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NOTE

Mixed Volumes and Slices of the Cube

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We give a combinatorial interpretation for the mixed volumes of two adjacent slices from the unit cube in terms of a refinement of the Eulerian numbers. © 1998 Academic Press

Let C^d denote the d-dimensional unit cube, that is,

$$C^d = \{\mathbf{x} \in \mathbb{R}^d : 0 \leqslant x_i \leqslant 1\}.$$

All the volumes and mixed volumes which appear in this note will be normalized so that the volume of C^d is given by

$$V(C^d) = d!.$$

We consider slices from the unit cube C^d where the cutting hyperplanes are orthogonal to the vector $(1, ..., 1)$. The *kth slice*, denoted by T^d_k, is the set

$$T^d_k = \left\{\mathbf{x} \in C^d : k-1 \leqslant \sum_{i=1}^{d} x_i \leqslant k\right\}.$$

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The *Eulerian number* $A_{d,k}$ is the number of permutations in the symmetric group S_d that have exactly $k-1$ descents. A result due to Laplace [5, p. 257ff] is an expression for the volume of T_k^d in terms of the Eulerian numbers:

$$V(T_k^d) = A_{d,k}. \tag{1}$$

Foata asked if there was a combinatorial proof of the identity in Eq. (1); see [3]. Stanley [8] provided a short and elegant argument.

We generalize Laplace's result to expressing the mixed volumes of two adjacent slices from the unit cube in terms of a refinement of the Eulerian numbers.

The *Minkowski sum* of two subsets K and L of $\mathbb{R}^d$ is the set

$$K + L = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in K, \mathbf{y} \in L\}.$$

For λ a real number, the *dilation* of K by λ is the set

$$\lambda \cdot K = \{\lambda \cdot \mathbf{x} : \mathbf{x} \in K\}.$$

A *convex body* is a convex and compact subset of $\mathbb{R}^d$. Convexity and compactness are preserved under Minkowski sum and dilation. Throughout we will reserve the letters K, L, and K_i to denote d-dimensional convex bodies. Similarly, λ, μ and λ_i denote non-negative scalars.

Given m convex bodies $K_1, ..., K_m$ in d-dimensional Euclidean space, consider the volume of the Minkowski linear combination $\lambda_1 \cdot K_1 + \cdots + \lambda_m \cdot K_m$. A classic result due to Minkowski [6] is that the volume $V(\lambda_1 \cdot K_1 + \cdots + \lambda_m \cdot K_m)$ is a homogeneous polynomial of degree d in the variables $\lambda_1, ..., \lambda_m$. That is, we may write

$$V(\lambda_1 \cdot K_1 + \cdots + \lambda_m \cdot K_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_d=1}^{m} a_{i_1, ..., i_d} \cdot \lambda_{i_1} \cdots \lambda_{i_d}. \tag{2}$$

The coefficients of this polynomial are called the *mixed volumes*; see [7, 12]. In fact, there is a symmetric d-ary function V on convex bodies such that $V(K_{i_1}, ..., K_{i_d}) = a_{i_1, ..., i_d}$. Thus, Eq. (2) may be written as

$$V(\lambda_1 \cdot K_1 + \cdots + \lambda_m \cdot K_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_d=1}^{m} V(K_{i_1}, ..., K_{i_d}) \cdot \lambda_{i_1} \cdots \lambda_{i_d}.$$

For shorthand we will write

$$V(K_1, i_1; ...; K_j, i_j) = V(\underbrace{K_1, ..., K_1}_{i_1}, ..., \underbrace{K_j, ..., K_j}_{i_j}),$$

where $i_1 + \cdots + i_j = d$. When the multiplicity i_m of some K_m is equal to 1, we will omit writing the multiplicity. For non-negative real numbers λ and μ, we can then write

$$V(\lambda \cdot K + \mu \cdot L) = \sum_{i=0}^{d} V(K, i; L, d-i) \cdot \binom{d}{i} \lambda^i \mu^{d-i}. \tag{3}$$

In general not much is known about computing mixed volumes. For K a convex body in d dimensions, $V(K, d)$ equals the volume of K. In the case where B is the unit ball in d dimensions, it is known that $V(K, d-1; B)$ gives the surface area of K up to a constant factor.

