Negative q-Stirling numbers

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Abstract. The notion of the negative q-binomial was recently introduced by Fu, Reiner, Stanton and Thiem. Mirroring the negative q-binomial, we show the classical q-Stirling numbers of the second kind can be expressed as a pair of statistics on a subset of restricted growth words. The resulting expressions are polynomials in q and $(1+q)$. We extend this enumerative result via a decomposition of the Stirling poset, as well as a homological version of Stembridge's $q = -1$ phenomenon. A parallel enumerative, poset theoretic and homological study for the q-Stirling numbers of the first kind is done beginning with de Médicis and Leroux's rook placement formulation. Letting $t = 1 + q$ we give a bijective combinatorial argument à la Viennot showing the (q, t)-Stirling numbers of the first and second kind are orthogonal.

Résumé. La notion de la q-binomial négatif était introduit par Fu, Reiner, Stanton et Thiem. Réfléchissant la q-binomial négatif, nous démontrons que les classiques q-nombres de Stirling de deuxième espèce peuvent exprimés comme une paire des statistiques sur un sous-ensemble des mots qui a de croissance restreinte. Les expressions résultants sont les polynômes en q et $1+q$. Nous étendons cet résultat énumérative via d'une décomposition de la poset de Stirling, ainsi que d'une version homologique du $q = -1$ phénomène de Stembridge. Un parallèle énumérative, poset théoretique et étude homologique des q-nombres de Stirling de première espèce se fait en commençant par le formulation du placement des tours par suite des auteurs de Médicis et Leroux. On laisse $t = 1 + q$ ce que donner les arguments combinatoires et bijectives à la Viennot que démontrent que les (q, t)-nombres de Stirling de première et deuxième espèces sont orthogonaux.

Keywords: q-analogues, discrete Morse Theory, poset decomposition, algebraic complex, homology, orthogonality.

1 Introduction

The notion of the *negative q-binomial* has been recently introduced by Fu, Reiner, Stanton and Thiem [7]. It is defined by substituting $-q$ for q in the Gaussian coefficient and adjusting the sign:

$$\begin{bmatrix} n \\ k \end{bmatrix}'_q = (-1)^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q}.$$

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Recall $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ is the Gaussian polynomial, that is, the familiar q-analogue of the binomial coefficient, where $[m]_q = 1+q+\cdots+q^{m-1}$ and $[m]_q! = [1]_q[2]_q\cdots[m]_q$. At first blush, the substitution of $-q$ for q may seem naive. However, it is not unmotivated. Substituting values for q, such as roots of unity, non-roots of unity, or even zero in the theory of quantum groups can yield vital information.

Two well-known combinatorial interpretations of the the Gaussian polynomial are (i) it counts inversions in $\Omega(n,k) = \{0^{n-k}, 1^k\}$, that is, the set of all 0-1 bit strings consisting of $(n-k)$ zeroes and k ones, and (ii) it enumerates the number of k-dimensional subspaces from an n-dimensional vector space over a finite field with q elements. The negative q-binomial enjoys similar properties: (i) it can be expressed as a generalized inversion number of a *subset* $\Omega(n,k)'$ of 0-1 bit strings in $\Omega(n,k)$:

$$\begin{bmatrix} n \\ k \end{bmatrix}'_q = \sum_{\omega\in\Omega(n,k)'} \mathrm{wt}(\omega) = \sum_{\omega\in\Omega(n,k)'} q^{a(\omega)}(q-1)^{p(\omega)}, \quad 0 \le k \le n, \tag{1.1}$$

for statistics $a(\omega)$ and $p(\omega)$ [7, Theorem 1], (ii) it counts a certain subset of the k-dimensional subspaces of $\mathbb{F}_q^n$ [7, Section 6.2], (iii) it reveals a representation theory connection with unitary subspaces and a two-variable version exhibits a cyclic sieving phenomenon [7, Sections 4, 5].

The goal of this paper is to develop negative q-Stirling numbers of the first and second kind and discuss their poset and topological implications. The full version of this paper may be found on the arXiv, as well as the authors' websites.

