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The r-cubical Lattice and a Generalization of the cd-index

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In this paper we generalize the **cd**-index of the cubical lattice to an **r**-**cd**-index, which we denote by $\boldsymbol{\Psi}(\mathbf{r})$. The coefficients of $\boldsymbol{\Psi}(\mathbf{r})$ enumerate augmented André **r**-signed permutations, a generalization of Purtill's work relating the **cd**-index of the cubical lattice and signed André permutations. As an application we use the **r**-**cd**-index to determine that the extremal configuration which maximizes the Möbius function of arbitrary rank selections, where all the r_i's are greater than one, is the odd alternating ranks, $\{1, 3, 5, \ldots\}$.

1. INTRODUCTION

The main purpose of this paper is to develop a generalization of the **cd**-index for the **r**-cubical lattice $C^{\mathbf{r}}$. This lattice is a natural generalization of the cubical lattice, that is, the face lattice of a cube. The cubical lattice of order n may be described by taking the nth power of the poset M_2 in Figure 1 and then adjoining a minimal element. The **r**-cubical lattice is constructed in a similar manner, where we instead take a product of posets $M_{\mathbf{r}}$ of the form represented in Figure 1. Such a lattice was first studied by Metropolis, Rota, Strehl and White in [12]. They were interested in Dilworth decompositions of the **r**-cubical lattice.

The **ab**-index is a non-commutative polynomial which encodes the Möbius function of rank selections from a poset P, i.e., its flag h-vector or beta invariant. Equivalently, the **ab**-index encodes the flag f-vector of P. Fine observed that when P is an Eulerian poset, the **ab**-index can be written uniquely in the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ (see [2]). This new polynomial is called the **cd**-index.

The importance of the **cd**-index is that it explicitly describes the generalized Dehn–Sommerville equations, also known as the Bayer–Billera relations [1]. Purtill obtained recursion formulas for the **cd**-index of the boolean algebra B_n and the cubical lattice C_n. In order to do, he showed that the coefficients of each **cd**-index enumerate André permutations and signed André permutations, respectively. André permutations were first studied by Foata and Schützenberger [8, 9]. We show that these two recurrences are easy to prove by using the standard R-labeling of B_n and C_n (see the arguments before equations (1) and (3)).

As a corollary, Purtill concluded that the **cd**-index of B_n and C_n have positive coefficients. Using a shelling argument, Stanley extended this result to showing that the **cd**-index has non-negative coefficients when P is the face poset of a shellable regular CW-sphere. This class of posets includes face lattices of convex polytopes.

Although the **r**-cubical lattice is not an Eulerian poset, we are still able to form its **ab**-index. We give a recursion for its **ab**-index in terms of the non-commutative variables $\bar{\mathbf{c}}_s = \mathbf{a} + (s-1)\cdot\mathbf{b}$, $\bar{\mathbf{d}}_s = \mathbf{ab} + (s-1)\cdot\mathbf{ba}$, $\mathbf{c}$ and $\mathbf{d}$. Since this recursion coincides with Purtill's **cd**-index recurrence for the cubical lattice, that is, when $\mathbf{r} = (2, \ldots, 2)$, we call it the **r**-**cd**-index, $\boldsymbol{\Psi}(C^{\mathbf{r}}) = \boldsymbol{\Psi}(\mathbf{r})$.

In Section 6 we extend Purtill's notion of signed André permutations to augmented André **r**-signed permutations. We show that the coefficients of $\boldsymbol{\Psi}(\mathbf{r})$ have a combinatorial interpretation, that is, they enumerate augmented André **r**-signed permutations.

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FIGURE 1. The Hasse diagrams of the poset M_r with $r = 2$ and $r = 5$.

Finally, in Section 7 we maximize the beta invariant of the **r**-cubical lattice over arbitrary rank selections. We do this by showing that the **ab**-index of $C^{\mathbf{r}}$, $\boldsymbol{\Psi}(\mathbf{r})$, has the strictly increasing alternating property. More precisely, we prove that the coefficient of the **ab**-word vw is larger than the coefficient of the **ab**-word vw^* when v ends in a different letter than w begins with and where w^* is obtained from w by uniformly exchanging the **a**'s and **b**'s. Thus, we can view these inequalities as the edges of an n-cube. Niven established these kind of inequalities when he determined that the largest class of permutations in the symmetric group having a fixed descent set is the alternating permutations [14]. Our inequalities imply that the coefficient of the alternating **ab**-word $\mathbf{baba}\cdots$ in $\boldsymbol{\Psi}(\mathbf{r})$ is the largest. In other words, the set of ranks $\{1, 3, 5, \ldots\}$ is the unique extremal configuration for the **r**-cubical lattice.

2. THE **ab**-INDEX

In this section we give a brief introduction to the **ab**-index and the **cd**-index. For all terminology and notation related to the **cd**-index, we will follow [23]. For poset terminology, we refer the reader to [22].

Let P be a finite, graded poset of rank $n + 1$ with $\hat{0}$ and $\hat{1}$. Denote the rank function of P by ρ. For $S \subseteq [n] = \{1, 2, \ldots, n\}$, we define the *S-rank-selected subposet* to be $P(S) = \{x \in P: \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. Let $\alpha(S) = \alpha_P(S)$ denote the number of maximal chains in $P(S)$ and let the *beta invariant* $\beta(S) = \beta_P(S)$ be defined by $\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|}\alpha(T)$.

To encode the beta invariant of the poset P, we begin by defining a monomial in the non-commutative variables **a** and **b** by $u_S = u_1 \cdots u_n$, where u_i is **a** if $i \notin S$ and u_i is **b** if $i \in S$. (Later when we work with permutations, it will be helpful to think of **a** as "ascent" and **b** as "descent".) As an example, if $n = 5$ and $S = \{1, 4, 5\}$, then $u_S = \mathbf{baabb}$. Form a non-commutative polynomial, called the **ab**-*index,* by

$$\boldsymbol{\Psi}(P) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

The degree of both **a** and **b** is defined to be one so that $\boldsymbol{\Psi}(P)$ is homogeneous of degree n.

For an **ab**-word w we denote its length by $|w|$. Let 1 denote the unique word of length 0. Also, the *complement* of the word w is the word formed by uniformly exchanging the letters **a** and **b**. We denote the complement of w by w^*.

Fine (refer to [2]) observed that if P is an Eulerian poset, then $\boldsymbol{\Psi}(P)$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the **cd**-*index.* See Stanley [23] for an elementary proof of the existence of the **cd**-index for Eulerian posets. Since both **c** and **d** are symmetric in **a** and **b**, this implies the well-known property that for an Eulerian poset P of rank $n + 1$, $\beta_P(S) = \beta_P(\bar{S})$, where $\bar{S}$ denotes the complement of S in the set $[n]$. In terms of a word w and its complement, this means that the coefficient of w is equal to the coefficient of w^* in $\boldsymbol{\Psi}(P)$.

DEFINITION 2.1. Let $\mathscr{L}$ be a linear combination of **ab**-words of length n. That is,

$\mathcal{L} = \sum_{z:\,|z|=n} c(z) \cdot z$. We say that $\mathcal{L}$ has the *weakly increasing alternating property* if the following two conditions hold:
(1) If v and w is a pair of words so that the last letter of the word v is different from the first letter of the word w and $|v| + |w| = n$, then $c(vw) \geqslant c(vw^*)$.
(2) If w is a word of length n that begins with **b**, then $c(w) \geqslant c(w^*)$.
If all of the above inequalities are strict, then we say that $\mathcal{L}$ has the *strictly increasing alternating property.*

It is easy to see that if $\mathcal{L}$ has the strictly and $\mathcal{K}$ has the weakly increasing alternating property, then their sum $\mathcal{L} + \mathcal{K}$ has the strictly increasing alternating property. Observe that $\mathcal{L}$ having the strictly increasing alternating property implies that the largest coefficient in $\mathcal{L}$ is the coefficient in front of the alternating word **baba** $\cdots$.

We say that a linear combination of **ab**-words, $\mathcal{L} = \sum_{z:\,|z|=n} c(z) \cdot z$, *is self-complementary* if, for all words w, $c(w) = c(w^*)$.

LEMMA 2.2. *If a linear combination $\mathcal{L}$ of* **ab**-*words of length n can be expressed as a* **cd**-*index with non-negative coefficients, then $\mathcal{L}$ has the weakly increasing alternating property and is self-complementary. Moreover, if $\mathcal{L}$ can be expressed as a* **cd**-*index with positive coeffiicients then the inequality $c(vw) \geqslant c(vw^*)$, where the last letter of the word v is different from the first letter of the word w, is a strict inequality. Hence, the* ***ab***-*words with largest coefficient are the two alternating words* **aba** $\cdots$ *and* **bab** $\cdots$.

