{i\in I} i\cdot a_i = n} \prod_{i\in I} \frac{1}{a_i!} \cdot \left(-\mu_I(i) \frac{x^i}{M(i)\cdot i!}\right)^{a_i} \\
&= \sum_{n\geq 1} \sum_{\sum_{i\in I} i\cdot a_i = n} (-1)^{\sum_{i\in I} a_i} \cdot (\mu_I(1)^{a_1} \cdots \mu_I(n)^{a_n}) \\
&\cdot \quad \frac{M(n)\cdot n!}{(M(1)\cdot 1!)^{a_1}\cdot a_1! \cdots (M(n)\cdot n!)^{a_n}\cdot a_n!} \frac{x^n}{M(n)\cdot n!} \\
&= \sum_{n\geq 1} (1-m_n) \cdot \frac{x^n}{M(n)\cdot n!} \\
&= \sum_{n\geq 1} \frac{x^n}{M(n)\cdot n!} - \sum_{n\notin I} m_n \cdot \frac{x^n}{M(n)\cdot n!}.
\end{aligned}
$$

The result now follows. □

Let I be a subset of the positive integers $\mathbb{P}$ and J be a subset of the natural numbers $\mathbb{N}$. For an exponential Dowling structure $\mathbf{R} = (R_0, R_1, \ldots)$, define the *restricted poset* $R_n^{I,J}$ to be all elements x in R_n whose type $(b; a_1, \ldots, a_n)$ satisfies $b \in J$ and $a_i > 0$ implies $i \in I$.

For $n \in J$ define $\mu_{I,J}(n)$ to be the Möbius function of the poset $R_n^{I,J} \cup \{\hat{0}\}$, that is, $R_n^{I,J}$ with a minimal element $\hat{0}$ adjoined. For $n \notin J$ let $\mu_{I,J}(n) = 0$. Define for any non-negative integer n

$$
p_n = \sum_{x\in R_n^{I,J}\cup\{\hat{0}\}} \mu_{I,J}(\hat{0}, x) = 1 + \sum_{x\in R_n^{I,J}} \mu_{I,J}(\hat{0}, x).
$$

Observe that for $n \in J$ we have $p_n = 0$ since the poset $R_n^{I,J} \cup \{\hat{0}\}$ has a maximal element. For $n \notin J$ we have

$$
\begin{aligned}
p_n &= 1 + \sum_{(b;a_1,\ldots,a_n)} (-1)^{a_1+\cdots+a_n} \cdot \frac{N(n)\cdot s^n \cdot n!}{N(b)\cdot s^b\cdot b!\cdot (M(1)\cdot s\cdot 1!)^{a_1}\cdot a_1! \cdots (M(n)\cdot s\cdot n!)^{a_n}\cdot a_n!} \\
&\qquad \cdot \mu_{I,J}(b)\cdot \mu_I(1)^{a_1} \cdots \mu_I(n)^{a_n},
\end{aligned}
$$

where the sum is over all types $(b; a_1, \ldots, a_n)$ where $b \in J$, $a_i > 0$ implies $i \in I$, and $b + \sum_{i\in I} i\cdot a_i = n$.

Theorem 4.2

$$
\sum_{b\in J} \mu_{I,J}(b) \cdot \frac{x^b}{N(b)\cdot b!} = \frac{-\sum_{n\geq 0} \frac{x^n}{N(n)\cdot n!} + \sum_{n\notin J} p_n \cdot \frac{x^n}{N(n)\cdot n!}}{\left(\sum_{n\geq 0} \frac{(s\cdot x)^n}{M(n)\cdot n!} - \sum_{n\notin I} m_n \cdot \frac{(s\cdot x)^n}{M(n)\cdot n!}\right)^{1/s}}
$$