2 RG-words

Recall a *set partition* on the n elements $\{1,2,\ldots,n\}$ is a decomposition into mutually disjoint nonempty sets called blocks. Unless otherwise indicated, throughout all set partitions will be written in standard form, that is, a partition into k blocks will be denoted by $\pi = B_1/B_2/\cdots/B_k$, where the blocks are ordered so that $\min(B_1) < \cdots < \min(B_k)$. We denote the set of all partitions of $\{1,\ldots,n\}$ by Π_n.

Given a partition $\pi \in \Pi_n$, we encode it using a *restricted growth word* $w(\pi) = w_1\cdots w_n$, where $w_i = j$ if the element i occurs in the jth block B_j of π. For example, the partition $\pi = 14/236/57$ has RG-word $w = 1221323$. Restricted growth words are also known as restricted growth functions. They have been studied by Hutchinson [11], Milne [16, 17] and Rota [20].

Two facts about RG-words follow immediately from using the standard form for set partitions.

Proposition 2.1 *The following properties are satisfied by RG-words:*

1. *Any RG-word begins with the element* 1.
2. *For an RG-word* ω *let* $\epsilon(j)$ *be the smallest index such that* $\omega_{\epsilon(j)} = j$. *Then* $\epsilon(1) < \epsilon(2) < \cdots$.

The *q-Stirling numbers of the second kind* are defined by

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q \cdot S_q[n-1,k], \text{ for } 0 \le k \le n, \tag{2.1}$$

with boundary conditions $S_q[n,0] = \delta_{n,0}$ and $S_q[0,k] = \delta_{0,k}$, where $\delta_{i,j}$ is the usual Kronecker delta function. Setting $q = 1$ gives the familiar Stirling number of the second kind $S(n,k)$ which enumerates the number of partitions $\pi \in \Pi_n$ with exactly k blocks. There is a long history of studying set partition statistics and q-Stirling numbers; see for example [1, 2, 4, 8, 9, 15, 17, 20, 27].

We begin by presenting a new statistic on RG-words which generate the q-Stirling numbers of the second kind. This differs from inversion type statistics others have used [15, 16, 21, 27]. Let $\mathcal{R}(n,k)$ denote the set of all RG-words of length n with maximum letter k. For $w \in \mathcal{R}(n,k)$, form the weight $\mathrm{wt}(w) = \Pi_{i=1}^{n} \mathrm{wt}_i(w)$, where for $m_i = \max\{w_1, \ldots, w_i\}$ let $\mathrm{wt}_1(w) = 1$ and for $2 \le i \le n$, let

$$\mathrm{wt}_i(w) = \begin{cases} q^{w_i - 1} & \text{if } w_i \le m_{i-1}, \\ 1 & \text{if } w_i > m_{i-1}. \end{cases} \tag{2.2}$$

For example, $\mathrm{wt}(1221323) = 1 \cdot 1 \cdot q^1 \cdot 1 \cdot 1 \cdot q^1 \cdot q^2 = q^4$. As a comment, the wt statistic differs from that of the ls statistic used in [27, page 29] which has an extra factor of $q^{\binom{k}{2}}$.

Lemma 2.2 *The q-Stirling number of the second kind is given by*

$$S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} \mathrm{wt}(w).$$

3 Allowable RG-words

Mirroring the negative q-binomial, in this section we define a subset of RG-words and statistics $A(\pi)$ and $B(\pi)$ which generate the classical q-Stirling number of the second kind as a polynomial in q and $(1+q)$. We will see in Sections 4 and 5 that this has *poset* and *topological* implications.

Definition 3.1 *An RG-word $\pi \in \mathcal{R}(n,k)$ is* allowable *if it is of the form*

$$\pi = u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdots,$$

where u_{2i-1} is a word on the alphabet $\{1, 3, \ldots, 2i-1\}$. Denote by $\mathcal{A}(n,k)$ the set of all allowable RG-words in $\mathcal{R}(n,k)$.