LEMMA 2.3. *Let $\mathcal{L}$ be a linear combination of* **ab**-*words of length n and let $\mathcal{K}$ be a linear combination of* **ab**-*words of length m. If both $\mathcal{L}$ and $\mathcal{K}$ have the weakly increasing alternating property and $\mathcal{K}$ is self-complementary, then $\mathcal{L} \cdot \mathcal{K}$ also has the weakly increasing alternating property.*

Stanley [23] proved that the **cd**-index of the face poset of a shellable regular CW-sphere has non-negative coefficients. Thus by Lemma 2.2 we conclude that the **ab**-index of such posets has the weakly increasing alternating property. Since convex polytopes are shellable regular CW-spheres, we know that face lattices of convex polytopes have the weakly increasing alternating property.

3. R-LABELINGS

An *edge-labeling* λ of a locally finite poset P is a map which assigns to each edge in the Hasse diagram of P an element from some poset Λ. For us, Λ will always be a linearly ordered poset. In this case we say that λ is a *linear edge labeling* (see [7] for a further study of linear edge labelings). If x and y is an edge in the poset, that is, y covers x in P, then we denote the label on this edge by $\lambda(x, y)$. A maximal chain $x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ in an interval $[x, y]$ in P is called *rising* if the labels are weakly increasing with respect to the order of the poset Λ, that is, $\lambda(x_0, x_1) \leqslant_\Lambda \lambda(x_1, x_2) \leqslant_\Lambda \cdots \leqslant_\Lambda \lambda(x_{k-1}, x_k)$. An edge-labeling is called an *R-labeling* if for every interval $[x, y]$ in P there is a unique rising maximal chain in $[x, y]$.

Let P be a poset of rank $n+1$ with R-labeling λ. For a maximal chain $c = \{\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{n+1} = \hat{1}\}$ in P, the *descent set* of the chain c is $D(c) = \{i\colon \lambda(x_{i-1}, x_i) >_\Lambda \lambda(x_i, x_{i+1})\}$. Observe that $D(c)$ is a subset of the set $[n]$.

A result of Björner and Stanley [5, Theorem 2.7] says that if P is a graded poset of rank $n+1$, $S \subseteq [n]$, and P admits an R-labeling, then $\beta(S)$ equals the number of maximal chains in P having descent set S with respect to the given R-labeling λ. Thus we may compute the **ab**-index by considering an R-labeling of the poset.

LEMMA 3.1. *Let P be a graded poset of rank $n+1$. If λ is an R-labeling of P, then the* **ab***-index of P is equal to*

$$\boldsymbol{\Psi}(P) = \sum_{c} u_{D(c)},$$

where the sum is over all maximal chains c.

As an example, we give the standard R-labeling for the boolean algebra. Viewing B_n as the poset of all the subsets of $[n]$ ordered by inclusion, label the edge $A \subset B$ with the unique element in $B - A$. Observe the maximal chains in B_n correspond to permutations of the set $[n]$. It is now easy to give a recursion for the **ab**-index of the boolean algebra. Consider permutations on the set $[n+2]$, and let $i+1$ be the position at which the element 1 or $n+2$ occurs first, reading from right to left. Note that there are i elements from the set $\{2, \ldots, n+1\}$ to the right of this position. If $i = 0$ then these permutations are enumerated by $\boldsymbol{\Psi}(B_{n+1}) \cdot \mathbf{c}$. If $1 \leqslant i \leqslant n$, they are enumerated by $\binom{n}{i} \cdot \boldsymbol{\Psi}(B_{n+1-i}) \cdot \mathbf{d} \cdot \boldsymbol{\Psi}(B_i)$. Thus

$$\boldsymbol{\Psi}(B_{n+2}) = \boldsymbol{\Psi}(B_{n+1}) \cdot \mathbf{c} + \sum_{i=1}^{n} \binom{n}{i} \cdot \boldsymbol{\Psi}(B_{n+1-i}) \cdot \mathbf{d} \cdot \Psi(B_i), \tag{1}$$

where $\boldsymbol{\Psi}(B_1) = 1$. This formula was established by Purtill in [18, Corollary 5.8] using André permutations.

Hence, by equation (1), we may compute

$$\begin{aligned} \boldsymbol{\Psi}(B_2) &= \mathbf{c}, & \boldsymbol{\Psi}(B_4) &= \mathbf{c}^3 + 2 \cdot \mathbf{cd} + 2 \cdot \mathbf{dc}, \\ \boldsymbol{\Psi}(B_3) &= \mathbf{c}^2 + \mathbf{d}, & \boldsymbol{\Psi}(B_5) &= \mathbf{c}^4 + 3 \cdot \mathbf{c}^2\mathbf{d} + 5 \cdot \mathbf{cdc} + 3 \cdot \mathbf{dc}^2 + 4 \cdot \mathbf{d}^2. \end{aligned}$$

By the recursion (1) it is easy to see that the coefficients of each **cd**-monomial in $\boldsymbol{\Psi}(B_n)$ are positive. Thus, by Lemma 2.2, we conclude the following.

THEOREM 3.2 (Sagan, Yeh and Ziegler [21]). *For arbitrary rank selections S from the boolean algebra B_n, the two unique extremal configurations for maximizing the beta invariant $\beta(S)$ are the following rank selections*:

$$\{1, 3, 5, \ldots\} \cap [n-1] \quad \textit{and} \quad \{2, 4, 6, \ldots\} \cap [n-1].$$

This theorem is implicit in the work of Niven [14] and de Bruijn [6], who studied permutations with a given descent set.

4. THE r-CUBICAL LATTICE

For r a positive integer, let M_r denote the poset formed from an r-element antichain and a maximal element $\hat{1}$, where each element of the antichain is covered by the maximal element $\hat{1}$. See Figure 1 for two examples.

For a sequence of positive integers $\mathbf{r} = (r_1, \ldots, r_n)$, define the (multi-indexed) **r**-*cubical lattice* $C^{\mathbf{r}}$ to be $M_{r_1} \times \cdots \times M_{r_n} \cup \{\hat{0}\}$. This is a graded poset of rank $n+1$. Indeed, this is a lattice, since it is a finite join-semilattice. When $\mathbf{r} = (r, \ldots, r)$, we will denote the poset by C_n^r, and simply call it the *r-cubical lattice.* When $\mathbf{r} = (2, \ldots, 2)$ the **r**-cubical lattice is the cubical lattice C_n, that is, the face lattice of the n-dimensional cube.

Another way in which to view the **r**-cubical lattice is to consider finite sequences $A = (A_1, A_2, \ldots, A_{\max(\mathbf{r})})$ of subsets from the set $[n] = \{1, 2, \ldots, n\}$, such that $A_j \cap A_k = \varnothing$

when $j \neq k$, and $i \notin A_j$ when $j > r_i$. Define the order relation by $A \leqslant B$ if $A_i \supseteq B_i$ for all $i = 1, 2, \ldots,$ and adjoin a minimal element $\hat{0}$.

The Whitney numbers of the second kind for $C^{\mathbf{r}}$ are given by elementary symmetric functions. That is, the number of elements of rank $n + 1 - k$ in the **r**-cubical lattice is the kth elementary symmetric function $e_k(r_1, \ldots, r_n)$ in the variables $r_1, r_2, \ldots, r_n$, for $k = 0, \ldots, n$.

The **r**-cubical lattice has a very nice R-labeling described as follows: for the cover relation $A \prec B$, where $A \neq \hat{0}$, label the corresponding edge in the Hasse diagram by (i, a), where i is the unique index such that $A_i \neq B_i$, and let a be the singleton element in $A_i - B_i$. Also, for the relation $\hat{0} \prec B$, let the label be the special element G. Hence, the set of labels T_n are $T_n = \{G\} \cup \{(i, j): 1 \leqslant j \leqslant m,\ 1 \leqslant i \leqslant r_j\}$. Also, define the set T'_n to be $T'_n = T_n - \{G\}$.

So far we have not given a linear order on the set of labels T_n. We now do this. Choose any linear order Λ which satisfies the following condition:

$$(i, j) <_{\Lambda} G \Rightarrow i < r_j \qquad \text{and} \qquad (i, j) >_{\Lambda} G \Rightarrow i = r_j. \tag{2}$$

This means that the labels above the element G in the ordering Λ are those of the form (r_j, j). It is now straightforward to prove the following.