Proof: By similar reasoning as in the proof of Theorem 4.1, we have

$$
\exp\left(-\sum_{i\in I} \mu_I(i) \cdot \frac{x^i}{M(i)\cdot s\cdot i!}\right) = \sum_{(a_1,\ldots,a_n)} \prod_{i\in I} \frac{1}{a_i!} \cdot \left(-\frac{\mu_I(i)\cdot x^i}{M(i)\cdot s\cdot i!}\right)^{a_i}.
$$

Multiplying with $\sum_{b\in J}\mu_{I,J}(b)\cdot\frac{x^b}{N(b)\cdot s^b\cdot b!}$ and expanding, we obtain

$$\begin{aligned}
&\left(\sum_{b\in J}\mu_{I,J}(b)\cdot\frac{x^b}{N(b)\cdot s^b\cdot b!}\right)\cdot\exp\left(-\sum_{i\in I}\mu_I(i)\cdot\frac{x^i}{M(i)\cdot s\cdot i!}\right)\\
=\;&\sum_{n\geq 0}\sum_{b\in J}\sum_{(a_1,\ldots,a_{n-b})}\left(\frac{\mu_{I,J}(b)\cdot x^b}{N(b)\cdot s^b\cdot b!}\right)\cdot\prod_{i\in I}\frac{1}{a_i!}\cdot\left(-\frac{\mu_I(i)\cdot x^i}{M(i)\cdot s\cdot i!}\right)^{a_i}\\
=\;&\sum_{n\geq 0}(p_n-1)\cdot\frac{x^n}{N(n)\cdot s^n\cdot n!}
\end{aligned}$$

Substituting $x\longmapsto sx$, we can rewrite this equation as

$$\begin{aligned}
&\sum_{b\in J}\mu_{I,J}(b)\cdot\frac{x^b}{N(b)\cdot b!}\\
=\;&\left(\sum_{n\geq 0}(p_n-1)\cdot\frac{x^n}{N(n)\cdot n!}\right)\cdot\exp\left(\frac{1}{s}\cdot\sum_{i\in I}\mu_I(i)\cdot\frac{(s\cdot x)^i}{M(i)\cdot i!}\right)\\
=\;&\left(\sum_{n\geq 0}(p_n-1)\cdot\frac{x^n}{N(n)\cdot n!}\right)\cdot\exp\left(\sum_{i\in I}\mu_I(i)\cdot\frac{(s\cdot x)^i}{M(i)\cdot i!}\right)^{1/s}
\end{aligned}$$

By applying Theorem 4.1 to the last term, the result follows. □

As a corollary to Theorems 4.1 and 4.2, we have

Corollary 4.3 *Let $I\subseteq\mathbb{P}$ be a semigroup and $J\subseteq\mathbb{N}$ such that $I+J\subseteq J$. Then the Möbius function of the restricted poset $Q_n^I\cup\{\hat{0}\}$ and $R_n^{I,J}\cup\{\hat{0}\}$ respectively has the generating function:*

$$\sum_{n\in I}\mu_I(n)\cdot\frac{x^n}{M(n)\cdot n!} = -\ln\left(\sum_{n\in I\cup\{0\}}\frac{x^n}{M(n)\cdot n!}\right), \tag{4.1}$$

$$\sum_{n\in J}\mu_{I,J}(n)\cdot\frac{x^n}{N(n)\cdot n!} = -\left(\sum_{n\in J}\frac{x^n}{N(n)\cdot n!}\right)\cdot\left(\sum_{n\in I\cup\{0\}}\frac{(s\cdot x)^n}{M(n)\cdot n!}\right)^{-1/s}. \tag{4.2}$$

Proof: The semigroup condition implies that the poset Q_n^I is empty when $n\notin I$ and hence $m_n=1$. Similarly, the other condition implies that the poset R_n^I is empty when $n\notin J$, so $p_n=1$. □

Let $\mathbf{D}$ be the Dowling structure consisting of the Dowling lattices, that is, $\mathbf{D}=(L_0,L_1,\ldots)$.