For an RG-word $\pi = \pi_1 \cdots \pi_n$ define the weight $\mathrm{wt}'(\pi) = \Pi_{i=1}^{n} \mathrm{wt}'_i(\pi)$, where for $m_i = \max\{\pi_1, \ldots, \pi_i\}$

$$\mathrm{wt}'_i(\pi) = \begin{cases} q^{\pi_i - 1} \cdot (1+q) & \text{if } \pi_i < m_{i-1}, \\ q^{\pi_i - 1} & \text{if } \pi_i = m_{i-1}, \\ 1 & \text{if } \pi_i > m_{i-1} \text{ or } i = 1. \end{cases} \tag{3.1}$$

For completeness, we decompose the weight statistic wt' into two statistics on RG-words. Let

$$A_i(\pi) = \begin{cases} \pi_i - 1 & \text{if } \pi_i \le m_{i-1}, \\ 0 & \text{if } \pi_i > m_{i-1} \text{ or } i = 1. \end{cases}, B_i(\pi) = \begin{cases} 1 & \text{if } \pi_i < m_{i-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Define

$$A(\pi) = \sum_{i=1}^{n} A_i(\pi) \quad \text{and} \quad B(\pi) = \sum_{i=1}^{n} B_i(\pi).$$

Theorem 3.2 *The q-Stirling number of the second kind can be expressed as a weighting over the set of allowable RG-words as follows:*

$$S_q[n,k] = \sum_{\pi \in \mathcal{A}(n,k)} \mathrm{wt}'(\pi) = \sum_{\pi \in \mathcal{A}(n,k)} q^{A(\pi)} (1+q)^{B(\pi)}. \tag{3.3}$$

Proposition 3.3 *The number of allowable words satisfies the recurrence*

$$|\mathcal{A}(n,k)| = |\mathcal{A}(n-1,k-1)| + \left\lceil \frac{k}{2} \right\rceil |\mathcal{A}(n-1,k)|.$$

Topological implications of Theorem 3.2 will be discussed in Section 5.

4 The Stirling poset of the second kind

In order to understand the q-Stirling numbers more deeply, we give a poset structure on $\mathcal{R}(n,k)$ which we call *the Stirling poset of the second kind*, denoted by $\Pi(n,k)$, as follows. For $\pi,\omega \in \mathcal{R}(n,k)$ let $\pi = \pi_1\pi_2\cdots\pi_n \prec \omega$ if $\omega = \pi_1\pi_2\cdots(\pi_i+1)\cdots\pi_n$ for some index i. See Figure 1 for an example. It is clear that if $\pi \prec \omega$ then $\mathrm{wt}(\omega) = q \cdot \mathrm{wt}(\pi)$, where the weight is defined as in (2.2). Thus the Stirling poset is graded. For basic terminology regarding posets, we refer the reader to Stanley's treatise [23].

As a remark, Park has a notion of the Stirling poset which arises from the theory of P-partitions [18]. It has no connection with the Stirling posets in this paper.

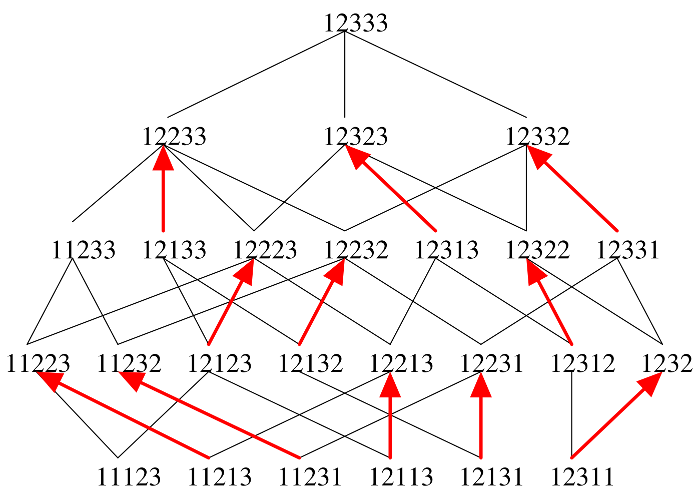

Fig. 1: The matching of the Stirling poset $\Pi(5,3)$. The matched elements are indicated by arrows. The unmatched elements have weight $1+q^2+q^4$.