LEMMA 4.1. *Let Λ be a linear order on the set T_n satisfying condition* (2). *Then the above-described edge-labeling for the* **r**-*cubical lattice is an R-labeling.*

EXAMPLE 4.2. A linear order on T_n satisfying condition (2) is the following. Define $G <_{\Lambda} (r_1, 1) <_{\Lambda} (r_2, 2) <_{\Lambda} \cdots <_{\Lambda} (r_n, n)$. Order the labels of the form (i, j), where $i < r_j$, by $(i_1, j_1) <_{\Lambda} (i_2, j_2)$ if $j_1 > j_2$, or if $j_1 = j_2$ and $i_1 < i_2$. Finally, we say $(i, j) <_{\Lambda} G$ if $i < r_j$. Observe that the largest element in the linear order Λ is (r_n, n) and the $r_n - 1$ smallest elements are $(1, n) <_{\Lambda} \cdots <_{\Lambda} (r_n - 1, n)$. The linear order Λ satisfies condition (2), and thus is an R-labeling of the **r**-cubical lattice.

5. AUGMENTED **r**-SIGNED PERMUTATIONS

DEFINITION 5.1. Let N be a finite set of cardinality n and let $\mathbf{r}$ be a vector which is indexed by the set N, that is, $\mathbf{r} = (r_i)_{i \in N}$. An *augmented* **r**-*signed permutation* σ on the set N is a list of the form

$$(G, (i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)),$$

where $1 \leqslant i_m \leqslant r_{j_m}$ and $(j_1, j_2, \ldots, j_m)$ is a permutation of the elements in the set N. We will write $\sigma_0 = G$ and $\sigma_k = (i_k, j_k)$.

We view the elements $i_1, \ldots, i_n$ as signs; hence the name **r**-signed permutation. Since we list the special element G first, we say that the permutation is augmented. Thus if we exclude the special element G, we may say that the permutation is *non-augmented.* Usually, we will consider the set $N = [n] = \{1, 2, \ldots, n\}$. For $0 \leqslant i \leqslant j \leqslant n$, we let $[i, j] = \{i, i + 1, \ldots, j\}$. We use the notation $\sigma|_{[i,j]}$ to denote the restricted permutation $\sigma|_{[i,j]} = (\sigma_i, \sigma_{i+1}, \ldots, \sigma_j)$.

Let Λ be a linear order on the set T_n. The descent set of an augmented **r**-signed permutation, $\sigma = (G = \sigma_0, \sigma_1, \ldots, \sigma_n)$, with respect to Λ is the set $D_{\Lambda}(\sigma) = \{i: \sigma_{i-1} >_{\Lambda} \sigma_i\}$. The same definition also applies to non-augmented **r**-signed permutations.

The maximal chains in the **r**-cubical lattice correspond to augmented **r**-signed permutations on the set $[n]$. Thus the number of augmented **r**-signed permutations having a certain descent set is equal to the number of maximal chains with this same descent set.

LEMMA 5.2. *Let Λ be a linear order on the set T'_n. The number of non-augmented* **r**-*signed permutations on the set $[n]$ with descent set $S \subseteq [n-1]$ is equal to the number of permutations in the symmetric group on n elements with descent set S times $r_1 \cdot r_2 \cdots r_m$.*

PROOF. Consider first the following linear order Γ on T'_n: $(i_1, j_1) \leq_\Gamma (i_2, j_2)$ if and only if $j_1 < j_2$ or, $j_1 = j_2$ and $i_1 \leq i_2$. It is easy to see that the lemma holds for this linear order Γ.

We will prove the lemma for any other linear order by changing a linear order on the set T'_n into another linear order by transposing adjacent entries, and showing the number of non-augmented **r**-signed permutations having descent S will remain constant. Thus it is enough to consider two linear orders Λ and Λ' on the set T'_n which only differ in that two adjacent elements, x and y, are transposed. That is, we have $x <_\Lambda y$ and $y <_{\Lambda'} x$.

Let P and P' be the set of non-augmented **r**-signed permutations having descent S in the linear order Λ, respectively Λ'. Let σ be a non-augmented **r**-signed permutation in the set $P - P'$. Since $\sigma \in P$, but $\sigma \notin P'$, we know that changing the underlying order from Λ to Λ' affects the descent set of σ. Hence σ has the form $\sigma = (\sigma_1, \ldots, \sigma_i, z_1, z_2, \sigma_{i+3}, \ldots, \sigma_n)$, where $\{z_1, z_2\} = \{x, y\}$. Consider $\sigma' = (\sigma_1, \ldots, \sigma_i, z_2, z_1, \sigma_{i+3}, \ldots, \sigma_n)$. It is easy to see that σ' lies in the set $P' - P$. Moreover, the mapping $\sigma \mapsto \sigma'$ is bijective. Hence $|P - P'| = |P' - P|$, which implies that $|P| = |P'|$. Thus the proof is complete. $\square$

We denote the **ab**-index of the **r**-cubical lattice by $\boldsymbol{\Psi}(C^{\mathbf{r}}) = \boldsymbol{\Psi}(\mathbf{r}) = \boldsymbol{\Psi}(r_1, \ldots, r_n)$. For a vector $\mathbf{r} = (r_1, \ldots, r_n)$ and a positive integer s, we write $(\mathbf{r}, s)$ for the vector $(r_1, \ldots, r_n, s)$. Let $\bar{\mathbf{c}}_s = \mathbf{a} + (s-1) \cdot \mathbf{b}$, and $\bar{\mathbf{d}}_s = \mathbf{ab} + (s-1) \cdot \mathbf{ba}$. For N a finite subset of $\mathbb{P} = \{1, 2, \ldots\}$, define the vector $\mathbf{r}_N$ by $(r_{m_1}, \ldots, r_{m_n})$, where $N = \{m_1, \ldots, m_n\}$. Another useful notation is $\Pi(N) = \prod_{m \in N} r_m$.

PROPOSITION 5.3. *The* **ab**-*index of the* $(\mathbf{r}, s)$-*cubical lattice satisfies the following recurrence:*

$$\boldsymbol{\Psi}(\mathbf{r}, s) = \boldsymbol{\Psi}(\mathbf{r}) \cdot \bar{\mathbf{c}}_s + \sum_{\substack{I + J = [n] \\ I \neq \varnothing}} \Pi(I) \cdot \boldsymbol{\Psi}(\mathbf{r}_J) \cdot \bar{\mathbf{d}}_s \cdot \boldsymbol{\Psi}(B_{|I|}),$$

where $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ *and* $\boldsymbol{\Psi}(C^{\varnothing}) = 1$.

EXAMPLE 5.4. With the above recurrence we may compute the following:

$$\begin{aligned}
\boldsymbol{\Psi}(C^{\varnothing}) &= 1, \\
\boldsymbol{\Psi}(C^{p}) &= \bar{\mathbf{c}}_p, \\
\boldsymbol{\Psi}(C^{p,q}) &= \bar{\mathbf{c}}_p \bar{\mathbf{c}}_q + p \cdot \bar{\mathbf{d}}_q, \\
\boldsymbol{\Psi}(C^{p,q,r}) &= \bar{\mathbf{c}}_p \bar{\mathbf{c}}_q \bar{\mathbf{c}}_r + p \cdot \bar{\mathbf{d}}_q \bar{\mathbf{c}}_r + p \cdot \bar{\mathbf{c}}_q \bar{\mathbf{d}}_r + q \cdot \bar{\mathbf{c}}_p \bar{\mathbf{d}}_r + pq \cdot \bar{\mathbf{d}}_r \mathbf{c}.
\end{aligned}$$

PROOF OF PROPOSITION 5.3. Fix the linear order Λ on the set T_{n+1} of labels to be the one described in Example 4.2. Consider an augmented $(\mathbf{r}, s)$-signed permutation

$\sigma = (G = \sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ on the set $[n+1]$. Let k be the index such that $\sigma_k = (i, n+1)$, that is, the position of the element $n+1$. Let J be the set of elements from $[n]$ that appears before the element $n+1$ in the permutation, and let I be the set of elements from $[n]$ that appears after. We can decompose the permutation σ into an augmented $\mathbf{r}_J$-signed permutation on the set J and a non-augmented $\mathbf{r}_I$-signed permutation on the set I. Namely, the augmented $\mathbf{r}_J$-signed permutation is described by $\sigma|_{[0,k-1]}$ and the non-augmented $\mathbf{r}_I$-signed permutation is $\sigma|_{[k+1,n+1]}$.