Proposition 4.4 *For the exponential Dowling structure $\mathbf{D}^{(r,k)}$ we have*

$$\sum_{n\geq 0}\mu\left(D_n^{(r,k)}\cup\{\hat{0}\}\right)\cdot\frac{x^{rn+k}}{(rn+k)!}=\left(\sum_{n\geq 0}\frac{x^{rn+k}}{(rn+k)!}\right)\cdot\left(\sum_{n\geq 0}\frac{(s\cdot x)^{rn}}{(rn)!}\right)^{-1/s}.$$

This can be proven from Corollary 4.3 using $I = r \cdot \mathbb{P}$ and $J = k + r \cdot \mathbb{N}$. This also follows from Corollary 3.8 by using the Dowling structure $\mathbf{D}^{(r,k)}$.

When $r = 1$ we have the following corollary.

Corollary 4.5 *Let $k \geq 1$. Then the Möbius function of the poset $D_n^{(1,k)} \cup \{\hat{0}\}$ is given by*

$$\mu\left(D_n^{(1,k)} \cup \{\hat{0}\}\right) = (-1)^n \cdot \binom{n+k-1}{k-1}.$$

Furthermore, the Möbius function does not depend on the order s.

Proof: We have

$$\sum_{n \geq 0} \mu\left(D_n^{(1,k)} \cup \{\hat{0}\}\right) \cdot \frac{x^{n+k}}{(n+k)!} = \left(\sum_{n \geq 0} \frac{x^{n+k}}{(n+k)!}\right) \cdot \exp(-x).$$

Differentiate with respect to x gives

$$\begin{aligned} \sum_{n \geq 0} \mu\left(D_n^{(1,k)} \cup \{\hat{0}\}\right) \cdot \frac{x^{n+k-1}}{(n+k-1)!} &= \left(\sum_{n \geq 0} \frac{x^{n+k-1}}{(n+k-1)!}\right) \cdot \exp(-x) - \left(\sum_{n \geq 0} \frac{x^{n+k}}{(n+k)!}\right) \cdot \exp(-x) \\ &= \frac{x^{k-1}}{(k-1)!} \cdot \exp(-x) \\ &= \sum_{n \geq 0} (-1)^n \cdot \binom{n+k-1}{k-1} \cdot \frac{x^{n+k-1}}{(n+k-1)!}. \end{aligned}$$ $\square$

When $r = 2$ we can express the generating function for the Möbius function in terms of hyperbolic functions. We have two cases, depending on whether k is even or odd.

Corollary 4.6 *The Möbius function of the poset $D_n^{(2,k)} \cup \{\hat{0}\}$ is given by*

$$\sum_{n \geq 0} \mu\left(D_n^{(2,2j)} \cup \{\hat{0}\}\right) \cdot \frac{x^{2n+2j}}{(2n+2j)!} = \left(\cosh(x) - \sum_{i=0}^{j-1} \frac{x^{2i}}{(2i)!}\right) \cdot \operatorname{sech}(s \cdot x)^{1/s}, \tag{4.3}$$

$$\sum_{n \geq 0} \mu\left(D_n^{(2,2j+1)} \cup \{\hat{0}\}\right) \cdot \frac{x^{2n+2j+1}}{(2n+2j+1)!} = \left(\sinh(x) - \sum_{i=0}^{j-1} \frac{x^{2i+1}}{(2i+1)!}\right) \cdot \operatorname{sech}(s \cdot x)^{1/s}. \tag{4.4}$$