In order to state the main result of this note, recall that a *descent* in a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_d$ in the symmetric group S_d is a position i such that $\sigma_i > \sigma_{i+1}$.

THEOREM 1. *The mixed volume $V(T_k^d, d-i; T_{k+1}^d, i)$ is equal to the number of permutations in the symmetric group S_{d+1} with k descents and ending with the element $i+1$.*

Let X_k^d be the subset of $(\lambda + 1) \cdot C^d$ given by

$$X_k^d = \left\{ \mathbf{x} \in (\lambda + 1) \cdot C^d \colon (k-1)\, \lambda + k \leqslant \sum_{i=1}^{d} x_i \leqslant k\lambda + k + 1 \right\}.$$

The set $(\lambda + 1) \cdot C^d$ is a d-dimensional cube with side length $\lambda + 1$. Hence X_k^d is a slice from the dilated cube $(\lambda + 1) \cdot C^d$.

Since the dimension d will stay fixed in what follows, we will drop the superscript d.

LEMMA 2. *The sets X_k and $\lambda \cdot T_k + T_{k+1}$ are equal.*

Proof. It is easy to check that any point in $\lambda \cdot T_k + T_{k+1}$ satisfies the inequalities that define X_k. Hence the set $\lambda \cdot T_k + T_{k+1}$ is a subset of X_k. To prove the reverse inclusion, observe that both sets are convex polytopes. Hence it is enough to prove that the vertices of X_k belong to $\lambda \cdot T_k + T_{k+1}$.

Up to permutation of the coordinates, X_k has three kinds of vertices, namely

(a) $(\underbrace{\lambda + 1, ..., \lambda + 1}_{k-1}, 1, \underbrace{0, ..., 0}_{d-k})$,

(b) $(\underbrace{\lambda + 1, ..., \lambda + 1}_{k}, \underbrace{0, ..., 0}_{d-k})$, and

(c) $(\underbrace{\lambda + 1, ..., \lambda + 1}_{k}, 1, \underbrace{0, ..., 0}_{d-k-1})$.

The vertices of type (b) are also vertices of the dilated cube $(\lambda+1)\cdot C^d$. The vertices of type (a), respectively (c), lie on the hyperplane $x_1+\cdots+x_d=(k-1)\,\lambda+k$, respectively $x_1+\cdots+x_d=k\lambda+k+1$. Note that the vertices of type (a) and (c) lie on the edges of the dilated cube.

It is now straightforward to verify that each of these vertices belongs to $\lambda\cdot T_k+T_{k+1}$. Hence we conclude that $X_k=\lambda\cdot T_k+T_{k+1}$. ∎

An *indexed permutation* of length d and with indices in $\{0, 1, ..., n-1\}$ is an ordinary permutation in the symmetric group S_d where each letter has been assigned an integer between 0 and $n-1$. Indexed permutations, or **r**-signed permutations, are a generalization of permutations; see [1, 2, 11]. We will follow the notation in [11]. The set of all such indexed permutations is denoted by S_d^n. As an example, $3_3 4_1 2_1 5_0 1_3$ is an element of S_5^4 (and of S_5^n for any $n\geqslant 4$).

We say that a descent occurs at position $j<d$ in an indexed permutation $\pi=(a_1)_{s_1}(a_2)_{s_2}\cdots(a_d)_{s_d}\in S_d^n$ if (s_j, a_j) is greater than (s_{j+1}, a_{j+1}) in lexicographic order. Moreover, $j=d$ is a descent in π if $s_d>0$. The total number of descents in an indexed permutation π is denoted des(π). For example, the descents in $\pi=3_3 4_1 2_1 5_0 1_3$ occur at positions 1, 2, 3, and 5, and so des$(\pi)=4$.

For the rest of this note, let λ be a non-negative integer and $n=\lambda+1$. The following is proved in [11, Theorem 50].