There are two cases, namely $I = \emptyset$ and $I \neq \emptyset$. Consider the first case. If i, the sign of the element $n+1$, is equal to s, then the permutation will end with an ascent. If $i < s$, then the permutation will end with a descent. Thus these permutations are enumerated by $\boldsymbol{\Psi}(\mathbf{r}) \cdot (\mathbf{a} + (s-1) \cdot \mathbf{b}) = \boldsymbol{\Psi}(\mathbf{r}) \cdot \bar{\mathbf{c}}_s$.

Now consider the second case, $I \neq \emptyset$. If $i = s$, then there is an ascent–descent between the permutation $\sigma|_{[0,k-1]}$, and the permutation $\sigma|_{[k+1,n+1]}$. Similarly, if $i < s$ then there is a descent–ascent between the two parts. Hence we have the term

$$\boldsymbol{\Psi}(\mathbf{r}_J) \cdot (\mathbf{ab} + (s-1) \cdot \mathbf{ba}) \cdot \Pi(I) \cdot \boldsymbol{\Psi}(B_{|I|}) = \Pi(I) \cdot \boldsymbol{\Psi}(\mathbf{r}_J) \cdot \bar{\mathbf{d}}_s \cdot \boldsymbol{\Psi}(B_{|I|}).$$

By summing over all decompositions $I + J = [n]$, the proposition follows. □

When we set $\mathbf{r} = (2, \ldots, 2)$ in Proposition 5.3, we obtain

$$\boldsymbol{\Psi}(C_{n+1}) = \boldsymbol{\Psi}(C_n) \cdot \mathbf{c} + \sum_{i=1}^{n} \binom{n}{i} \cdot 2^i \cdot \boldsymbol{\Psi}(C_{n-i}) \cdot \mathbf{d} \cdot \boldsymbol{\Psi}(B_i), \tag{3}$$

where $\boldsymbol{\Psi}(C_0) = 1$. This identity was first established by Purtill [18, Corollary 5.12]. However, the proof given here is a direct argument.

EXAMPLE 5.5. Specializing Example 5.4 to the cubical lattice, we obtain the following:

$$\begin{aligned}
\boldsymbol{\Psi}(C_0) &= 1,\\
\boldsymbol{\Psi}(C_1) &= \mathbf{c},\\
\boldsymbol{\Psi}(C_2) &= \mathbf{c}^2 + 2 \cdot \mathbf{d},\\
\boldsymbol{\Psi}(C_3) &= \mathbf{c}^3 + 4 \cdot \mathbf{cd} + 6 \cdot \mathbf{dc}.
\end{aligned}$$

By the recursion (3) it is easy to see that the coefficients of each **cd**-monomial in $\boldsymbol{\Psi}(C_n)$ are positive. Thus by Lemma 2.2 we conclude the following:

THEOREM 5.6 (Readdy [20]). *For arbitrary rank selections s from the cubical lattice C_n, the two unique extremal configurations which maximize the beta-invariant $\beta(S)$ are the following rank selections:*

$$\{1, 3, 5, \ldots\} \cap [n] \qquad \textit{and} \qquad \{2, 4, 6, \ldots\} \cap [n].$$

6. ANDRÉ PERMUTATIONS

Purtill showed a relation between the **cd**-index of the cubical lattice and André signed permutations [18]. In this section we define augmented André **r**-signed permutations and obtain a relation between these permutations and the **r**-**cd**-index of the **r**-cubical lattice. We study two sets of **r**-signed permutations, $\mathscr{A}^{\mathbf{r}}$ and $\mathscr{N}_0^{\mathbf{r}}$. The set $\mathscr{A}^{\mathbf{r}}$ corresponds to the **r**-cubical lattice and the set $\mathscr{N}_0^{\mathbf{r}}$ to the boolean algebra. We also enumerate the number of augmented André r-signed permutations. When we set $\mathbf{r} = (2, \ldots, 2)$, the results of this section specialize to Purtill's work.

Define the two sets T and T' by $T' = \{(i, j): j \in \mathbb{P},\ 1 \leqslant i \leqslant r_j\}$ and $T = T' \cup \{G\}$. Observe that the entries of $\mathbf{r}$-signed permutations are elements of T. Throughout what follows in this section we fix Λ, a linear order on the set T. (Note that in the proof of Proposition 6.5 we will use another linear order on the set T'.) Define $G <_\Lambda (r_1, 1) <_\Lambda (r_2, 2) <_\Lambda \cdots$. Order the labels of the form (i, j), where $i < r_j$, by $(i_1, j_1) <_\Lambda (i_2, j_2)$ if $j_1 > j_2$, or if $j_1 = j_2$ and $i_1 < i_2$. Finally, say that $(i, j) <_\Lambda G$ if and only if $i < r_j$. This is an extension of the linear order used in Example 4.2.

Definition 6.1. Let $\mathbf{r}$ be a vector which is indexed by a finite set N, with $|N| = n > 0$, that is, $\mathbf{r} = (r_i)_{i \in N}$. We may assume that $N \subseteq \mathbb{P}$. We say that an augmented $\mathbf{r}$-signed permutation $\sigma = (G = \sigma_0, \sigma_1, \ldots, \sigma_n)$ on the set N is an *augmented André* $\mathbf{r}$*-signed permutation* if the following two conditions are satisfied.
(1) For all $1 \leqslant i < j \leqslant n$, if $\sigma_{i-1} = \max_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$ and $\sigma_j = \min_\Lambda\{\sigma_{i-1}, \sigma_i, \sigma_{j-1}, \sigma_j\}$, then there exists a k, with $i < k < j$, such that $\sigma_{i-1} <_\Lambda \sigma_k$.
(2) For $x = \max N$, $(r_x, x) = \sigma_m$ for some $1 \leqslant m \leqslant n$ and $\sigma|_{[0, m-1]}$ is an augmented André $\mathbf{r}_J$-signed permutation on the set J, where $J = \{y \in N: (z, y) = \sigma_k$ for some $1 \leqslant k \leqslant m - 1\}$.
The permutation (G) is defined to be an augmented André $\mathbf{r}$-signed permutation on the set $N = \varnothing$.

Let $\sigma = (G = \sigma_0, \sigma_1, \ldots, \sigma_n)$ be an augmented $\mathbf{r}$-signed permutation on a set N of cardinality n. We say that σ has a *double descent* if there is an index i, where $1 \leqslant i \leqslant n - 1$, such that σ has a descent at the ith and $(i + 1)$st positions. In other words, i and $i + 1$ are contained in the descent set $D_\Lambda(\sigma)$ of σ. Observe that condition (1) of Definition 6.1 implies that the permutation σ has no double descents.

A non-augmented $\mathbf{r}$-signed permutation satisfying condition (1) in Definition 6.1 is called a *non-augmented André* $\mathbf{r}$*-signed permutation.* (Note that for the non-augmented case we need to reformulate the beginning of condition (1) as, 'For all $2 \leqslant i < j \leqslant n \ldots$'.) We denote the set of all augmented André $\mathbf{r}$-signed permutations by $\mathscr{A}^{\mathbf{r}}$ and the set of all non-augmented André $\mathbf{r}$-signed permutations by $\mathscr{N}^{\mathbf{r}}$. Furthermore, we denote the set of all non-augmented André $\mathbf{r}$-signed permutations which begin with its smallest element (with respect to the linear order Λ) by $\mathscr{N}^{\mathbf{r}}_0$. That is,

$$\mathscr{N}^{\mathbf{r}}_0 = \{(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathscr{N}^{\mathbf{r}}: \sigma_1 = \min_\Lambda\{\sigma_1, \sigma_2, \ldots, \sigma_n\}\}.$$

We will mainly work with the sets $\mathscr{A}^{\mathbf{r}}$ and $\mathscr{N}^{\mathbf{r}}_0$.

The following two lemmas describe how André $\mathbf{r}$-signed permutations behave under restriction. For ease in notation, we define $N(\sigma, i, j) = \{y \in N: (z, y) = \sigma_k$ for some $i \leqslant k \leqslant j\}$.