We next review Kozlov's formulation of a Morse matching [14]. This will enable us to find a natural decomposition of the Stirling poset of the second kind, and to later be able to draw homological conclusions. A *partial matching* on a poset P is a matching on the underlying graph of the Hasse diagram of P, that is, a subset $M \subseteq P \times P$ satisfying (i) the ordered pair $(a,b) \in M$ implies $a \prec b$, and (ii) each element $a \in P$ belongs to at most one element in M. When $(a,b) \in M$, we write $u(a) = b$ and $d(b) = a$. A partial matching on P is *acyclic* if there does not exist a cycle

$$b_1 \succ d(b_1) \prec b_2 \succ d(b_2) \prec \cdots \prec b_n \succ d(b_n) \prec b_1$$

with $n \geq 2$, and the elements $b_1, \ldots, b_n$ distinct.

We define a matching M on the Stirling poset $\Pi(n,k)$ in the following manner. Let π_i be the first entry in $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{R}(n,k)$ such that π is weakly decreasing, that is, $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_{i-1} \geq \pi_i$ and where we require the inequality $\pi_{i-1} \geq \pi_i$ to be strict unless both π_{i-1} and π_i are even. We have two subcases. If π_i is even then let $d(\pi) = \pi_1\pi_2\cdots\pi_{i-1}(\pi_i - 1)\pi_{i+1}\cdots\pi_n$. Immediately we have $\mathrm{wt}(d(\pi)) = q^{-1} \cdot \mathrm{wt}(\pi)$. Otherwise, if π_i is odd then let $u(\pi) = \pi_1\pi_2\cdots\pi_{i-1}(\pi_i + 1)\pi_{i+1}\cdots\pi_n$ and we have $\mathrm{wt}(u(\pi)) = q \cdot \mathrm{wt}(\pi)$. If π has no index where it is weakly decreasing, then π is unmatched in the poset. Again, we refer to Figure 1.

Lemma 4.1 *For the partial matching M described on the poset $\Pi(n,k)$ the unmatched words $U(n,k)$ are of the form*

$$\pi = \begin{cases} u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdot 6 \cdots u_{k-1} \cdot k, & \text{for } k \text{ even} \\ u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdot 6 \cdots k-1 \cdot u_k, & \text{for } k \text{ odd} \end{cases} \tag{4.1}$$

where $u_{2i-1} = (2i-1)^{j_i}$, that is, u_{2i-1} is a word consisting of $j_i \geq 1$ copies of the odd integer $2i-1$.

As an example, the unmatched words in $\mathcal{R}(5,3)$ are 12333, 11233 and 11123.

To show this matching on $\Pi(n,k)$ is acyclic, we need the following result of Kozlov [14, Theorem 11.2].

Theorem 4.2 (Kozlov) *A partial matching on P is acyclic if and only if there exists a linear extension L of P such that the elements a and $u(a)$ follow consecutively in L.*

By defining a linear extension of the partial order on $\Pi(n,k)$ and considering a case-by-case analysis, we have the following result.

Theorem 4.3 *The matching M described for $\Pi(n,k)$ is an acyclic matching.*

5 Decomposition of the Stirling poset

We next decompose the Stirling poset $\Pi(n,k)$ into Boolean algebras indexed by the allowable words. This gives a poset explanation for the factorization of the q-Stirling number $S_q[n,k]$ in terms of powers of q and $(1+q)$. To state this decomposition, we need one definition. For $\omega \in \mathcal{A}(n,k)$ an allowable word let $\mathrm{Inv}(\omega) = \{\omega_i : \omega_j > \omega_i \text{ for some } j < i\}$ be the set of all entries in ω that contribute to an inversion. Such an entry ω_i must be odd since in a given allowable word any entry occurring to the left of an even entry must be strictly less than it.

Theorem 5.1 *The Stirling poset of the second kind $\Pi(n,k)$ can be decomposed as a disjoint union of Boolean algebras, where each corresponds to an allowable word, that is,*

$$\Pi(n,k) \cong \bigcup_{w \in \mathcal{A}(n,k)} B_{|\mathrm{Inv}(w)|}.$$

Furthermore, if $w \in \mathcal{A}(n,k)$ has weight $\mathrm{wt}'(w) = q^i(1+q)^j$, then the rank of w is i, and w is the minimal element of a Boolean algebra on j elements occurring in $\Pi(n,k)$.

