Enumerative and asymptotic analysis of a moduli space

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To Doron Zeilberger on the occasion of his 60th birthday

Abstract

We study exact and asymptotic enumerative aspects of the Hilbert series of the cohomology ring of the moduli space of stable pointed curves of genus zero. This manifold is related to the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations of string theory.

Keywords: asymptotic enumeration, cohomology of moduli space, graded Hilbert series, integral operator, Lambert W function. MSC: primary 05E40, secondary 05A16.

1 Introduction

The study of moduli spaces has been a source of much research activity in physics, topology and algebraic topology. See for example [10, 12]. The classical example is the moduli space of smooth n-pointed stable curves of genus g, denoted $\overline{M}_{g,n}$. This moduli space was introduced by Deligne, Mumford and Knudsen [4, 9, 16] and gives a natural compactification of Mumford's [15] moduli space of nonsingular curves of genus g. To show how rich this family is, the case $\overline{M}_{1,n}$ corresponds to elliptic curves when $n = 1$ and to abelian varieties for $n > 1$.

In this paper we focus on combinatorial aspects of the Hilbert series of the cohomology ring of the moduli space of stable pointed curves of genus zero. We show its graded Hilbert series satisfies an integral operator identity. This is used to give asymptotic behavior, and in some cases, exact values, of the coefficients themselves. We then study the total dimension, that is, the sum of the coefficients of the Hilbert series. Its asymptotic behavior surprisingly involves the Lambert W function, which has applications to classical tree enumeration, signal processing and fluid mechanics.

2 Asymptotics and exact formulas for the cohomology ring of the moduli space

We begin by considering exact enumerative data of the Hilbert series of the cohomology ring of the moduli space $\overline{M}_{0,n}$. For completeness, we give the physicist's definition of this cohomology ring based on Keel's work. See [11, Section 0.10] and [8]. For $n \geq 3$ and k a field of characteristic zero, denote by P_n the set of *stable* 2*-partitions of* $\{1, \ldots, n\}$, that is, the set of all unordered partitions of the elements $\{1, \ldots, n\}$ into two blocks $\sigma = S_1|S_2$ with $|S_i| \geq 2$. For each $\sigma \in P_n$, the element x_σ corresponds to a cohomology class of $H^*(\overline{M}_{0,n})$. For $\sigma = S_1|S_2$ and $\tau = T_1|T_2$ from the index set P_n, let $a(\sigma, \tau)$ be the number of nonempty pairwise distinct sets among $S_i \cap T_j$, where $1 \leq i, j \leq 2$. Define the ideal I_n in the polynomial ring $\mathrm{k}[x_\sigma : \sigma \in P_n]$ to be generated by the relations:

(a)	(b)
1	1
1 1	2
1 5 1	7
1 16 16 1	34
1 42 127 42 1	213
1 99 715 715 99 1	1630
1 219 3292 7723 3292 219 1	14747

Table 1: The coefficients of the Hilbert series of the cohomology ring of the moduli space $\overline{M}_{0,n}$ for $n = 0, \ldots, 6$: (a) The triangular array of the graded dimensions $\alpha_{i,j}$ for $0 \leq i + j \leq 6$ and (b) table for the total dimension σ_n for $0 \leq n \leq 6$.

1. (linear relations) For i, j, k, l distinct:
$$R_{ijkl} : \quad \sum_{ij\sigma kl} x_\sigma - \sum_{kj\tau il} x_\tau, \tag{2.1}$$
where the summand $ij\sigma kl$ means to sum over all stable 2-compositions $\sigma = S_1|S_2$ with the elements $i, j \in S_1$ and the elements $k, l \in S_2$.

2. (quadratic relations) For each pair σ and τ with $a(\sigma, \tau) = 4$,
$$x_\sigma \cdot x_\tau. \tag{2.2}$$

The cohomology ring of the moduli space of n-pointed stable curves of genus 0 is given by the quotient ring $H^*(\overline{M}_{0,n}) = \mathrm{k}[x_\sigma : \sigma \in P_n]/I_n$.

See Table 1 for some values. One immediately sees the coefficients satisfy Poincaré duality.

Proposition 2.1 *The coefficients of the Hilbert series of the moduli space $\overline{M}_{0,n}$ are symmetric.*

This follows from the fact that the genus g moduli space $\overline{M}_{g,n}$ is a smooth, complete and compact variety [4, 9, 16]. For a proof in the genus 0 case, see [8]. By Proposition 2.1 we thus can write the Hilbert series as
$$\mathcal{H}(H^*(\overline{M}_{0,n})) = \sum_{(i,j)} \alpha_{i,j} t^i,$$
where the sum is over all pairs (i, j) of non-negative integers with $i + j = n$.

We next present a recursion for the coefficients of the Hilbert series of the cohomology ring of the moduli space $\overline{M}_{0,n}$. The recursion we give here is a symmetrized version of Keel's [8, page 550].