Lemma 6.2. *Let $\sigma = (G, \sigma_1, \ldots, \sigma_n)$ be an augmented André* $\mathbf{r}$*-signed permutation on an index set N of cardinality n. Let $0 \leqslant j \leqslant n$, and let J be the set $N(\sigma, 1, j)$. Then the restriction $\sigma|_{[0,j]}$ is an augmented André $\mathbf{r}_J$-signed permutation on the index set J. Furthermore, let $1 \leqslant i \leqslant j \leqslant n$ and let K be the set $N(\sigma, i, j)$. Then the restriction $\sigma|_{[i,j]}$ is a non-augmented André $\mathbf{r}_K$-signed permutation on the index set K.*

Similarly, let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a non-augmented André $\mathbf{r}$*-signed permutation on an index set N of cardinality n. Let $1 \leqslant i \leqslant j \leqslant n$ and let K be the set $N(\sigma, i, j)$. Then the restriction $\sigma|_{[i,j]}$ is a non-augmented André $\mathbf{r}_K$-signed permutation on the index set K.*

The proof of Lemma 6.2 follows from the definitions.

Corollary 6.3. *If $\sigma = (G, \sigma_1, \ldots, \sigma_n)$ is an augmented André* $\mathbf{r}$*-signed* *permutation, then $G <_{\Lambda} \sigma_1$. In other words, every augmented André* **r***-signed permutation begins with an ascent.*

Lemma 6.4. *Let $\sigma = (G, \sigma_1, \ldots, \sigma_n)$ be an augmented André* **r***-signed permutation on an index set N of cardinality n. Assume that $x = \max N$, and $\sigma_m = (r_x, x)$. Let I be the set $N(\sigma, m+1, n)$. Then the restriction $\sigma|_{[m+1,n]} = (\sigma_{m+1}, \ldots, \sigma_n)$ is a non-augmented André $\mathbf{r}_I$-signed permutation on the index set I. Moreover, $\sigma|_{[m+1,n]}$ belongs to the set $\mathcal{N}_0^{\mathbf{r}_I}$.*

Similarly, let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a non-augmented André **r***-signed permutation on an index set N of cardinality n. Assume that $\sigma_m = \max_{\Lambda}\{\sigma_1, \ldots, \sigma_n\}$. Let I be the set $N(\sigma, m+1, n)$. Then the restriction $\sigma|_{[m+1,n]} = (\sigma_{m+1}, \ldots, \sigma_n)$ is a non-augmented André $\mathbf{r}_I$-signed permutation on the index set I. Moreover, $\sigma|_{[m+1,n]}$ belongs to the set $\mathcal{N}_0^{\mathbf{r}_I}$.*

For σ a non-augmented André **r**-signed permutation of an n-set, the *variation* of σ is given by $U(\sigma) = u_S$, where S is the descent set of σ taken with respect to Λ and u_S is the **ab**-word defined in Section 2. The *reduced variation* of $\sigma \in \mathcal{N}_0^{\mathbf{r}}$, which we denote by $V(\sigma)$, is formed by replacing each **ab** in $U(\sigma)$ with **d** and then replacing each remaining **a** by a **c**. Observe that this is always possible since an element in $\mathcal{N}_0^{\mathbf{r}}$ does not begin with a descent and cannot have any double descents.

We recursively define the reduced variation $V(\sigma)$ for an augmented André **r**-signed permutation σ on the set N by recursion. Assume that N has cardinality n. If $\sigma_m = (r_x, x) = (s, x)$, where $x = \max N$, then

$$V(\sigma) = \begin{cases} V(\sigma|_{[0,m-1]}) \cdot \bar{\mathbf{d}}_s \cdot V(\sigma|_{[m+1,n]}) & \text{if } m < n, \\ V(\sigma|_{[0,n-1]}) \cdot \bar{\mathbf{c}}_s & \text{if } m = n, \end{cases}$$

with $V(G) = 1$. This definition makes sense since $\sigma|_{[m+1,n]}$ belongs to the set $\mathcal{N}_0^{\mathbf{r}_I}$.

Proposition 6.5. *For $\mathbf{r} = (r_1, \ldots, r_n)$, the following equality holds:*

$$\sum_{\sigma \in \mathcal{N}_0^{\mathbf{r}}} V(\sigma) = \Pi(N) \cdot \boldsymbol{\Psi}(B_n).$$

We denote the sum by $V(\mathcal{N}_0^{\mathbf{r}})$.

Proof. This proof is similar to the proof of Lemma 5.2. First, one may easily show that $V(\mathcal{N}_0^{\mathbf{r}})$ does not depend on the linear order of the set of labels T'_n. That is, if we transpose two adjacent entries in a linear order, the sum of the reduced variation of non-augmented André **r**-signed permutations which belong to $\mathcal{N}_0^{\mathbf{r}}$ will remain the same. Thus we may consider the following linear order Γ on T'_n: $(i_1, j_1) \leqslant_{\Gamma} (i_2, j_2)$ if and only if $j_1 < j_2$ or, $j_1 = j_2$ and $i_1 \leqslant i_2$.

We find a recursion formula for $V(\mathcal{N}_0^{\mathbf{r}})$ by looking at where the largest element occurs in each non-augmented André **r**-signed permutation. Let the index set be $[n+1]$ and denote r_{n+1} by s.

Claim 6.6. *There exists a bijection between the two sets*

$$\mathcal{N}_0^{\mathbf{r},s} \qquad \textit{and} \qquad \mathcal{N}_0^{\mathbf{r}} \times [s] \cup \bigcup_{\substack{I+J=[n] \\ 1 \in J, I \neq \varnothing}} \mathcal{N}_0^{\mathbf{r}_J} \times [s] \times \mathcal{N}_0^{\mathbf{r}_I},$$

where all the unions are disjoint and $\times$ denotes the Cartesian product.

The proof of the claim is very similar to the proof of Proposition 6.7. Observe that $1 \in J$, since the element $\sigma_1 = \min_\Gamma(\sigma_1, \sigma_2, \ldots, \sigma_{n+1})$ is of the form $(i, 1)$.

Summing the reduced variation over $\mathcal{N}_0^{\mathbf{r},s}$ we find that

$$V(\mathcal{N}_0^{\mathbf{r},s}) = V(\mathcal{N}_0^{\mathbf{r}}) \cdot s \cdot \mathbf{c} + \sum_{\substack{I+J=[n] \\ 1 \in J, I \neq \varnothing}} V(\mathcal{N}_0^{\mathbf{r}_I}) \cdot s \cdot \mathbf{d} \cdot V(\mathcal{N}_0^{\mathbf{r}_J}).$$

By equation (1) one sees that the quantity $\Pi(N) \cdot \boldsymbol{\Psi}(B_n)$ satisfies the same recursion and initial conditions as $V(\mathcal{N}_0^{\mathbf{r}})$. □

Proposition 6.7. *There exists a bijection between the two sets*

$$\mathscr{A}^{\mathbf{r},s} \qquad \text{and} \qquad \mathscr{A}^{\mathbf{r}} \cup \bigcup_{\substack{I+J=[n] \\ I \neq \varnothing}} \mathscr{A}^{\mathbf{r}_J} \times \mathcal{N}_0^{\mathbf{r}_I},$$

where all the unions are disjoint and $\times$ *denotes the Cartesian product.*

Proof. Let the index set be $[n+1]$ and denote r_{n+1} by s. We break the augmented André $(\mathbf{r}, s)$-signed permutations at the point at which the largest element $(r_{n+1}, n+1)$ occurs. By doing so, we have the following map:

$$F: \mathscr{A}^{\mathbf{r},s} \to \mathscr{A}^{\mathbf{r}} \cup \bigcup_{\substack{I+J=[n] \\ I \neq \varnothing}} \mathscr{A}^{\mathbf{r}_J} \times \mathcal{N}_0^{\mathbf{r}_I}.$$

To see that F is bijective, it is enough to prove that F has an inverse. Given $\sigma' = (G, \sigma_1, \ldots, \sigma_{m-1}) \in \mathscr{A}^{\mathbf{r}_J}$ and $\sigma'' = (\sigma_{m+1}, \ldots, \sigma_{n+1}) \in \mathcal{N}_0^{\mathbf{r}_I}$, let $\sigma = (\sigma', (r_{n+1}, n+1), \sigma'')$. It is easy to see that σ satisfies condition (2) in Definition 6.1. To show that σ also satisfies condition (1), it is enough to consider the following two cases. First, when $i < m$ and $m < j$, let $k = m$ in condition (1). The remaining case is when $i - 1 = m$ and $m + 1 < j$. However, this case will not occur, since σ'' belongs to $\mathcal{N}_0^{\mathbf{r}_I}$. Hence σ is an augmented André permutation, and lies in the set $\mathscr{A}^{\mathbf{r},s}$. Thus we conclude that F is a bijection. □

Theorem 6.8. *For* $\mathbf{r} = (r_1, \ldots, r_n)$, *the following equality holds:*

$$\boldsymbol{\Psi}(\mathbf{r}) = \sum_{\sigma \in \mathscr{A}^{\mathbf{r}}} V(\sigma).$$

We denote this sum by $V(\mathscr{A}^{\mathbf{r}})$ *and call it the* non-commutative augmented André $\mathbf{r}$-signed polynomial.