PROPOSITION 3 [11]. *The volume of the slice X_k is equal to the number of indexed permutations in S_d^n with k descents.*

Let $A_{d+1,k,i}$ be the number of permutations in the symmetric group S_{d+1} ending in i and having k descents. The following lemma follows from the proof of Theorem 44 in [11]. See also [10, Exercise 3.71(d)] for the case $\lambda=1$

LEMMA 4 [11]. *The number of indexed permutations in S_d^n with k descents and with exactly i positive indices is given by $A_{d+1,k,d+1-i}\cdot\binom{d}{i}\lambda^i$.*

Proof. We prove this identity with an explicit bijection. Let P be a subset of $\{1, ..., d\}$ of cardinality i. For each element a in P, choose an index $s(a)$ from the set $\{1, ..., \lambda\}$. Let the index for all elements not in P be 0. Observe that the choice of the set P and the indices can be done in $\binom{d}{i}\lambda^i$ possible ways.

Choose a permutation $\sigma=\sigma_1\cdots\sigma_{d+1}$ in the symmetric group S_{d+1} such that $\sigma_{d+1}=d+1-i$. We will use σ and the indices chosen in the previous paragraph to construct an indexed permutation π which has exactly i positive indices. Moreover, the number of descents of the indexed permutation π will be equal to the number of descents of σ.

Consider the set $\{1_{s(1)}, 2_{s(2)}, ..., d_{s(d)}\}$. Sort this set in lexicographic order, that is, the element $a_{s(a)}$ comes before the element $b_{s(b)}$ if the pair $(s(a), a)$ is less than the pair $(s(b), b)$ in the lexicographic order. Let $q(j)$ denote the jth smallest element in the set. Let π be the indexed permutation $\pi = (\pi(1), ..., \pi(d))$, where

$$\pi(j) = \begin{cases} q(\sigma_j) & \text{if} \quad \sigma_j < d+1-i, \\ q(\sigma_j - 1) & \text{if} \quad \sigma_j > d+1-i. \end{cases}$$

Observe that if k is a position where σ has a descent then π also has a descent at this position and conversely. Hence the number of descents is preserved.

It is straightforward to verify that any indexed permutation may be built this way, that is, this is a bijection. ∎

We now give the proof of Theorem 1. By Lemma 4 we obtain the following expression for the number of indexed permutations in S_d^n with k descents.

$$|\{\pi \in S_d^n\colon \operatorname{des}(\pi) = k\}| = \sum_{i=0}^{d} A_{d+1, k, d+1-i} \cdot \binom{d}{i} \lambda^i. \tag{4}$$

Consider now the volume of the set $X_k = \lambda \cdot T_k + T_{k+1}$. By Eq. (3) we may express this volume as

$$V(\lambda \cdot T_k + T_{k+1}) = \sum_{i=0}^{d} V(T_k, i; T_{k+1}, d-i) \cdot \binom{d}{i} \lambda^i. \tag{5}$$

By Proposition 3 and Lemma 2, the left-hand sides of Eqs. (4) and (5) are equal. Comparing coefficients of $\binom{d}{i} \lambda^i$ in the right-hand sides of these equations, we obtain Theorem 1.

The Aleksandrov–Fenchel inequalities for mixed volumes state that the sequence

$$V(K_1; ...; K_{d-m}; K, i; L, m-i), \qquad i = 0, 1, ..., m,$$

is log-concave; see [7, 12]. Thus, as a corollary we obtain:

Corollary 5. *The sequence $A_{d+1, k, i}$, for $i = 1, ..., d+1$, is log-concave.*

Compare this result with Stanley's Corollary 3.3 in [9]: Fix a subset S of $\{1, ..., d\}$ and an integer j between 1 and $d+1$. Let ω_i be the number of permutations σ in S_{d+1} with descent set S and $\sigma_j = i$. Then the sequence $\omega_1, ..., \omega_{d+1}$ is log-concave. Motivated by his corollary, Stanley has asked the following question. Let $A_i = A_{d+1, k, j, i}$ be the number of permutations σ in S_{d+1} with k descents and $\sigma_j = i$. Is the sequence $A_1, ..., A_{d+1}$ log-concave?

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