Theorem 2.2 *The coefficients of the Hilbert series series of the cohomology ring of the moduli space $\overline{M}_{0,n}$ satisfy the recursion:*
$$\alpha_{i+1,j+1} = \alpha_{i+1,j} + \alpha_{i,j+1} + \frac{1}{2} \sum_{p=0}^{i} \sum_{q=0}^{j} \binom{i+j+4}{p+q+2} \alpha_{p,q} \cdot \alpha_{i-p,j-q}, \tag{2.3}$$
where $i, j \geq 0$ and with the initial conditions $\alpha_{i,0} = \alpha_{0,i} = 1$ for $i \geq 0$.

$$
\begin{aligned}
f_0(x) &= e^x - x - 1 \\
f_1(x) &= 2e^{2x} - \left(\frac{1}{2}x^2 + 2x + 2\right) e^x \\
f_2(x) &= \frac{27}{2}e^{3x} - \left(4x^2 + 20x + 22\right) e^{2x} + \left(\frac{1}{8}x^4 + \frac{11}{6}x^3 + \frac{17}{2}x^2 + 15x + \frac{17}{2}\right) e^x \\
f_3(x) &= \frac{512}{3}e^{4x} - \left(\frac{243}{4}x^2 + 324x + 378\right) e^{3x} + \left(4x^4 + \frac{160}{3}x^3 + 240x^2 + 432x + 262\right) e^{2x} \\
&\quad - \left(\frac{1}{48}x^6 + \frac{2}{3}x^5 + \frac{185}{24}x^4 + 41x^3 + \frac{423}{4}x^2 + 126x + \frac{164}{3}\right) e^x
\end{aligned}
$$

Table 2: Expressions for $f_j(x)$, $j = 0, \ldots, 3$.

Define the exponential generating function

$$f_j(x) = \sum_{i \geq 0} \alpha_{i,j} \frac{x^{i+2}}{(i+2)!}.$$

These are series formed by taking diagonal entries in the $\alpha_{i,j}$ triangle with the power of t slightly shifted. See Table 2 for the first few values of $f_j(x)$. Let I be the integral operator

$$I(f(x)) = \int_0^x f(t)\ dt$$

and let I^j denote the jth iterated integral operator. The following integral operator identity holds among these series.

Lemma 2.3 *The generating function $f_j(x)$ satisfies the integral operator identity*

$$I^{j+1}(f_{j+1}(x)) = I^{j+2}(f_{j+1}(x)) + I^{j+1}(f_j(x)) + \frac{1}{2}\sum_{q=0}^{j} I^q(f_q(x)) \cdot I^{j-q}(f_{j-q}(x)), \tag{2.4}$$

where f_0 is given by $f_0 = e^x - x - 1$.

Proof: Multiply the identity (2.3) with $x^{i+j+4}/(i+j+4)!$ and sum over $i \geq 0$ to give

$$\sum_{i \geq 0} \frac{\alpha_{i+1,j+1}}{(i+j+4)!} \cdot x^{i+j+4} = \sum_{i \geq 0} \frac{\alpha_{i,j+1}}{(i+j+4)!} \cdot x^{i+j+4} + \sum_{i \geq 0} \frac{\alpha_{i+1,j}}{(i+j+4)!} \cdot x^{i+j+4} \tag{2.5}$$

$$+ \frac{1}{2}\sum_{i \geq 0}\sum_{p=0}^{i}\sum_{q=0}^{j} \frac{\alpha_{p,q}}{(p+q+2)!} \cdot \frac{\alpha_{i-p,j-q}}{(i-p+j-q+2)!} \cdot x^{i+j+4}. \tag{2.6}$$

The left-hand side of (2.5) can be written as

$$
\begin{aligned}
\sum_{i \geq 0} \frac{\alpha_{i+1,j+1}}{(i+j+4)!} \cdot x^{i+j+4} &= \sum_{i \geq 0} \alpha_{i,j+1} \cdot \frac{x^{i+j+3}}{(i+j+3)!} - \alpha_{0,j+1} \cdot \frac{x^{j+3}}{(j+3)!} \\
&= I^{j+1}(f_{j+1}(x)) - \frac{x^{j+3}}{(j+3)!}.
\end{aligned}
$$

In a similar manner, the right-hand side of (2.5) becomes

$$\sum_{i\geq 0}(\alpha_{i,j+1}+\alpha_{i+1,j})\cdot\frac{x^{i+j+4}}{(i+j+4)!} \quad = \quad I^{j+2}(f_{j+1}(x))+I^{j+1}(f_j(x))-\frac{x^{j+3}}{(j+3)!}.$$

Finally, the triple sum in (2.6) can be written as

$$\sum_{q=0}^{j}\left(\sum_{p\geq 0}\alpha_{p,q}\cdot\frac{x^{p+q+2}}{(p+q+2)!}\right)\cdot\left(\sum_{p\geq 0}\alpha_{p,j-q}\cdot\frac{x^{p+j-q+2}}{(p+j-q+2)!}\right) \quad = \quad \sum_{q=0}^{j} I^q(f_q(x))\cdot I^{j-q}(f_{j-q}(x)).$$

The result now follows by combining these identities. □

Notice the degrees of the polynomials preceding the e^{kx} terms in $f_j(x)$ depend only on j and k. This motivates one to define the *degree sequence* for functions having the expansion $f(x)=\sum_{j=0}^{k}p_j(x)\cdot e^{jx}$ where the $p_j(x)$ are polynomials in x by $\deg(f)=(\deg(p_j(x)))_{j=0}^{k}$. Here we define the degree of the zero polynomial to be $\deg(0)=-\infty$.

It is straightforward to verify the following proposition.