Proof. It is enough to show that the non-commutative augmented André $\mathbf{r}$-signed polynomial satisfies the same recurrence as the one given for $\boldsymbol{\Psi}(\mathbf{r})$ in Proposition 5.3. The recursion formula will follow by the bijection given in Proposition 6.7. Summing the reduced variation over $\mathscr{A}^{\mathbf{r},s}$ we find that

$$V(\mathscr{A}^{\mathbf{r},s}) = V(\mathscr{A}^{\mathbf{r}}) \cdot \bar{\mathbf{c}}_s + \sum_{\substack{I+J=[n] \\ I \neq \varnothing}} V(\mathscr{A}^{\mathbf{r}_J}) \cdot \bar{\mathbf{d}}_s \cdot V(\mathcal{N}_0^{\mathbf{r}_I}). \qquad \square$$

We will end this section by enumerating augmented André r-signed permutations. That is, we set $\mathbf{r} = (r, \ldots, r)$, where $\mathbf{r}$ has length n. By Proposition 6.5 we know that $V(\mathcal{N}_0^{\mathbf{r}}) = r^n \cdot \boldsymbol{\Psi}(B_n)$. Hence, by the fact that the exponential generating function of the number of alternating permutations is given by $\sec(x) + \tan(x)$, we may derive that the exponential generating function for the number of non-augmented André **r**-signed permutations is $\sec(rx)+\tan(rx)$. By Theorem 6.8 we obtain a recursion for the number of augmented André **r**-signed permutations. Solving this recursion we obtain the following:

THEOREM 6.9. *The exponential generating function of the number of augmented André r-signed permutations is*

$$\sum_{n\geqslant 0} g_n \cdot \frac{x^n}{n!} = \sqrt[r]{\frac{1}{1-\sin(rx)}}.$$

By applying the Hardy–Littlewood–Karamata Tauberian Theorem (see [3]) to this generating function we have the following:

LEMMA 6.10. *When $n \to \infty$, the number of augmented André r-signed permutations has the asymptotics*

$$g_n \sim \frac{1}{\Gamma(2/r)} \cdot \left(\frac{8}{\pi^2}\right)^{1/r} \cdot n^{(2/r)-1} \cdot \left(\frac{2r}{\pi}\right)^n \cdot n!.$$

7. ARBITRARY RANK SELECTIONS

In this section we consider the problem of maximizing the beta invariant of the **r**-cubical lattice over arbitrary rank selections. We will do so by showing that $\boldsymbol{\Psi}(C^{\mathbf{r}_N})$ has the strictly increasing alternating property.

We will assume that $r_1, r_2, \ldots$ are all positive integers greater than or equal to 2. For N a finite subset of $\mathbb{P} = \{1, 2, \ldots\}$, $|N| = n$, we define the vector $\mathbf{r}_N$ by $(r_{m_1}, \ldots, r_{m_n})$, where $N = \{m_1, \ldots, m_n\}$. For an **ab**-word w of length n, we define $\beta(w, N)$ to be the coefficient of w in the **ab**-index $\boldsymbol{\Psi}(C^{\mathbf{r}_N})$. Thus $\beta(w, N)$ is a symmetric function in the variables $r_{m_1}, \ldots, r_{m_n}$. Also, we let $\beta_B(w)$ be the coefficient of w in the **ab**-index of the boolean algebra, $\boldsymbol{\Psi}(B_{|w|+1})$. Thus we have the two identities:

$$\boldsymbol{\Psi}(C^{\mathbf{r}_N}) = \sum_w \beta(w, N) \cdot w \qquad \text{and} \qquad \boldsymbol{\Psi}(B_{n+1}) = \sum_w \beta_B(w) \cdot w,$$

where w ranges over all **ab**-words of length n. Observe that since the coefficients $\beta(w, N)$ enumerate augmented **r**-signed permutations, we know that they are non-negative. We also have the following recursion for them:

LEMMA 7.1. *Let E be a linear map from symmetric functions in the variables $r_1, \ldots, r_n$ to symmetric functions in the variables $r_1, \ldots, r_{n+1}$ such that*

$$E(e_i(r_1, \ldots, r_n)) = e_i(r_1, \ldots, r_{n+1}).$$

Then $\beta(w, [n])$ may be computed by the following relations:

$$\begin{aligned}
\beta(1, \varnothing) &= 1, \\
\beta(\mathbf{a}w, [n+1]) &= E(\beta(w, [n])), \\
\beta(\mathbf{b}w, [n+1]) &= \beta_B(w) \cdot e_{n+1}(r_1, \ldots, r_{n+1}) - E(\beta(w, [n])),
\end{aligned}$$

where w has length n.

This lemma implies that $\beta(w, N)$ may be written as a linear combination of the elementary symmetric functions in the variables $r_{m_1}, \ldots, r_{m_n}$. Thus $\beta(w, N)$, viewed as a function of r_{m_i}, will be a polynomial of degree one.

An alternative way to compute $\beta(w, [n])$ is by a determinant. MacMahon gave a determinantal identity for the number of permutations in the symmetric group having a fixed descent set [13, Article 157]. An equivalent determinant was given by Niven [14]. By generalizing this determinant, we obtain a closed form formula for $\beta(w, [n])$. Here e_k is the kth elementary symmetric function $e_k(r_1, \ldots, r_n)$.

LEMMA 7.2. *Let n be a positive integer and let $k_1, \ldots, k_r$ be a sequence of integers such that $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n$. Let S be the set $\{n+1-k_r, \ldots, n+1-k_1\} \subseteq [n]$. Then*

$$\beta(u_S, [n]) = \det \begin{bmatrix} 1 & \binom{k_1}{k_1} & \cdots & \binom{k_1}{k_r} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{k_r}{k_1} & \cdots & \binom{k_r}{k_r} \\ 1 & e_{k_1} & \cdots & e_{k_r} \end{bmatrix}.$$

For an **ab**-word $w = w_1 w_2 \cdots w_n$ of length n, define, for $0 \leqslant i \leqslant n$, $w^{(i)} = w_1 \cdots w_i$ and $w^{((i))} = w_{n-i+1} \cdots w_n$. Also, we will need the following two notions:

$$S_{\mathbf{ab}}(w) = \{i \colon w = w^{(i)}\, \mathbf{ab}\, w^{((n-i-2))}\},$$
$$S_{\mathbf{ba}}(w) = \{i \colon w = w^{(i)}\, \mathbf{ba}\, w^{((n-i-2))}\},$$

where n is the length of w. Observe that $S_{\mathbf{ab}}(w), S_{\mathbf{ba}}(w) \subseteq \{0, 1, \ldots, n-2\}$.

Since the **cd**-index of the boolean algebra has positive coefficients (this may be verified by equation (1)), we deduce the following strict inequality:

$$\beta_B(vw^*) > \beta_B(vw) \qquad \text{when } v^{((1))} = w^{(1)}.$$

This fact will be useful to us later. Similarly, for the cubical lattice, that is when $\mathbf{r} = (2, \ldots, 2)$, we also know that the **cd**-index has positive coefficients. Thus the same strict inequality holds. Since the cubical lattice is Eulerian, we know that $\beta(w, [n])$ attains a maximum exactly when w is alternating, that is, when $w = \underbrace{\mathbf{baba}\cdots}_{n}$ or $w = \underbrace{\mathbf{abab}\cdots}_{n}$.