See Figure 2 for an example of this decomposition.

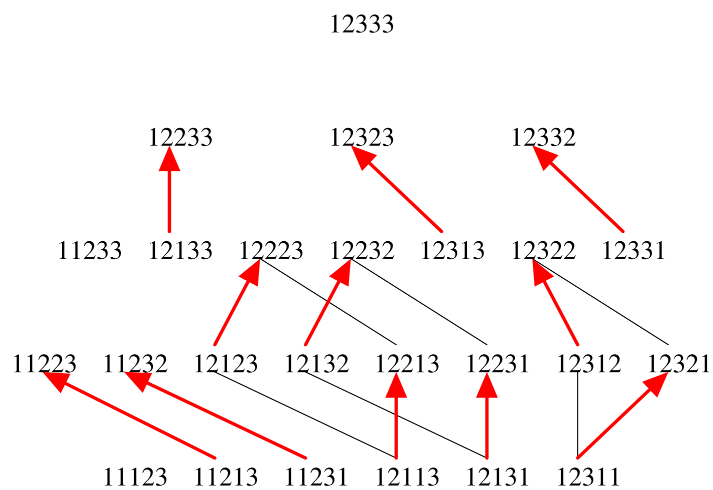

Fig. 2: The decomposition of the Stirling poset $\Pi(5,3)$ into Boolean algebras B_0, B_1 and B_2. Based on the ranks of the minimal elements in each Boolean algebra, one can read the weight of the poset as $S_q[5,3] = 1 + 2(1+q) + 3(1+q)^2 + q^2 + 3q^2(1+q) + q^4$.

6 Homological $q = -1$ phenomenon

Stembridge's $q = -1$ phenomenon [24, 25], and the more general cyclic sieving phenomenon of Reiner, Stanton and White [19] counts symmetry classes in combinatorial objects by evaluating their q-generating series at a primitive root of unity. Recently Hersh, Shareshian and Stanton [10] have given a homological interpretation of the $q = -1$ phenomenon by viewing it as an Euler characteristic computation on a chain complex supported by a poset. In the best scenario, the homology is concentrated in dimensions of the same parity and one can identify a homology basis. For further information about algebraic discrete Morse theory, see [12, 13, 22].

With this preparation, we have the graded poset $\Pi(n,k)$ supporting an algebraic complex $(\mathcal{C}, \partial)$ and a boundary map ∂. The aforementioned matching for $\Pi(n,k)$ (Theorem 4.3) is a discrete Morse matching for this complex. Hence using standard discrete Morse theory [6], we can give a basis for the homology.

The boundary map ∂ is defined as follows: for any $\omega \in \Pi(n,k)$, let $E(\omega) = \{\omega_{i_1} \ldots, \omega_{i_j} : i_1 < \cdots < i_j, \omega_{i_k}$ is even with $\omega_r \geq \omega_{i_k}$ for some $r < i_k\}$ be the set of all repeated even entries in ω arranged by index. Then $\partial(\omega) = \sum_{\omega_{i_r} \in E(\omega)} (-1)^{r-1} \omega_1 \cdots \omega_{i_r - 1} (\omega_{i_r} - 1) \omega_{i_r + 1} \cdots \omega_n$. If $E(\omega)$ is empty (in which case $\omega \in \mathcal{A}(n,K)$), define $\partial(\omega) = 0$.

We first need to verify that ∂ is actually a boundary map.

Lemma 6.1 *The map ∂ is a boundary map on the algebraic complex $(\mathcal{C}, \partial)$ with the poset $\Pi(n,k)$ as support.*

Lemma 6.2 *The weighted generating function of the unmatched words $U(n,k)$ in $\Pi(n,k)$ is given by the*

q^2-binomial coefficient

$$\sum_{u\in U(n,k)} \mathrm{wt}(u) = \begin{bmatrix} n-1-\lfloor\frac{k}{2}\rfloor \\ \lfloor\frac{k-1}{2}\rfloor \end{bmatrix}_{q^2}.$$

Notice that when we substitute $q^2 = 1$, the q^2-binomial coefficient reduces to the number of unmatched words.