Proposition 2.4 *Let f and g have series expansions $f=\sum_{j=0}^{k}p_j(x)\cdot e^{jx}$ and $g=\sum_{j=0}^{m}q_j(x)\cdot e^{jx}$ with $\deg(f)=(a_0,a_1,\ldots,a_k)$ and $\deg(g)=(b_0,b_1,\ldots,b_m)$. Then*

(a) $\deg(f+g)\leq(c_0,\ldots,c_n)$, *where* $c_i=\max(a_i,b_i)$ *and* $n=\max(k,m)$.

(b) $\deg(f\cdot g)\leq(d_0,\ldots,d_{k+m})$, *where* $d_i=\max\limits_{p+q=i}(a_p+b_q)$.

(c) $\deg(\frac{df}{dx})=(a_0-1,a_1,\ldots,a_k)$ *with the convention* $0-1=-\infty$.

(d) If f is non-zero then

$$\deg(I(f))=(b_0,a_1,\ldots,a_k)$$

with $b_0=\max(a_0+1,0)$.

(e) When $f\equiv 0$, that is, $\deg(f)=(-\infty,-\infty,\ldots,-\infty)$ *then*

$$\deg(I(f))=(0,-\infty,\ldots,-\infty).$$

(f) The solution y to the differential equation

$$y'=y+f(x)$$

has degree sequence

$$\deg(y)=(a_0,\max(a_1+1,0),a_2,\ldots,a_k).$$

The degree sequence properties are now used to derive an upper bound for the degree sequences of the series f_j.

Proposition 2.5 *For $j \geq 1$, the series $f_j(x)$ has the expansion*

$$f_j(x) = \sum_{k=1}^{j+1} p_{j,k}(x) e^{kx},$$

where $p_{j,k}(x)$ is a polynomial of degree at most $2(j-k+1)$, that is,

$$\deg(f_j(x)) \leq (-\infty, 2j, 2(j-1), \ldots, 2, 0).$$

Proof: It is straightforward to verify that $f_0 = e^x - 1 - x$ and $f_1 = 2e^x - (\frac{1}{2}x^2 + 2x + 2)e^x$. Hence $\deg(f_0) = (1, 0)$ and $\deg(f_1) = (-\infty, 2, 0)$.

The proof is by induction on j, and we have just verified the induction basis $j = 1$. Apply the differential operator $\frac{d^{j+2}}{dx^{j+2}}$ to equation (2.4) to obtain

$$\frac{d}{dx}(f_{j+1}(x)) = f_{j+1}(x) + \frac{d^{j+2}}{dx^{j+2}} \left(I^{j+1}(f_j(x)) + \frac{1}{2} \sum_{q=0}^{j} I^q(f_q(x)) \cdot I^{j-q}(f_{j-q}(x)) \right). \tag{2.7}$$

Assume that $\deg(f_k) \leq (-\infty, 2k, 2k-2, \ldots, 2, 0)$ for all $1 \leq k \leq j$. By Proposition 2.4 we have

(i) $\deg(I^{j+1}(f_j)) \leq (j, 2j, 2j-2, \ldots, 2, 0)$,

(ii) $\deg(I^q(f_q) \cdot I^{j-q}(f_{j-q})) \leq (j-2, j + \max(q, j-q) - 1, 2j, 2j-2, \ldots, 2, 0)$ for $1 \leq q \leq j-1$,

(iii) $\deg(I^j(f_j) \cdot f_0) \leq (j, 2j+1, 2j, 2j-2, \ldots, 2, 0)$.

Adding these terms together, we obtain the upper bound for the corresponding degree sequence to be $(j, 2j+1, 2j, 2j-2, \ldots, 2, 0)$. Applying the operator d^{j+2}/dx^{j+2}, we have the upper bound $(-\infty, 2j+1, 2j, 2j-2, \ldots, 2, 0)$ for the rightmost term in the differential equation (2.7). Hence by Proposition 2.4 (f), the degree sequence of f_{j+1} has the desired form. □

In order to prove the next two theorems, we will need a lemma.

Lemma 2.6 *The following two identities hold for $t \geq 1$:*

$$\sum_{\substack{c+d=t \\ c,d \geq 1}} \binom{t}{c, d} c^{c-1} \cdot d^{d-1} = 2 \cdot (t-1) \cdot t^{t-2} \tag{2.8}$$

$$\sum_{\substack{c+d=t \\ c,d \geq 1}} \binom{t}{c, d} c^{c} \cdot d^{d-1} = (t-1) \cdot t^{t-1} \tag{2.9}$$

Proof: Observe the left-hand side of equation (2.8) enumerates the number of pairs of rooted labeled trees on a t-element set, that is, the coefficient of $x^t/t!$ in the generating function $f(x)^2$, where $f(x) = \sum_{t \geq 1} t^{t-1} \cdot x^t/t!$. Noticing that $f(x)$ satisfies the functional equation $f(x) = x \cdot e^{f(x)}$, the coefficient can also be determined by Lagrange inversion formula [20, Theorem 5.4.2] as follows:

$$\left[\frac{x^t}{t!}\right] f(x)^2 = t! \cdot \frac{2}{t} \left[x^{t-2}\right] (e^x)^t = 2 \cdot (t-1)! \cdot \frac{t^{t-2}}{(t-2)!} = 2 \cdot (t-1) \cdot t^{t-2}.$$

One can also prove equation (2.8) using the fact the Abel polynomials $p_n(x) = x \cdot (x - na)^{n-1}$ form a *sequence of binomial type*, that is, they satisfy the relation $p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$. See [19]. The desired identity then follows from the substitution $x = y = 1$, $a = -1$ and $n = t - 2$.