5 Permutations and partitions with restricted block sizes

For a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ in the symmetric group $\mathfrak{S}_n$ define the descent set of σ to be the set $\{i : \sigma_i < \sigma_{i+1}\}$. An equivalent notion is the *descent word* of σ, which is the **ab**-word $u = u_1 u_2 \cdots u_{n-1}$

of degree $n-1$ where $u_i = \mathbf{a}$ if $\sigma_i < \sigma_{i+1}$ and $u_i = \mathbf{b}$ otherwise. For an **ab**-word u of length $n-1$ let $\beta(u)$ be the number of permutations σ in $\mathfrak{S}_n$ with descent word u. Similarly, define the q-analogue $\beta_q(u)$ to be the sum

$$\beta_q(u) = \sum_{\sigma} q^{\mathrm{inv}(\sigma)},$$

where the sum ranges over all permutations σ in $\mathfrak{S}_n$ with descent word u and $\mathrm{inv}(\sigma)$ is the number of inversions of σ. Let $[n]$ denote $1 + q + \cdots + q^{n-1}$ and $[n]! = [1] \cdot [2] \cdots [n]$. Finally, let $\left[{n \atop k}\right]$ denote the Gaussian coefficient $[n]!/([k]! \cdot [n-k]!)$.

Lemma 5.1 *For two* **ab***-words u and v of degree $n-1$, respectively $m-1$, the following identity holds:*

$$\begin{bmatrix} n+m \\ n \end{bmatrix} \cdot \beta_q(u) \cdot \beta_q(v) = \beta_q(u \cdot \mathbf{a} \cdot v) + \beta_q(u \cdot \mathbf{b} \cdot v)\,.$$

This is "the Multiplication Theorem" due to MacMahon [15, Article 159]. Using this identity, we obtain the following lemma for Eulerian generating functions.

Lemma 5.2 *Let $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ be two sequences of* **ab***-words such that the nth word has degree $n-1$. Then the following Eulerian generating function identity holds:*

$$\begin{aligned} &\left(\sum_{n\geq 1} c_n \cdot \beta_q(u_n) \cdot \frac{x^n}{[n]!}\right) \cdot \left(\sum_{n\geq 1} d_n \cdot \beta_q(v_n) \cdot \frac{x^n}{[n]!}\right) \\ &= \sum_{n\geq 2} \sum_{\substack{i+j=n \\ i,j\geq 1}} c_i \cdot d_j \cdot (\beta_q(u_i \cdot \mathbf{a} \cdot v_j) + \beta_q(u_i \cdot \mathbf{b} \cdot v_j)) \cdot \frac{x^n}{[n]!}. \end{aligned}$$

Now we obtain the following proposition. In the special case when $w = \mathbf{a}^i$, where $0 \leq i \leq r-1$, the result is due to Stanley [18]. See also [20, Section 3.16].

Proposition 5.3 *Let w be an* **ab***-word of degree $k-1$. Then Eulerian generating function for the descent statistic $\beta_q((\mathbf{a}^{r-1}\mathbf{b})^n \cdot w)$ is given by*

$$\sum_{n\geq 0} (-1)^n \cdot \beta_q\Big((\mathbf{a}^{r-1}\mathbf{b})^n \cdot w\Big) \cdot \frac{x^{rn+k}}{[rn+k]!} = \frac{\displaystyle\sum_{n\geq 0} \beta_q(\mathbf{a}^{rn} \cdot w) \cdot \frac{x^{rn+k}}{[rn+k]!}}{\displaystyle\sum_{n\geq 0} \frac{x^{rn}}{[rn]!}}.$$

Proof: Consider the following product of generating functions:

$$\left(\sum_{n\geq 1} \beta_q\Big(\mathbf{a}^{rn-1}\Big) \cdot \frac{x^{rn}}{[rn]!}\right) \cdot \left(\sum_{n\geq 0} (-1)^n \cdot \beta_q\Big((\mathbf{a}^{r-1}\mathbf{b})^n \cdot w\Big) \cdot \frac{x^{rn+k}}{[rn+k]!}\right)$$

$$= \sum_{n\geq 0}\left(\sum_{\substack{i+j=n\\ i\geq 1}}(-1)^j\cdot\Big(\beta_q\Big(\mathbf{a}^{ri}(\mathbf{a}^{r-1}\mathbf{b})^j\cdot w\Big)+\beta_q\Big(\mathbf{a}^{r(i-1)}(\mathbf{a}^{r-1}\mathbf{b})^{j+1}\cdot w\Big)\Big)\right)\cdot\frac{x^{rn+k}}{[rn+k]!}$$

$$= \sum_{n\geq 0}\Big(\beta_q(\mathbf{a}^{rn}\cdot w)+(-1)^{n-1}\cdot\beta_q\Big((\mathbf{a}^{r-1}\mathbf{b})^n\cdot w\Big)\Big)\cdot\frac{x^{rn+k}}{[rn+k]!}.$$