We will need a lemma due to Hersh, Shareshian and Stanton [10, Lemma 3.2]. This is part (ii) of the original statement of the lemma.

Lemma 6.3 (Hersh–Shareshian–Stanton) *Let P be a graded poset supporting an algebraic complex (C, ∂) and assume P has a Morse matching M such that for all $q = M(p)$ with $q < p$, one has $\partial_{p,q} \in \mathbb{F}^x$. If all unmatched poset elements occur in ranks of the same parity, then $\dim H_i(C, d) = |P_i^{\mathrm{un}M}|$, that is, the number of unmatched elements of rank i.*

Theorem 6.4 *For the algebraic complex $(\mathcal{C}, \partial)$ supported by the Stirling poset $\Pi(n, k)$, the basis for homology is given by the* increasing *allowable RG-words in $\mathcal{A}(n, k)$. Furthermore, we have*

$$\sum_{i\geq 0} (\dim H_i) q^i = \begin{bmatrix} n-1-\lfloor\frac{k}{2}\rfloor \\ \lfloor\frac{k-1}{2}\rfloor \end{bmatrix}_{q^2}.$$

7 q-Stirling numbers of the first kind: Combinatorial interpretation

The (unsigned) q-Stirling number of the first kind is defined by the recurrence formula

$$c[n, k] = c[n-1, k-1] + [n-1]_q \cdot c[n-1, k], \tag{7.1}$$

where $c[n, 0] = \delta_{n,0}$ and $[m]_q = 1 + q + \cdots + q^{m-1}$.

One way to express q-Stirling numbers of the first kind is via rook placements. This is due to de Médicis and Leroux [3]. This is predated by work of Garsia and Remmel [8], who expressed q-Stirling numbers of the second kind using rook placements.

Definition 7.1 *Let $\mathcal{P}(m, n)$ be the set of all ways to place n rooks onto a staircase chessboard of length m such that no two rooks are in the same column. Moreover, for any rook placement $T \in \mathcal{P}(m, n)$, denote by $\mathrm{s}(T)$ the number of squares to the south of the rooks in T.*

Theorem 7.2 (de Médicis–Leroux) *The q-Stirling number of the first kind is given by*

$$c[n, k] = \sum_{T\in\mathcal{P}(n-1,n-k)} q^{\mathrm{s}(T)},$$

where the sum is over all rook placements of $n - k$ rooks on a staircase board of length $n - 1$.

We now define a subset $\mathcal{Q}(n-1, n-k)$ of rook placements in $\mathcal{P}(n-1, n-k)$ so that the q-Stirling number of the first kind $c[n, k]$ can be expressed as a statistic on the subset involving q and $q + 1$.

Definition 7.3 *Given any staircase chessboard, assign it a chequered pattern such that every other antidiagonal strip of squares is shaded, beginning with the lowest antidiagonal. Let $\mathcal{Q}(m, n) = \{T \in \mathcal{P}(m, n) :$ all rooks are placed in shaded squares$\}$. For any $T \in \mathcal{Q}(m, n)$, let $\mathrm{r}(T)$ denote the number of rooks in T that are not in the first row. For $T \in \mathcal{Q}(m, n)$, define the weight to be $\mathrm{wt}(T) = q^{\mathrm{s}(T)}(1+q)^{\mathrm{r}(T)}$.*

Theorem 7.4 *The q-Stirling number of the first kind is given by*

$$c[n,k] = \sum_{T \in \mathcal{Q}(n-1,n-k)} \mathrm{wt}(T) = \sum_{T \in \mathcal{Q}(n-1,n-k)} q^{\mathrm{s}(T)}(1+q)^{\mathrm{r}(T)},$$

where the sum is over all rook placements of $n-k$ rooks on an alternating shaded staircase board of length $n-1$.