To prove (2.9), multiply (2.8) by $c + d = t$ and use that the two resulting sums are equal. □

We now show the bounds given in Proposition 2.5 are sharp.

Theorem 2.7 *For $s \geq 0$, $t \geq 1$, the coefficient of $x^{2s} e^{tx}$ in the series f_{s+t-1} is given by*

$$[x^{2s} e^{tx}] f_{s+t-1} = \frac{(-1)^s}{2^s \cdot s!} \cdot \frac{t^{2(s+t-1)}}{t!}. \tag{2.10}$$

Furthermore, the degree sequence of the series f_j for $j \geq 1$ is equal to

$$\deg(f_j(x)) = (-\infty, 2j, 2(j-1), \ldots, 2, 0).$$

Proof: Let $\gamma_{s,t} = [x^{2s} e^{tx}] f_{s+t-1}$. Note $\gamma_{s,0} = 0$ for $s \geq 0$. Observe that for $t \geq 1$

$$[x^{2s} e^{tx}](I^k(f_{s+t-1})) = \frac{\gamma_{s,t}}{t^k}.$$

Extract the coefficient of $x^{2s} e^{tx}$ from Lemma 2.3 in the case $j + 1 = s + t - 1$ to give

$$\begin{aligned} \frac{\gamma_{s,t}}{t^{s+t-1}} &= \frac{\gamma_{s,t}}{t^{s+t}} + 0 + \frac{1}{2} \cdot 2 \cdot [x^{2s} e^{tx}] f_0 \cdot I^{s+t-2}(f_{s+t-2}) \\ &+ \frac{1}{2} \sum_{q=1}^{s+t-3} [x^{2s} e^{tx}] I^q(f_q(x)) \cdot I^{s+t-2-q}(f_{s+t-2-q}(x)). \end{aligned}$$

We obtain

$$\frac{\gamma_{s,t}}{t^{s+t-1}} \cdot \left(1 - \frac{1}{t}\right) = \frac{1}{2} \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d \geq 1}} \frac{\gamma_{a,c}}{c^{a+c-1}} \cdot \frac{\gamma_{b,d}}{d^{b+d-1}} \tag{2.11}$$

because the sum of the inequalities $a+c \leq q+1$ and $b+d \leq s+t-2-q+1$ implies $a+b+c+d \leq s+t$. Furthermore, since $a + b + c + d = s + t$, these two inequalities are actually equalities, giving the index of summation as above.

Equation (2.11) can be viewed as a recursion for the coefficients $\gamma_{s,t}$. Hence we can prove the theorem by induction on the quantity $s + t$. The induction basis is $s = 0$ and $t = 1$, which is straightforward to verify. The induction step is to verify

$$\frac{(-1)^s}{2^s \cdot s!} \frac{t^{s+t-1}}{t!} \left(1 - \frac{1}{t}\right) = \frac{1}{2} \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d \geq 1}} \frac{(-1)^a}{2^a a!} \cdot \frac{c^{a+c-1}}{c!} \cdot \frac{(-1)^b}{2^b \cdot b!} \cdot \frac{d^{b+d-1}}{d!}.$$

Multiply by $2 \cdot (-2)^s \cdot s! \cdot t!$ to obtain

$$\begin{aligned} 2(t-1) \cdot t^{s+t-2} &= \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d \geq 1}} \binom{s}{a, b} \binom{t}{c, d} c^{a+c-1} \cdot d^{b+d-1} \\ &= \sum_{\substack{c+d=t \\ c,d \geq 1}} \binom{t}{c, d} c^{c-1} \cdot d^{d-1} \sum_{a+b=s} \binom{s}{a, b} c^a \cdot d^b \\ &= t^s \cdot \sum_{\substack{c+d=t \\ c,d \geq 1}} \binom{t}{c, d} c^{c-1} \cdot d^{d-1}, \end{aligned}$$

which is true by equation (2.8) in Lemma 2.6.

The equality in the degree sequence follows from the inequality in Proposition 2.5 and the fact the coefficients $\gamma_{s,t}$ are non-zero. □

Corollary 2.8 *The asymptotic behavior of $\alpha_{i,j}$ as $i \to \infty$ is given by*

$$\alpha_{i,j} \sim \frac{(j+1)^{2j+1}}{j!} \cdot (j+1)^i.$$

By a similar argument, we can find the next coefficient in the $f_j(x)$.

Theorem 2.9 *For $s, t \geq 1$, the coefficient of $x^{2s-1}e^{tx}$ in the series f_{s+t-1} is given by*

$$[x^{2s-1}e^{tx}]\, f_{s+t-1}(x) = (-1)^s \cdot \frac{(5s+9t-8) \cdot t^{2s+2t-4}}{3 \cdot 2^{s-1}(s-1)!(t-1)!}. \tag{2.12}$$

Proof: Using integration by parts we have that for $t \neq 0$

$$I\left((\gamma \cdot x^{2s} + \delta \cdot x^{2s-1}) \cdot e^{tx}\right) = \left(\frac{\gamma}{t} \cdot x^{2s} + \left(\frac{\delta}{t} - 2s \cdot \frac{\gamma}{t^2}\right) \cdot x^{2s-1}\right) \cdot e^{tx} + \text{lower order terms.}$$

By iterating operator I k times, we obtain

$$I^k\left((\gamma \cdot x^{2s} + \delta \cdot x^{2s-1}) \cdot e^{tx}\right) = \left(\frac{\gamma}{t^k} \cdot x^{2s} + \left(\frac{\delta}{t^k} - 2 \cdot k \cdot s \cdot \frac{\gamma}{t^{k+1}}\right) \cdot x^{2s-1}\right) \cdot e^{tx} + \text{lower order terms.}$$

Let $\delta_{s,t}$ denote the coefficient $\delta_{s,t} = [x^{2s-1}e^{tx}]f_{s+t-1}(x)$. Note that $\delta_{1,0} = -1$, $\delta_{s,0} = 0$ for $s \geq 2$ and $\delta_{0,t} = 0$ for $t \geq 0$.