Now add $\sum_{n\geq 0}(-1)^n\cdot\beta_q((\mathbf{a}^{r-1}\mathbf{b})^n\cdot w)\cdot x^{rn+k}/[rn+k]!$ to both sides and the desired identity is established. $\square$

For r a positive integer and n a non-negative integer let $m = rn$. Define the poset Π^r_m to be the collection of all partitions π of the set $\{1,\ldots,m\}$ such that each block size is divisible by r together with a minimal element $\hat{0}$ adjoined. This is the well-known and well-studied *r-divisible partition lattice.* See [6, 17, 19, 26]. Other restrictions of the partition lattice and the Dowling lattice can be found in [2, 11, 12].

A natural extension of the r-divisible partition lattice is the following. For r a positive integer, and n and j non-negative integers, let $m = rn + j$. Define the poset $\Pi^{r,j}_m$ to be the collection of all partitions π of the set $\{1,\ldots,m\}$ such that

(i) a block B of π containing the element m must have cardinality at least j,

(ii) a block B of π not containing the element m must have cardinality divisible by r,

together with a minimal element $\hat{0}$ adjoined to the poset. We order all such partitions in the usual way by refinement. For instance, $\Pi^{1,1}_m$ is the classical partition lattice Π_m with $\hat{0}$ adjoined. Observe that the poset $\Pi^{r,j}_m - \{\hat{0}\}$ is a filter (upper order ideal) of the partition lattice Π_m. Hence $\Pi^{r,j}_m$ is a finite semi-join lattice and we can conclude that it is a lattice. The same argument holds for Π^r_m.

By combining Propositions 4.4 and 5.3, we obtain the next result.

Theorem 5.4 *Let r and k be positive integers and n a non-negative integer and let $m = rn + k + 1$. Then the Möbius function of the lattice $\Pi^{r,k+1}_m$ is given by the sign $(-1)^n$ times the number of permutations on $m-1$ elements with the descent set $\{r, 2r, \ldots, nr\}$, that is,*

$$\mu(\Pi^{r,k+1}_m) = (-1)^n\cdot\beta\left((\mathbf{a}^{r-1}\mathbf{b})^n\cdot\mathbf{a}^{k-1}\right).$$

Proof: Begin to observe that $\Pi^{r,k+1}_m$ is isomorphic to the poset $D^{(r,k)}_n$ when $s = 1$. Namely, remove the element m from the block B that contains this element and rename this block to be the zero block. The result follows now by observing that setting $w = \mathbf{a}^{k-1}$ and $q = 1$ in Proposition 5.3 gives the same generating function as setting $s = 1$ in Proposition 4.4. $\square$

For completeness, we also consider the case $j = 1$.

Theorem 5.5 *Let r and n be positive integers and let $m = rn + 1$. Then the Möbius function of the lattice $\Pi_m^{r,1}$ is 0.*

Proof: This follows directly from Proposition 4.4 by setting $k = 0$ and $s = 1$. A direct combinatorial argument is the following. Each of the atoms of the lattice $\Pi_m^{r,1}$ has the element m in a singleton block. The same holds for the join of all the atoms and hence the join of all the atoms is not the maximal element $\hat{1}$ of the lattice. Thus by Corollary 3.9.5 in [20] the result is obtained. □

Setting $k = r - 1$ in Theorem 5.4, we obtain the following corollary due to Stanley [19].