LEMMA 7.3. *Let $1 \leqslant k \leqslant n$. Let w be an* **ab**-*word of length $k-1$, and v be an* **ab**-*word of length $n-k$. Then*

$$\beta_B(w) \cdot \sum_{\substack{I+J=[n] \\ |J|=n-k}} \Pi(I) \cdot \beta(v, J) = \beta(v\, \mathbf{a}\, w, [n]) + \beta(v\, \mathbf{b}\, w, [n]).$$

PROOF. The right-hand side of the equality enumerates augmented **r**-signed permutations having descent set corresponding to the **ab**-words $v\,\mathbf{a}\,w$ or $v\,\mathbf{b}\,w$. Thus we are counting augmented **r**-signed permutations with either an ascent or a descent at position $n-k+1$. Hence we can enumerate such permutations by first choosing an augmented **r**-signed permutation on the entries $J \subseteq [n]$ with descent set corresponding to v, where J has cardinality $n-k$. Then we can independently choose a non-augmented permutation on the entries $[n]-J=I$ with descent set corresponding to w. By Lemma 5.2 this may be done in $\beta_B(w)\cdot\prod_{i\in I} r_i=\beta_B(w)\cdot\Pi(I)$ possible ways. $\square$

There is a similar statement for the boolean algebra. As in Lemma 7.3, let w be an **ab**-word of length $k-1$ and let v be an **ab**-word of length $n-k$. Then

$$\binom{n+1}{k}\cdot\beta_B(w)\cdot\beta_B(v)=\beta_B(v\,\mathbf{a}\,w)+\beta_B(v\,\mathbf{b}\,w).$$

This statement was already known to MacMahon in his study of Simon Newcomb's problem. He called it 'The Multiplication Theorem' [13, Article 159].

LEMMA 7.4. *Let w be an* **ab**-*word of length $k+1$ that begins with* **b**. *For any function f on* **ab**-*words of length k, the following identity holds*:

$$\begin{aligned} f(w^{((k))}) &= f(w^{(k)})\cdot\begin{Bmatrix} 1 & \text{if } w^{((1))}=\mathbf{b} \\ -1 & \text{if } w^{((1))}=\mathbf{a}\end{Bmatrix} \\ &\quad+\sum_{j\in S_{\mathbf{ba}}(w)} f(w^{(j)}\,\mathbf{a}\,w^{((k-j-1))})+f(w^{(j)}\,\mathbf{b}\,w^{((k-j-1))}) \\ &\quad-\sum_{j\in S_{\mathbf{ab}}(w)} f(w^{(j)}\,\mathbf{a}\,w^{((k-j-1))})+f(w^{(j)}\,\mathbf{b}\,w^{((k-j-1))}).\end{aligned}$$

THEOREM 7.5. *Let $\mathbf{r}=(r_1,\ldots,r_n)$, where $r_1,\ldots,r_n\geqslant 2$, and at least one entry is greater than or equal to* 3. *Then the* **ab**-*index of the* **r**-*cubical lattice $C^{\mathbf{r}}$ has the strictly increasing alternating property.*

COROLLARY 7.6. *Let $\mathbf{r}=(r_1,\ldots,r_n)$, where $r_1,\ldots,r_n\geqslant 2$, and at least one entry is greater than or equal to* 3. *For arbitrary rank selections $S\subseteq[n]$ of the* **r**-*cubical lattice $C^{\mathbf{r}}$, $\beta(S)$ attains a unique maximum when we take S to be $S=\{1,3,5,\ldots\}\cap[n]$.*

When $\mathbf{r}=(2,\ldots,2)$ the lattice $C^{\mathbf{r}}$ is the cubical lattice C_n. As was observed in Theorem 5.6, this lattice has two extremal configurations, namely $\{1,3,5,\ldots\}\cap[n]$ and $\{2,4,6,\ldots\}\cap[n]$. When at least one of the r_i's equals 1, Theorem 7.5 does not hold.

PROOF OF THEOREM 7.5. By symmetry in the r_i's, we may assume that $r_1\geqslant 3$. The proof is by induction on n. When $n=1$, all we need to check is that $\beta(\mathbf{b},\{1\})>\beta(\mathbf{a},\{1\})$, which is indeed true since $r_1-1>1$.

Let us now assume that the theorem holds for all values less than or equal to n, and that we would like to prove it for $n+1$. Say that $r_{n+1}=s$. We now consider the $(r_1,\ldots,r_n,s)$-cubical lattice, where $s\geqslant 2$. Let $\mathscr{K}$ denote the coefficient of the linear term in s in the expression $\boldsymbol{\Psi}(\mathbf{r},s)$. Then we may write

$$\boldsymbol{\Psi}(\mathbf{r},s)=\boldsymbol{\Psi}(\mathbf{r},2)+(s-2)\cdot\mathscr{K}.$$

The theorem will follow once we are able to show that $\boldsymbol{\Psi}(\mathbf{r},2)$ has the strictly increasing alternating property and $\mathscr{K}$ has the weakly increasing alternating property.

We begin by showing that $\boldsymbol{\Psi}(\mathbf{r}, 2)$ has the strictly increasing alternating property. Recall the recursion formula for $\boldsymbol{\Psi}(\mathbf{r}, s)$ in Proposition 5.3. Observe that $\bar{\mathbf{c}}_2 = \mathbf{a} + \mathbf{b} = \mathbf{c}$ and $\bar{\mathbf{d}}_2 = \mathbf{ab} + \mathbf{ba} = \mathbf{d}$. We have

$$\boldsymbol{\Psi}(\mathbf{r}, 2) = \boldsymbol{\Psi}(\mathbf{r}) \cdot \mathbf{c} + \sum_{\substack{I+J=[n] \\ I \neq \varnothing}} \Pi(I) \cdot \boldsymbol{\Psi}(\mathbf{r}_J) \cdot \mathbf{d} \cdot \boldsymbol{\Psi}(B_{|I|}). \tag{4}$$

By Lemma 2.3 we know that each term in equation (4) has the weakly increasing alternating property. Hence this sum has the weakly increasing alternating property. We claim it also has the strictly increasing property. Let v and w be words such that $|v| + |w| = n$. Assume that $v = 1$ and $w^{(1)} = \mathbf{b}$, or $v^{((1))} \neq w^{(1)}$. Consider the coefficients of vw and vw^* in the two terms

$$\mathbf{d} \cdot \boldsymbol{\Psi}(B_n) \qquad \text{and} \qquad (\mathbf{a} + (r_1 - 1)\mathbf{b}) \cdot \mathbf{d} \cdot \boldsymbol{\Psi}(B_{n-1}).$$

These terms correspond to $J = \varnothing$ and $J = \{r_1\}$ in $\boldsymbol{\Psi}(\mathbf{r}, 2)$. In the first term we know that strict inequality will hold except when $|v| = 0$ and $|v| = 2$. In the second term we know that strict inequality will hold except when $|v| = 1$ and $|v| = 3$. Since this covers all possibilities for the length of v, we know that $\boldsymbol{\Psi}(\mathbf{r}, 2)$ has the strictly increasing alternating property.

Recall that $\mathscr{K} = [s]\boldsymbol{\Psi}(\mathbf{r}, s)$, where $[s]$ denotes the coefficient of the linear term in the variable s. By the recursion formula in Proposition 5.3, we have

$$\mathscr{K} = \boldsymbol{\Psi}(\mathbf{r}) \cdot \mathbf{b} + \sum_{\substack{I+J=[n] \\ I \neq \varnothing}} \Pi(I) \cdot \boldsymbol{\Psi}(\mathbf{r}_J) \cdot \mathbf{ba} \cdot \boldsymbol{\Psi}(B_{|I|}).$$

We would like to prove that $\mathscr{K}$ has the weakly increasing alternating property. This follows from two claims.

CLAIM 7.7. *Let v and w be* **ab**-*words of lengths $n - k$ and $k + 1$ respectively. Assume that v ends with the letter* **a** *or v is empty, and that w begins with the letter* **b**. *Then the following inequality holds:*

$$[s]\beta(vw, [n+1]) \geqslant [s]\beta(vw^*, [n+1]).$$

CLAIM 7.8. *Let v and w be* **ab**-*words of lengths $n - k$ and $k + 1$ respectively. Assume that v ends with the letter* **b** *and that w begins with the letter* **b**. *Then the following inequality holds:*

$$[s]\beta(vw^*, [n+1]) \geqslant [s]\beta(vw, [n+1]).$$

PROOF OF CLAIM 7.7. We will write $y = vw$ and $z = vw^*$. It is easy to see that

$$S_{\mathbf{ba}}(y) = S_{\mathbf{ba}}(v) \cup (S_{\mathbf{ba}}(w) + |v|),$$

$$S_{\mathbf{ba}}(z) = S_{\mathbf{ba}}(v) \cup (S_{\mathbf{ab}}(w) + |v|),$$

where the '$+|v|$' indicates translation of the given set of integers by $|v|$ units. Hence we may derive the following expression for the linear term of s in $\beta(y, [n+1])$, where χ is the characteristic function.