Consider now the coefficient of $x^{2s-1}e^{tx}$ in Lemma 2.3 where $j = s+t-2$.

$$\begin{aligned} \frac{\delta_{s,t}}{t^{s+t-1}} - 2 \cdot (s+t-1) \cdot s \cdot \frac{\gamma_{s,t}}{t^{s+t}} &= \frac{\delta_{s,t}}{t^{s+t}} - 2 \cdot (s+t) \cdot s \cdot \frac{\gamma_{s,t}}{t^{s+t+1}} + 0 \\ &\quad + [x^{2s-1}e^{tx}]\frac{1}{2} \cdot \sum_{q=0}^{s+t-2} I^q(f_q(x)) \cdot I^{s+t-2-q}(f_{s+t-2-q}(x)) \end{aligned}$$

We now focus our attention on the last term in the right-hand side of the above identity. For shorthand, we call this term X.

$$\begin{aligned} X &= \frac{1}{2} \cdot \sum_{q=0}^{s+t-2} \sum_{a+b=s} \sum_{c+d=t} [x^{2a-1}e^{cx}]I^q(f_q(x)) \cdot [x^{2b}e^{dx}]I^{s+t-2-q}(f_{s+t-2-q}(x)) \\ &+ \frac{1}{2} \cdot \sum_{q=0}^{s+t-2} \sum_{a+b=s} \sum_{c+d=t} [x^{2a}e^{cx}]I^q(f_q(x)) \cdot [x^{2b-1}e^{dx}]I^{s+t-2-q}(f_{s+t-2-q}(x)). \end{aligned}$$

In order for $[x^{2a}e^{cx}]I^q(f_q(x))$ to be non-zero, we need $a+c \leq q+1$. Similarly, for $[x^{2a-1}e^{cx}]I^q(f_q(x))$ to be non-zero, we need $a+c \leq q+1$. Thus the variables of summation must satisfy $a+c \leq q+1$ and $b+d \leq s+t-q-1$. Adding these two inequalities gives an equality. Hence $q = a+c-1$

and we can remove the summation over q. Furthermore, the two double summation expressions are equal, so we obtain

$$\begin{aligned} X &= \sum_{a+b=s} \sum_{c+d=t} [x^{2a-1} e^{cx}] I^{a+c-1}(f_{a+c-1}(x)) \cdot [x^{2b} e^{dx}] I^{b+d-1}(f_{b+d-1}(x)) \\ &= -\frac{\gamma_{s-1,t}}{t^{s+t-2}} + \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d\geq 1}} \left(\frac{\delta_{a,c}}{c^{a+c-1}} - 2(a+c-1) \cdot a \cdot \frac{\gamma_{a,c}}{c^{a+c}} \right) \cdot \frac{\gamma_{b,d}}{d^{b+d-1}}. \end{aligned}$$

Observe the terms corresponding to $d = 0$ all vanish. The only surviving term corresponding to $c = 0$ is when $a = 1$, yielding the term $-\gamma_{s-1,t}/t^{s+t-2}$ above.

To summarize, we have

$$\begin{aligned} \frac{\delta_{s,t}}{t^{s+t-1}} - 2 \cdot (s+t-1) \cdot s \cdot \frac{\gamma_{s,t}}{t^{s+t}} &= \frac{\delta_{s,t}}{t^{s+t}} - 2 \cdot (s+t) \cdot s \cdot \frac{\gamma_{s,t}}{t^{s+t+1}} - \frac{\gamma_{s-1,t}}{t^{s+t-2}} \\ &\quad + \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d\geq 1}} \left(\frac{\delta_{a,c}}{c^{a+c-1}} - 2(a+c-1) \cdot a \cdot \frac{\gamma_{a,c}}{c^{a+c}} \right) \cdot \frac{\gamma_{b,d}}{d^{b+d-1}}. \end{aligned}$$

Using Theorem 2.7, we note

$$-\frac{\gamma_{s-1,t}}{t^{s+t-2}} = 2s \cdot \frac{\gamma_{s,t}}{t^{s+t}},$$

thus simplifying the identity to

$$\begin{aligned} &\frac{\delta_{s,t}}{t^{s+t-1}} \cdot \left(1 - \frac{1}{t}\right) - 2 \cdot (s+t) \cdot s \cdot \frac{\gamma_{s,t}}{t^{s+t}} \cdot \left(1 - \frac{1}{t}\right) \\ &= \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d\geq 1}} \left(\frac{\delta_{a,c}}{c^{a+c-1}} - 2(a+c-1) \cdot a \cdot \frac{\gamma_{a,c}}{c^{a+c}} \right) \cdot \frac{\gamma_{b,d}}{d^{b+d-1}}. \end{aligned}$$