Corollary 5.6 *For $r \geq 2$ and $m = rn$ the Möbius function of the r-divisible partition lattice Π_m^r is given by the sign $(-1)^{n-1}$ times the number of permutations of $rn - 1$ elements with the descent set $\{r, 2r, \ldots, (n-1)r\}$, that is,*

$$\mu(\Pi_m^r) = (-1)^{n-1} \cdot \beta\left((\mathbf{a}^{r-1}\mathbf{b})^{n-1} \cdot \mathbf{a}^{r-2}\right).$$

When $r = 2$ this corollary reduces to $(-1)^{n-1} \cdot E_{2n-1}$, where E_i denotes the ith Euler number. This result is originally due to G. S. Sylvester [25]. The odd indexed Euler numbers are known as the *tangent numbers* and the even indexed ones as the *secant numbers*. Setting $r = 2$ and $k = 2$ in Theorem 5.4 we obtain that the Möbius function of the partitions where all blocks have even size except the block containing the largest element, which has an odd size greater than or equal to three, is given by the secant numbers, that is, $(-1)^{n-1} \cdot E_{2n}$.

6 EL-labeling

It is a natural question to ask if the poset $\Pi_m^{r,j}$ occurring in Theorems 5.4 and 5.5 is EL-shellable. The answer is positive. An EL-labeling that works is the one using Wachs' EL-labeling [26] for the r-divisible partition lattice Π_m^r, which we state here for the extended partition lattice $\Pi_m^{r,j}$. Let r and j be positive integers and n a non-negative integer and let $m = rn + j$. Define the labeling λ as follows. First consider the edges in the Hasse diagram not adjacent to the minimal element $\hat{0}$. Let x and y be two elements in $\Pi_m^{r,k+1} - \{\hat{0}\}$ such that x is covered by y and B_1 and B_2 are the blocks of x that are merged to form the partition y. Assume that $\max(B_1) < \max(B_2)$. Set

$$\lambda(x, y) = \begin{cases} -\max(B_1) & \text{if } \max(B_1) > \min(B_2), \\ \max(B_2) & \text{otherwise.} \end{cases} \tag{6.1}$$

Now consider the edges between the minimal element $\hat{0}$ and the atoms. There are $M = (m-1)!/(n! \cdot r!^n \cdot (j-1)!)$ number of atoms. For each atom $a = \{B_1, B_2, \ldots, B_{n+1}\}$ order the blocks such that $\min(B_1) < \min(B_2) < \cdots < \min(B_{n+1})$. Let $\widetilde{a}$ be the permutation in $\mathfrak{S}_m$ that is obtained by going through the blocks in order and writing down the elements in each block in increasing order. For instance, for the atom $a = 16|23|459|78$ we obtain the permutation $\widetilde{a} = 162345978$. It is straightforward to see that different atoms give rise to different permutations by considering where the largest element m

is. Finally, order the atoms $a_1 < a_2 < \cdots < a_M$ such that the permutations $\widetilde{a_1} < \widetilde{a_2} < \cdots < \widetilde{a_M}$ are ordered in lexicographic order. Define the label of the edge from the minimal element to an atom by

$$\lambda(\hat{0}, a_i) = 0_i \tag{6.2}$$

Order the labels by

$$\{-m < -(m-1) < \cdots < -1 < 0_1 < 0_2 < \cdots < 0_M < 1 < \cdots < m\}.$$

Let $A_m^{r,j}$ be the collection of all permutations $\sigma \in \mathfrak{S}_m$ such that the descent set of σ is $\{r, 2r, \ldots, nr\}$ and $\sigma(m) = m$. Note that when $j = 1$ there are no such permutations since the condition $\sigma(m) = m$ forces nr to be an ascent. Given a permutation $\sigma \in A_m^{r,j}$, let $t_1, \ldots, t_n$ be the permutation of $1, \ldots, n$ such that

$$\sigma(rt_1) > \sigma(rt_2) > \cdots > \sigma(rt_n).$$

Define the maximal chain f_σ in $\Pi_m^{r,j}$ whose i-block partition is obtained by splitting σ at rt_1, rt_2, ..., rt_{i-1}. As an example, for $\sigma = 562418379$ where $r = 2$, $n = 3$, $j = 3$ and $m = 9$, we have the maximal chain

$$f_{562418379} = \{\hat{0} < 56|24|18|379 < 56|2418|379 < 562418|379 < 562418379 = \hat{1}\}.$$

Observe that different permutations in $A_m^{r,j}$ give different maximal chains.