$$\begin{aligned}
[s]\beta(y, [n+1]) &= \beta(y^{(n)}, [n]) \cdot \chi(y^{((1))} = \mathbf{b}) \\
&\quad + \sum_{\substack{I+J=[n] \\ |J| \in S_{\mathbf{ba}}(y)}} \Pi(I) \cdot \beta(y^{(|J|)}, J) \cdot \beta_B(y^{((|I|-1))}) \\
&= \beta(vw^{(k)}, [n]) \cdot \chi(w^{((1))} = \mathbf{b}) \\
&\quad + \sum_{\substack{I+J=[n] \\ |J| \in S_{\mathbf{ba}}(v)}} \Pi(I) \cdot \beta(v^{(|J|)}, J) \cdot \beta_B(v^{((|I|-k-2))}w) \\
&\quad + \sum_{\substack{I+J=[n] \\ |J| \in S_{\mathbf{ba}}(w)+n-k}} \Pi(I) \cdot \beta(vw^{(|J|-n+k)}, J) \cdot \beta_B(w^{((|I|-1))}).
\end{aligned}$$

Apply Lemma 7.3 to each term in the last sum.

$$\begin{aligned}
[s]\beta(y, [n+1]) &= \beta(vw^{(k)}, [n]) \cdot \chi(w^{((1))} = \mathbf{b}) \\
&\quad + \sum_{\substack{I+J=[n] \\ |J| \in S_{\mathbf{ba}}(v)}} \Pi(I) \cdot \beta(v^{(|J|)}, J) \cdot \beta_B(v^{((|I|-k-2))}w) \\
&\quad + \sum_{j \in S_{\mathbf{ba}}(w)} \beta(vw^{(j)}\, \mathbf{a}\, w^{((k-j-1))}, [n]) + \beta(vw^{(j)}\, \mathbf{b}\, w^{((k-j-1))}, [n]).
\end{aligned}$$

Similarly, the linear term of s in $\beta(z, [n+1])$ is given by

$$\begin{aligned}
[s]\beta(z, [n+1]) &= \beta(vw^{(k)*}, [n]) \cdot \chi(w^{((1))} = \mathbf{a}) \\
&\quad + \sum_{\substack{I+J=[n] \\ |J| \in S_{\mathbf{ba}}(v)}} \Pi(I) \cdot \beta(v^{(|J|)}, J) \cdot \beta_B(v^{((|I|-k-2))}w^*) \\
&\quad + \sum_{j \in S_{\mathbf{ab}}(w)} \beta(vw^{(j)*}\, \mathbf{a}\, w^{((k-j-1))*}, [n]) + \beta(vw^{(j)*}\, \mathbf{b}\, w^{((k-j-1))*}, [n]). \qquad (5)
\end{aligned}$$

We apply the induction hypothesis to the first term of equation (5). For its second term we use the fact that the **ab**-index of the boolean algebra has the increasing alternating property. Since $0 \notin S_{\mathbf{ab}}(w)$, we may also apply the induction hypothesis to the third term. Hence we have

$$\begin{aligned}
[s]\beta(z, [n+1]) &\leqslant \beta(vw^{(k)}, [n]) \cdot \chi(w^{((1))} = \mathbf{a}) \\
&\quad + \sum_{\substack{I+J=[n] \\ |J| \in S_{\mathbf{ba}}(v)}} \Pi(I) \cdot \beta(v^{(|J|)}, J) \cdot \beta_B(v^{((|I|-k-2))}w) \\
&\quad + \sum_{j \in S_{\mathbf{ab}}(w)} \beta(vw^{(j)}\, \mathbf{b}\, w^{((k-j-1))}, [n]) + \beta(vw^{(j)}\, \mathbf{a}\, w^{((k-j-1))}, [n]).
\end{aligned}$$

Thus the desired inequality $[s]\beta(y, [n+1]) \geqslant [s]\beta(z, [n+1])$ will follow if we can prove that

$$\begin{aligned}
0 &\leqslant \beta(vw^{(k)}, [n]) \cdot \chi(w^{((1))} = \mathbf{b}) \\
&\quad + \sum_{j \in S_{\mathbf{ba}}(w)} \beta(vw^{(j)}\, \mathbf{a}\, w^{((k-j-1))}, [n]) + \beta(vw^{(j)}\, \mathbf{b}\, w^{((k-j-1))}, [n]) \\
&\quad - \beta(vw^{(k)}, [n]) \cdot \chi(w^{((1))} = \mathbf{a}) \\
&\quad - \sum_{j \in S_{\mathbf{ab}}(w)} \beta(vw^{(j)}\, \mathbf{b}\, w^{((k-j-1))}, [n]) + \beta(vw^{(j)}\, \mathbf{a}\, w^{((k-j-1))}, [n]).
\end{aligned}$$

However, the right-hand side of this inequality is equal to $\beta(vw^{((k))}, [n])$ by applying

Lemma 7.4 with $f(u)=\beta(vu, [n])$. Since $\beta(vw^{((k))}, [n])$ is non-negative, the proof of this claim is complete. ■

The proof of the second claim is quite similar to the proof of Claim 7.7, and hence omitted. Thus the proof of the theorem is complete. □

As at the end of Section 6, it is interesting to determine the generating function and an asymptotic expression for the number of augmented r-signed permutations having **ab**-word **bab**$\cdots$. (Recall that this is the case when $\mathbf{r}=(r, \ldots, r)$.) Let $h_n = \beta(\mathbf{bab}\cdots, [n])$. This calculation has been done in [7], so we simply quote the two results. The exponential generating function and the asymptotics are given by

$$\sum_{n\geqslant 0} h_n \cdot \frac{x^n}{n!} = \frac{\sin((r-1)x)+\cos(x)}{\cos(rx)} \qquad \text{and} \qquad h_n \sim \frac{4}{\pi}\cdot\cos\left(\frac{\pi}{2r}\right)\cdot\left(\frac{2r}{\pi}\right)^n \cdot n!.$$

8. Concluding Remarks

There are many related problems to study. For instance, are there other posets which have an **r**-**cd**-index? More generally, are there other extensions of the **cd**-index? An example of a poset P having an **r**-**cd**-index is as follows. Let T_r be the poset on the set $\bigcup_{i=0}^{n} [r]^i$, where the i indicates Cartesian power. The cover relation in T_r is given by $(a_1, \ldots, a_{i-1}, a_i) \prec (a_1, \ldots, a_{i-1})$. The poset P defined by

$$P = \{(x, y): x \in B_n,\ y \in T_r,\ \rho(x) = \rho(y)\} \cup \{\hat{0}\},$$

has an **r**-**cd**-index. In fact, the poset P will have the same **ab**-index as C_n^r.

Pak and Postnikov [16] have found a multivariable generalization of André polynomials using a branching rule on k-ary (rather than binary) trees. Work of Pak [15] contains a different multivariable generalization which reduces to a recurrence for the André polynomial of the subspace lattice. His recurrence can be thought of as a q-analogue of the **cd**-index of the boolean algebra. It would be interesting to see if one could develop a **cd**-theory for q-analogues of posets. Billera and Liu [4] (also see [11]) have developed a general algebraic setting to view the **cd**-index. Is there a q-analogue of their theory?

What other classes of posets will have their **ab**-index satisfying the strictly increasing alternating property? A poset that seems to fulfill a similar condition is the partition lattice Π_n. Our data suggests that the **ab**-word with the largest coefficient in $\boldsymbol{\Psi}(\Pi_n)$ is the word $\underbrace{\mathbf{bab}\cdots}_{n-3}\mathbf{b}$. Is there a 'meta-theorem' which explains why alternating rank selections maximize the beta invariant?

Stanley proved that the **cd**-index of the face lattice of a convex polytope has non-negative coefficients [23, Corollary 2.2]. (For a more general statement, see [23, Theorem 2.2].) From this he observed that the beta invariant will reach its maximum value, for arbitrary rank selections, by taking alternating rank selections. However, uniqueness of this result (i.e., that the two alternating rank selections are the only extremal configurations) does not follow from his observation. By Lemma 2.2 it would be enough to show that the **cd**-index of the face lattice of a convex polytope has positive coefficients. Stanley has conjectured that among all Gorenstein* lattices of rank n, the boolean algebra B_n minimizes all the coefficients of the **cd**-index [24, Conjecture 2.7]. This conjecture implies that the **cd**-index of the face lattice of a convex polytope has positive coefficients.

The exponential generating function $\sqrt[r]{1/(1-rx)}$ enumerates r-multipermutations, see [17]. Notice that both this generating function and the one for the number of augmented André r-signed permutations are of the form $\sqrt[r]{f(rx)}$, where $f(x)$ is an exponential generating function. Is there a theory which explains generating functions of this form?

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