Observe that this identity is a recursion for $\delta_{s,t}$. Hence we prove the theorem by induction on $s+t$. The induction basis $s = t = 1$ is straightforward. The induction step is to verify the following identity:

$$\begin{aligned} &\left(\frac{(-1)^s \cdot (5s+9t-8) \cdot t^{s+t-3}}{3 \cdot 2^{s-1} \cdot (s-1)!(t-1)!} - 2 \cdot (s+t) \cdot s \cdot \frac{(-1)^s \cdot t^{s+t-2}}{2^s \cdot s!t!} \right) \cdot \left(1 - \frac{1}{t}\right) \\ &= \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d\geq 1}} \left(\frac{(-1)^a \cdot 2a \cdot (5a+9c-8) \cdot c^{a+c-2}}{3 \cdot 2^a \cdot a!c!} - \frac{(-1)^a \cdot 2a \cdot (a+c-1) \cdot c^{a+c-2}}{2^a \cdot a!c!} \right) \cdot \frac{(-1)^b d^{b+d-1}}{2^b \cdot b!d!} \\ &= \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d\geq 1}} \frac{(-1)^a \cdot 2a \cdot c^{a+c-2}}{2^a \cdot a!c!} \left(\frac{(5a+9c-8)}{3} - (a+c-1) \right) \cdot \frac{(-1)^b d^{b+d-1}}{2^b \cdot b!d!}. \end{aligned}$$

Multiplying by $3 \cdot (-2)^s \cdot s! \cdot t!$ and simplifying gives

$$\begin{aligned} (2s+6t-8) \cdot s \cdot t^{s+t-3} \cdot (t-1) &= \sum_{a+b=s} \sum_{\substack{c+d=t \\ c,d\geq 1}} \binom{s}{a,b} \binom{t}{c,d} a \cdot c^{a+c-2} \cdot d^{b+d-1} \cdot (2a+6c-5) \\ &= \sum_{\substack{c+d=t \\ c,d\geq 1}} \binom{t}{c,d} \cdot c^{c-2} \cdot d^{d-1} \sum_{a+b=s} a \cdot (2a+6c-5) \cdot \binom{s}{a,b} \cdot c^a \cdot d^b. \end{aligned}$$

By applying the operator $c \cdot \frac{\partial}{\partial c}$ once, respectively twice, to the binomial theorem, we obtain the two identities

$$\sum_{a+b=s} a \cdot \binom{s}{a, b} \cdot c^a \cdot d^b = s \cdot c \cdot (c+d)^{s-1}, \tag{2.13}$$

$$\sum_{a+b=s} a^2 \cdot \binom{s}{a, b} \cdot c^a \cdot d^b = s \cdot c \cdot (c+d)^{s-1} + s \cdot (s-1) \cdot c^2 \cdot (c+d)^{s-2}. \tag{2.14}$$

We now have

$$\begin{aligned}
(2s+6t-8) \cdot s \cdot t^{s+t-3} \cdot (t-1) &= \sum_{\substack{c+d=t \\ c,d \geq 1}} \binom{t}{c, d} \cdot c^{c-2} \cdot d^{d-1} \big((6c-5) \cdot s \cdot c \cdot (c+d)^{s-1} \\
&\quad +2 \cdot s \cdot c \cdot (c+d)^{s-1} + 2 \cdot s \cdot (s-1) \cdot c^2 \cdot (c+d)^{s-2}\big) \\
&= s \cdot t^{s-2} \cdot \sum_{\substack{c+d=t \\ c,d \geq 1}} \binom{t}{c, d} \cdot c^{c-1} \cdot d^{d-1} \left((6c-5) \cdot t + 2t + 2(s-1) \cdot c\right) \\
&= s \cdot t^{s-2} \cdot \left(-3t \cdot 2(t-1)t^{t-2} + (6t + 2(s-1)) \cdot (t-1)t^{t-1}\right) \\
&= s \cdot t^{s-2} \cdot (t-1) \cdot t^{t-1} \cdot (-6 + 6t + 2s - 2),
\end{aligned}$$

where we have applied identities (2.13) and (2.14) in the first step and Lemma 2.6 in the third step. This completes the induction. □

3 Asymptotic behavior of the total graded dimension

Mathematical physicists often refer to the sum of the coefficients of a finite Hilbert series as the "dimension" of a space. For example, using techniques from combinatorial commutative algebra [18], it was proved that the dimension of the Losev–Manin ring [13], that is, the cohomology ring arising from the Commutativity Equations of physics, is given by $n!$.

Let

$$\sigma_n = \sum_{i+j=n} \alpha_{i,j}$$

denote the total graded dimension of the cohomology of the moduli space $\overline{M}_{0,n}$.

Proposition 3.1 *The total graded dimension σ_n satisfies the recursion*

$$\sigma_{n+2} = 2\sigma_{n+1} + \frac{1}{2} \sum_{i+j=n} \binom{n+4}{i+2, j+2} \sigma_i \cdot \sigma_j, \tag{3.1}$$

with the initial condition $\sigma_0 = 1$.

Proof: Sum the recursion (2.3) for $\alpha_{p,q}$ in the case $p+q = n+2$. □

The Lambert W function is the multivalued complex function which satisfies $\mathrm{W}(z) \cdot e^{\mathrm{W}(z)} = z$. The principal branch $\mathrm{W}_0(z)$ and the -1 branch $\mathrm{W}_{-1}(z)$ are the only two branches which take on real values. The Lambert W function has applications to classical tree enumeration, signal processing and fluid mechanics. For more details, see [1, 3].