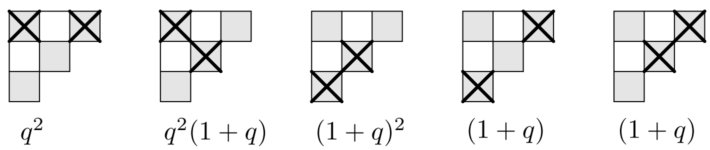

Fig. 3: Computing the q-Stirling number of the first kind $c[4,2]$ using $\mathcal{Q}(3,2)$.

Note that when we substitute $q=-1$ into the q-Stirling number of the first kind, the weight $\mathrm{wt}(T)$ of a rook placement T will be 0 if there is a rook in T that is not in the first row. Hence the Stirling number of the first kind $c[n,k]$ evaluated at $q=-1$ counts the number of rook placements in $\mathcal{Q}(n-1,n-k)$ such that all of the rooks occur in the first row. Thus we have

Corollary 7.5 *The q-Stirling number of the first kind $c[n,k]$ evaluated at $q=-1$ gives the number of rook placements in $\mathcal{Q}(n-1,n-k)$ where all of the rooks occur in the first row, that is,*

$$c[n,k]_{q=-1} = \binom{\lfloor \frac{n}{2} \rfloor}{n-k}.$$

8 Structure and topology of Stirling poset of the first kind

We define a poset structure on rook placements on a staircase shape board. For T and T' in $\mathcal{P}(m,n)$, let $T \prec T'$ if T' can be obtained from T by moving a rook to the left (west) or up (north) by one square. We call this poset *the Stirling poset of the first kind* and denote it by $\Gamma(m,n)$. We again wish to study its topological properties.

We define a matching on the poset as follows. Given $T \in \Gamma(m,n)$, let r be the first rook (reading from left to right) that is not in a shaded square in the first row. Then T is matched to T' where T' is obtained from T by moving r one square down if r is not in a shaded square, or one square up if r is in a shaded square but not in the first row. As the matching is defined, it is straightforward to check that the unmatched rook placements are the ones with all the rooks in the shaded squares appearing in the first row.

As an example, the matching on $\Gamma(3,2)$ is shown in Figure 4, where an upward arrow indicates a matching and the other edges indicate the remaining cover relations. Observe the unmatched rook placements are the ones with all the rooks occurring in the shaded squares in the first row. By the way a chessboard is shaded, the unmatched rook placements only appear in even ranks in the poset.

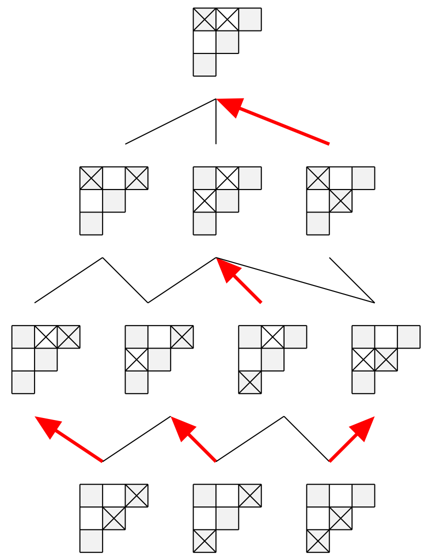

Fig. 4: The matching on $\Gamma(3,2)$. There is one unmatched rook placement on rank 2.

Theorem 8.1 *For the Stirling poset of the first kind $\Gamma(m,n)$, the generating function on unmatched words is*

$$\sum_{\substack{T\in\mathcal{P}(m,n)\\ T \text{ unmatched}}} \mathrm{wt}(T) = q^{n(n-1)} \begin{bmatrix} \lfloor \frac{m+1}{2} \rfloor \\ n \end{bmatrix}_{q^2} .$$

This theorem is a q-analogue of Corollary 7.5.

Using a similar linear extension argument as for $\Pi(n,k)$, we have the following result.

