POLYTOPES

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1. Lecture I: Introduction to Polytopes and Face Enumeration

Grünbaum and Shephard [40] remarked that there were three developments which foreshadowed the modern theory of convex polytopes.

(1) The publication of Euclid's Elements and the five Platonic solids. In modern terms, these are the regular 3-polytopes.
(2) Euler's Theorem which states that that

$$v - e + f = 2$$

holds for any 3-dimensional polytope, where v, e and f denote the number of vertices, edges and facets, respectively. In modern language,

$$f_0 - f_1 + f_2 = 2,$$

where f_i, $i = 0, 1, 2$, is the number of i-dimensional faces.
(3) The discovery of polytopes in dimensions greater or equal to four by Schläfli.

We will use these as a springboard to describe the theory of convex polytopes in the 21st century.

1.1. **Examples.**

Recall a set S in $\mathbb{R}^n$ is *convex* if the line segment connecting any two points in S is completely contained in the set S. In mathematical terms, given any $x_1, x_2 \in S$, the set of all points $\lambda \cdot x_1 + (1 - \lambda)x_2 \in S$ for $0 \leq \lambda \leq 1$. A *convex polytope* or *polytope* in n-dimensional Euclidean space $\mathbb{R}^n$ is defined as the convex hull of k points $x_1, \ldots, x_k$ in $\mathbb{R}^n$, that is, the intersection of all convex sets containing these points. Throughout we will assume all of the polytopes we work with are convex.

One can also define a polytope as the bounded intersection of a finite number of half-spaces in $\mathbb{R}^n$. These two descriptions can be seen to be equivalent by Fourier-Motzkin elimination [73]. A polytope is *n-dimensional*, and thus

Date: Women and Mathematics Program, Institute for Advanced Study, May 2013.

said to be a n-polytope, if it is homeomorphic to a closed n-dimensional ball $\mathbb{B}^n = \{(x_1, \ldots, x_r) : x_1^2 + \cdots + x_n^2 \leq 1, x_{n+1} = \cdots = x_r = 0\}$ in $\mathbb{R}^r$. Given a polytope P in $\mathbb{R}^n$ with supporting hyperplane H, that is, $P \cap H \neq \emptyset$, $P \cap H_+ \neq \emptyset$ and $P \cup H_- = \emptyset$, where H_+ and H_- are the half open regions determined by the hyperplane H, then we say $P \cap H$ is a *face*. Observe that a face of a polytope is a polytope in its own right.

We now give some examples of polytopes. Note that there are many ways to describe each of these polytopes geometrically. The importance for us is that they are *combinatorially equivalent*, that is, they have the same face incidences structure though are not necessarily affinely equivalent. As an example, compare the square with a trapezoid.

Example 1.1.1. Polygons. The n-gon in $\mathbb{R}^2$ consists of n vertices, $n \geq 3$, so that no vertex is contained in the convex hull of the other $n-1$ vertices. Note the n-gon has n edges, so we encode its facial data by the f-vector $(f_0, f_1) = (n, n)$.

For the next example, we need the notion of affinely independence. A set of points $x_1, \ldots, x_n$ is *affinely independent* if

$$\sum_{1 \leq i \leq n} \lambda_i x_i = 0 \quad \text{with} \quad \sum_{1 \leq i \leq n} \lambda_i = 0 \text{ implies } \lambda_1 = \cdots = \lambda_n = 0.$$

Here $\lambda_1, \ldots, \lambda_n$ are scalars.

Example 1.1.2. The n-simplex Δ_n. *The n-dimensional simplex* or *n-simplex* is the convex hull of any $n+1$ affinely independent points in $\mathbb{R}^n$. Equivalently, it can be described as the convex hull of the $n+1$ points $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ where $\mathbf{e}_i$ is the ith unit vector $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. It is convenient to intersect this polytope with the hyperplane $x_1 + \cdots + x_{n+1} = 1$ so that the n-simplex lies in $\mathbb{R}^n$. Its f-vector has entries $f_i = \binom{n+1}{i+1}$, for $i = 0, \ldots, n-1$. The n-simplex is our second example of a *simplicial polytope*, that is, a polytope where all of its facets ($(n-1)$-dimensional faces) are combinatorially equivalent to the $(n-1)$-simplex. Our first, although trivial example, is the n-gon.

Example 1.1.3. The n-dimensional hypercube (n-cube). This is the convex hull of the 2^n points $C_n = \text{conv}\{(x_1, \ldots, x_n) : x_i \in \{0, 1\}\}$. In $\mathbb{R}^2$ this is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Observe that every vertex of the n-cube is *simple*, that is, every vertex is adjacent to exactly n edges. The f-vector has entries $f_i = \binom{n}{i} \cdot 2^{n-i}$ for $i = 0, \ldots, n$.

Example 1.1.4. The n-dimensional cross-polytope. This is the convex hull of the $2n$ points $\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \ldots, \pm\mathbf{e}_n\}$ in $\mathbb{R}^n$. In $\mathbb{R}^3$ this is the octahedron.

Consider the f-vector of the 3-cube and the octahedron. They are respectively $(8, 12, 6)$ and $(6, 12, 8)$. These two polytopes are said to be *dual* or *polar*. More formally, a polytope P is dual to a polytope P^* if there is an inclusion-reversing bijection between the faces of P and P^*.