Theorem 6.1 *The labeling $(\lambda(x, y), -\rho(x))$ where λ is defined in equations (6.1) and (6.2), ρ denotes the rank function and the ordering is lexicographic on the pairs, is an EL-labeling for the poset $\Pi_m^{r,j}$. The falling maximal chains are given by $\{f_\sigma \,:\, \sigma \in A_m^{r,j}\}$.*

The proof that this labeling is an EL-labeling mimics the proof of Theorem 5.2 in Wachs' paper [26] and hence is omitted.

We distinguish between the cases $j = 1$ and $j \geq 2$ in the following two corollaries.

Corollary 6.2 *The chain complex of $\Pi_m^{r,1}$ is contractible.*

Corollary 6.3 *The chain complex of $\Pi_m^{r,j}$ is homotopy equivalent to a wedge of $\beta\left((\mathbf{a}^{r-1}\mathbf{b})^n \cdot \mathbf{a}^{j-2}\right)$ number of $(n-1)$-dimensional spheres. Hence all the poset homology of the poset $\Pi_m^{r,j}$ is concentrated in the top homology which has rank $\beta\left((\mathbf{a}^{r-1}\mathbf{b})^n \cdot \mathbf{a}^{j-2}\right)$*

7 Concluding remarks

Can more examples of exponential Dowling structures be given? For instance, find the Dowling extension of counting matrices with non-negative integer entries having a fixed row and column sum. See [21, Chapter 5].

Theorem 5.4 has been generalized in [10]. As we have seen in this theorem the generating function for the Möbius function of $D_n^{(r,k)} \cup \{\hat{0}\}$ in Proposition 4.4 in the case when the order s is equal to 1 has a permutation enumeration analogue. It would be interesting to find a permutation interpretation for this generating function for general values of the order s. Similar generating functions have appeared when enumerating classes of r-signed permutations. A few examples are $(\sin(px) + \cos((r-p)x)/\cos(rx)$ counting p-augmented r-signed permutations in [8], $\sqrt[r]{1/(1-\sin(rx))}$ counting augmented André r-signed permutations in [9], and $\sqrt[r]{1/(1-rx)}$ counting r-multipermutations in [16].

There are several other questions to raise. Is there a q-analogue of the partition lattice such that a natural q-analogue of Theorem 5.4 also holds? We only use the case $w = \mathbf{a}^{k-1}$ in Proposition 5.3. Are there other poset statistics that correspond to other **ab**-words w?

The symmetric group $\mathfrak{S}_{m-1}$ acts naturally on the lattice $\Pi_m^{r,j}$. Hence it also acts on the top homology group of $\Pi_m^{r,j}$. In a forthcoming paper we study the representation of this $\mathfrak{S}_{m-1}$ action.

Similar questions arise concerning the poset $D_n^{(r,k)} \cup \{\hat{0}\}$; see Proposition 4.4. Is this poset shellable? Is the homology of this poset concentrated in the top homology? Note that the wreath product $G \wr \mathfrak{S}_n$ acts on the Dowling lattice $L_n(G) = L_n$. Hence $G \wr \mathfrak{S}_n$ acts on the exponential Dowling structure $D_n^{(r,k)} \cup \{\hat{0}\}$. What can be said about the action of the wreath product $G \wr \mathfrak{S}_n$ on the homology group(s) of $D_n^{(r,k)} \cup \{\hat{0}\}$?

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