Theorem 8.2 *This matching on $\Gamma(m,n)$ is an acyclic matching.*

To apply Lemma 6.3, we need to define a boundary map on $\Gamma(m,n)$. Let $N(T)$ be the set of all rooks r_i in a rook placement $T \in \Gamma(m,n)$ that are not in shaded squares and $I(T)$ be the set of indices for the rooks in $N(T)$ arranged in increasing order, that is, $I(T) = \{i_j : r_{i_j} \in N(T) \text{ and } i_1 < i_2 < \cdots < i_{|N(T)|}\}$.

Lemma 8.3 *The map $\partial(T) = \sum_{r_{i_j}\in N(T)} (-1)^{j-1} T_{r_{i_j}}$ defined on the algebraic complex supported by the Stirling poset of the first kind $\Gamma(m,n)$ is a boundary map.*

Applying Theorem 8.1, Theorem 8.2 and Lemma 8.3 to Lemma 6.3, we have the following result.

Theorem 8.4 *For the algebraic complex $(\mathcal{C},\partial)$ supported by the Stirling poset of the first kind $\Gamma(m,n)$ we have*

$$\sum_{i\geq 0} \dim(H_i) q^i = q^{n(n-1)} \begin{bmatrix} \lfloor \frac{m+1}{2} \rfloor \\ n \end{bmatrix}_{q^2} .$$

9 Generating Function and Orthogonality

There are a number of two-variable Stirling numbers of the second kind using bistatistics on RG-words and rook placements. See [27] and the references therein. In this section, we give natural two variable (q,t)-analogues for Stirling numbers of the first and second kind, and their generating polynomials.

Definition 9.1 *Define the (q,t)-Stirling numbers of the first and second kind by*

$$s_{q,t}[n,k] = (-1)^{n-k} \cdot \sum_{T \in \mathcal{Q}(n-1,n-k)} q^{s(T)} \cdot t^{r(T)} \quad \text{and} \quad S_{q,t}[n,k] = \sum_{\pi \in \mathcal{A}(n,k)} q^{A(\pi)} \cdot t^{B(\pi)}, \tag{9.1}$$

where $t = q + 1$.

Recall the generating polynomials for the q-Stirling numbers are

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n,k] \cdot x^k \text{ , and } x^n = \sum_{k=0}^{n} S_q[n,k] \cdot (x)_{k,q} \text{ ,} \tag{9.2}$$

where $(x)_{k,q} = \Pi_{m=0}^{k-1}(x - [m]_q)$. See [8, pp. 270–271], as well as the overview article [5]. We generalize these to (q,t)-polynomials. Let $[k]_{q,t} = (q^{k-2} + q^{k-4} + \cdots + 1) \cdot t$ when k is even and $[k]_{q,t} = q^{k-1} + (q^{k-3} + q^{k-5} + \cdots + 1) \cdot t$ when k is odd.

Theorem 9.2 *The generating polynomials for the (q,t)-Stirling numbers are*

$$(x)_{n,q,t} = \sum_{k=0}^{n} s_{q,t}[n,k] \cdot x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_{q,t}[n,k] \cdot (x)_{k,q,t} \text{ ,} \tag{9.3}$$

where $(x)_{k,q,t} = \Pi_{m=0}^{k-1}(x - [m]_{q,t})$. When $k = 0$, define $(x)_{0,q,t} = 1$.

In [26] Viennot has some beautiful results which give combinatorial bijections for orthogonal polynomials. One well-known relation between the ordinary signed Stirling numbers of first kind and Stirling numbers of the second kind is their orthogonality. A bijective proof of the orthogonality of their q-analogues via 0–1 tableaux was given by de Médicis and Leroux [3, Proposition 3.1].

We show orthogonality holds combinatorially for the (q,t)-version of the Stirling numbers via a sign-reversing involution on ordered pairs of rook placements and RG-words.

Theorem 9.3 *The (q,t)-Stirling numbers are orthogonal, that is,*

$$\sum_{k=m}^{n} s_{q,t}[n,k] \cdot S_{q,t}[k,m] = \delta_{m,n} \text{ , and } \sum_{k=m}^{n} S_{q,t}[n,k] \cdot s_{q,t}[k,m] = \delta_{m,n} \text{ .} \tag{9.4}$$

Furthermore, this orthogonality holds bijectively.

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