For our purposes we will need to consider two branches of the Lambert W function which together form an analytic function near $-1/e$.

Lemma 3.2 *Let Ω be the function defined by*

$$\Omega(z) = \begin{cases} W_{-1}(z) & \mathrm{Im}(z) \geq 0, \\ W_1(z) & \mathrm{Im}(z) < 0. \end{cases}$$

Then Ω is analytic in $\mathbb{C} - (-\infty, -1/e] - [0, +\infty)$. The Puiseux series of Ω is given by

$$\Omega(z) = \sum_{k \geq 0} \mu_k p^k = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 - \cdots$$

where $p = -\sqrt{2(ez+1)}$ and the coefficients μ_k are computed by the recurrences

$$\begin{aligned} \mu_k &= \frac{k-1}{k+1}\left(\frac{\mu_{k-2}}{2} + \frac{\alpha_{k-2}}{4}\right) - \frac{\alpha_k}{2} - \frac{\mu_{k-1}}{k+1}, \\ \alpha_k &= \sum_{j=2}^{k-1} \mu_j \cdot \mu_{k+1-j}, \end{aligned}$$

with $\mu_0 = -1, \mu_1 = 1, \alpha_0 = 2$ and $\alpha_1 = -1$.

Proof: Analyticity follows from piecing together the analytic parts of the Lambert W function. The Puiseux series expansion is due to D. Coppersmith. See [3, Section 4]. □

Form the exponential generating function for the total graded dimension:

$$g(x) = \sum_{n \geq 0} \sigma_n \frac{x^{n+2}}{(n+2)!}.$$

Proposition 3.3 *The function $g(x)$ satisfies the differential equation*

$$g'(x) = \frac{x + 2g(x)}{1 - g(x)} \tag{3.2}$$

with boundary condition $g(0) = 0$.

Proof: Multiplying (3.1) by $x^n/(n+4)!$ and summing over $n \geq 0$ gives

$$\sum_{n \geq 0} \frac{\sigma_{n+2} \cdot x^n}{(n+4)!} = 2\sum_{n \geq 0} \frac{\sigma_{n+1} \cdot x^n}{(n+4)!} + \frac{1}{2}\left(\frac{1}{x^2} g(x)\right)^2. \tag{3.3}$$

Observe $g'(x) = \sum_{n \geq 0} \sigma_n \cdot x^{n+1}/(n+1)!$ and

$$I(g(x)) = x^4 \left(\sum_{m \geq 0} \sigma_{m+1} \cdot \frac{x^m}{(m+4)!}\right) + \frac{x^3}{3!}.$$

Thus we may rewrite (3.3) as

$$g(x) - \frac{x^2}{2} - \frac{2x^3}{3!} = 2\int_0^x g(t)dt - \frac{2x^3}{3!} + \frac{1}{2}g(x)^2. \tag{3.4}$$

Differentiating (3.4) gives the desired differential equation. $\Box$

This generating function is related to the Lambert W function.

Theorem 3.4 *The function $g(z)$ is given by*

$$g(z) = 1 - (z+2)\left(1 + \frac{1}{\Omega(-e^{-2}(z+2))}\right). \tag{3.5}$$

It is analytic in $\mathbb{C} - (-\infty, -2] - [e-2, +\infty)$.

Proof: Substitute $g(x) = 1 - (x+2)(u+1)$ in the differential equation (3.2). We obtain the equation

$$\frac{dx}{x+2} = -\left(\frac{1}{u} + \frac{1}{u^2}\right) du.$$

Integrating gives

$$\log|x+2| + C = -\log|u| + \frac{1}{u}.$$

The boundary condition $g(0) = 0$ implies $u(0) = -1/2$ and $C = -2$. Hence

$$-e^{-2}(x+2) = \frac{1}{u} \cdot e^{\frac{1}{u}}.$$

As the Lambert W function satisfies the functional equation $\mathrm{W}(z) \cdot e^{\mathrm{W}(z)} = z$, we have that $W(-(x+2)/e^2) = 1/u$. Recall that the value $x = 0$ implies $u = -1/2$, so $W(-2e^{-2}) = -2$. This implies W_{-1} is the correct real branch of the Lambert W function to select. See [3]. Hence we obtain the generating function $g(x) = 1 - (x+2)(1 + 1/\mathrm{W}_{-1}(-e^{-2}(x+2)))$. The analytic continuation to $\mathbb{C} - (-\infty, -2] - [e-2, +\infty)$ follows from Lemma 3.2. $\Box$

In order to obtain asymptotic behavior of the total graded dimension, one needs to apply results of Flajolet and Odlyzko. See [6, Section 4.2] and the overview article [17]. We follow Flajolet and Odlyzko's notation and terminology. Write $f(z) \sim g(z)$ as $z \to w$ to mean that $\frac{f(z)}{g(z)} \to 1$ as $z \to w$. For $r, \eta > 0$ and $0 < \varphi < \pi/2$ define

$$\Delta(r, \varphi, \eta) = \{z : \ |z| \le r + \eta, \ |\arg(z - r)| \ge \varphi\}.$$

A function $L(u)$ is of *slow variation at* ∞ if it satisfies (i) there exists a positive real number u_0 and an angle φ with $0 < \varphi < \pi/2$ such that $L(u) \neq 0$ and is analytic for $\{u : -(\pi - \varphi) \le \arg(u - u_0) \le \pi - \varphi\}$ and (ii) there exists a function $\epsilon(x)$ defined for $x \ge 0$ satisfying $\lim_{x\to+\infty} \epsilon(x) = 0$ and for all $\theta \in [-(\pi - \varphi), \pi - \varphi]$ and $u \ge u_0$ we have

$$\left|\frac{L(ue^{i\theta})}{L(u)} - 1\right| < \epsilon(u) \text{ and } \left|\frac{L(u\log^2(u))}{L(u)} - 1\right| < \epsilon(u).$$

The following asymptotic theorem appears in [6, Theorem 5].

Theorem 3.5 (Flajolet and Odlyzko) *Assume $f(z)$ is analytic on $\Delta(r,\varphi,\eta)-\{r\}$ and $L(u)$ is a function of slow variation at ∞. If $\alpha \in \mathbb{R}$ and $\alpha \notin \{0,1,2,\dots\}$ and*

$$f(z) \sim (r-z)^{\alpha} \cdot L\left(\frac{1}{r-z}\right)$$

uniformly as $z \to r$ for $z \in \Delta(r,\varphi,\eta)-\{r\}$, then

$$[z^n]f(z) \sim \frac{r^{-n}\cdot n^{-\alpha-1}}{\Gamma(-\alpha)} \cdot L(n).$$

Applying Theorem 3.5 to the Puiseux series of the analytic function Ω gives the following result.

Theorem 3.6 *The total dimension of the cohomology ring of the moduli space $\overline{M}_{0,n}$ has asymptotic behavior*

$$\sigma_n \sim \sqrt{\frac{e}{2\pi}} \cdot (e-2)^{-n-2} \cdot n^{-3/2} \cdot (n+2)! \quad \text{as } n \to \infty.$$

Proof: We will apply Theorem 3.5 to the function $f(z) = g(z) - g(e-2) = g(z) - 1$. By Lemma 3.2

$$\Omega(\zeta) = -1 + p + O\left(p^2\right),$$

where $p = -\sqrt{2(e\zeta+1)}$. By inverting this relation we have

$$\begin{aligned} \frac{1}{\Omega(\zeta)} &= -1 - p + O\left(p^2\right) \\ &= -1 + \sqrt{2e} \cdot \sqrt{\zeta + \frac{1}{e}} + O\left(\zeta + \frac{1}{e}\right). \end{aligned}$$

In the above, substitute $\zeta = -e^{-2}(z+2)$, add 1 and then multiply with $-(z+2) = -e + (e-2-z)$. We obtain

$$f(z) = -e \cdot \sqrt{2e} \cdot \sqrt{\frac{(e-2-z)}{e^2}} + O\left(e-2-z\right).$$

In other words,

$$f(z) \sim -\sqrt{2e} \cdot \sqrt{e-2-z}$$

uniformly as $z \to e-2$ for z in a deleted pie-shaped neighborhood $\Delta(r,\varphi,\eta)$ of $r = e-2$. By letting $r = e-2$, $\alpha = 1/2$ and L be the constant function $-\sqrt{2e}$, Theorem 3.5 applies. We conclude that

$$[x^n]g(x) = [x^n]f(x) \sim -\frac{(e-2)^{-n}\cdot n^{-3/2}}{\Gamma(-1/2)} \cdot \sqrt{2e} = \sqrt{\frac{e}{2\pi}} \cdot (e-2)^{-n} \cdot n^{-3/2}$$

as $n \longrightarrow \infty$. Since $[x^n]g(x) = \sigma_{n-2}/n!$, we obtain

$$\sigma_n \sim \sqrt{\frac{e}{2\pi}} \cdot (e-2)^{-n-2} \cdot (n+2)^{-3/2} \cdot (n+2)!.$$

Since $(n+2)^{-3/2} \sim n^{-3/2}$ as $n \to \infty$, the result follows. $\square$

An equivalent asymptotic expression appears without proof in [14, Chapter 4, page 194]. See [7] for a recursion for the total dimension of a generalization related to configuration spaces and [5] for associated results.

4 Concluding remarks

It remains to find the complete sequence of polynomials to describe the coefficients in the series f_j. We make the following conjecture.

Conjecture 4.1 *For k a non-negative integer and $s, t \geq k$, the coefficient of $x^{2s-k}e^{tx}$ in $f_{s+t-1}(x)$ is given by*

$$[x^{2s-k}e^{tx}]f_{s+t-1}(x) = (-1)^s \frac{t^{2s+2t-2k-2}}{2^{s-k} \cdot (s-k)!(t-k)!} \cdot Q_k(s,t), \tag{4.1}$$

where $Q_k(s,t)$ is a polynomial in the variables s and t of degree k.

As a special case we have already proved $Q_0(s,t) = 1$ and $Q_1(s,t) = (5s + 9t - 8)/3$.

In [8] Keel showed the moduli space $\overline{M}_{0,n}$ is related to n-pointed rooted trees of one-dimensional projective spaces. It is interesting to note that trees also emerge in our asymptotic study of the Hilbert series of the cohomology ring. See recent work of Chen, Gibney and Krashen [2] for further results on trees of higher-dimensional projective spaces.

5 Acknowledgements

The author would like to thank the Institute for Advanced Study where this paper was completed while the author was a Member of the School of Mathematics during 2010–2011, as well as Robert Corless and David Jeffrey for information about the Lambert W function.

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