Example 1.1.5. The permutahedron. This is the $(n-1)$-dimensional polytope defined by taking the convex hull of the $n!$ points $(\pi_1, \ldots, \pi_n)$ in $\mathbb{R}^n$, where $\pi = \pi_1 \cdots \pi_n$ is a permutation written in one-line notation from the symmetric group $\mathfrak{S}_n$ on n elements.

Example 1.1.6. The cyclic polytope. For fixed positive integers n and k the cyclic polytope $C_{n,k}$ is the convex hull of k distinct points on the moment curve $(t, t^2, \ldots, t^n)$.

Example 1.1.7. The Birkhoff polytope. The Birkhoff polytope is the set of all $n \times n$ doubly stochastic matrices, that is, all $n \times n$ matrices with non-negative entries and each row and column sum is 1. This is a polytope of dimension $(n-1)^2$. It is a nice application of Hall's Marriage Theorem that this polytope is the convex hull of the $n!$ permutation matrices.

1.2. **The face and flag vectors.**

The f-vector of a convex polytope is given by $(f_0, \ldots, f_{n-1})$, where f_i enumerates the number of i-dimensional faces in the n-dimensional polytope. It satisfies the *Euler–Poincaré relation*

$$f_0 - f_1 + f_2 - \cdots + (-1)^{n-1} \cdot f_{n-1} = 1 - (-1)^n. \tag{1.1}$$

equivalently,

$$\sum_{i=-1}^{n} (-1)^i f_i = 0, \tag{1.2}$$

where f_{-1} denotes the number of empty faces $(= 1)$ and $f_n = 1$ counts the entire polytope.

In 1906 Steinitz [67] completely characterized the f-vectors of 3-polytopes.

Theorem 1.2.1 (Steinitz)**.** *For a* 3*-dimensional polytope, the f-vector is uniquely determined by the values f_0 and f_2. The (f_0, f_2)-vector of every* 3*-dimensional polytope satisfies the following two inequalities:*

$$2(f_0 - 4) \geq f_2 - 4 \text{ and } f_0 - 4 \leq 2(f_2 - 4).$$

Furthermore, every lattice point in this cone has at least one 3*-dimensional polytope associated to it.*

The possible f-vectors lie in the lattice cone in the $f_0 f_2$-plane with apex at $(f_0, f_2) = (4, 4)$ and two rays emanating out of this point in the direction $(1, 2)$ and $(2, 1)$. The lattice points on these extremal rays are the simple and simplicial polytopes. See Exercise 1.5.4 for cubical 3-polytopes.

For polytopes of dimension greater than three the problem of characterizing their f-vectors is still open.

Open question 1.2.2. *Characterize f-vectors of d-polytopes where $d \geq 4$.*

S	f_S	h_S	u_s	$\mathbf{c}^3$	$10 \cdot \mathbf{dc}$	$6 \cdot \mathbf{cd}$
$\emptyset$	1	1	**aaa**	1	0	0
$\{0\}$	12	11	**baa**	1	10	0
$\{1\}$	18	17	**aba**	1	10	6
$\{2\}$	8	7	**aab**	1	0	6
$\{0, 1\}$	36	7	**bba**	1	0	6
$\{0, 2\}$	36	17	**bab**	1	10	6
$\{1, 2\}$	36	11	**abb**	1	10	0
$\{0, 1, 2\}$	72	1	**bbb**	1	0	0

TABLE 1. The flag f- and flag h-vectors, **ab**-index and **cd**-index of the hexagonal prism. The sum of the last three columns equals the flag h column, showing the **cd**-index of the hexagonal prism is $\mathbf{c}^3 + 10 \cdot \mathbf{dc} + 6 \cdot \mathbf{cd}$.

The f-vectors of simplicial polytopes have been completely characterized by work of McMullen [56], Billera and Lee [13] and Stanley [64]. See the lecture end-notes for further comments.

We now wish to keep track of not just the *number* of faces in a polytope, but also the face *incidences*. We encode this with the *flag f-vector* (f_S), where $S \subseteq \{0, \ldots, n-1\}$. More formally, for $S = \{s_1 < \cdots < s_k\} \subseteq \{0, \ldots, n-1\}$, define f_S to be the number of flags of faces

$$f_S = \#\{F_1 \subsetneq F_2 \cdots \subsetneq F_k\}$$

where $\dim(F_i) = s_i$. Observe that for an n-polytope the flag f-vector has 2^n entries. It also contains the f-vector data.

The *flag h-vector* $(h_S)_{S \subseteq \{0,\ldots,n-1\}}$ is defined by the invertible relation

$$h_S = \sum_{T \subseteq \{0,\ldots,n-1\}} (-1)^{|S-T|} f_T. \tag{1.3}$$

Equivalently, by the Möbius Inversion Theorem (MIT)

$$f_S = \sum_{T \subseteq \{0,\ldots,n-1\}} h_T. \tag{1.4}$$

See Table 1 for the computation of the flag f- and flag h-vectors of the hexagonal prism. Observe that the symmetry of the flag h-vector reduces the number of entries we have to keep track of from 2^3 to 2^2. This is true in general.

Theorem 1.2.3 (Stanley)**.** *For an n-polytope, and more generally, an Eulerian poset of rank n,*

$$h_S = h_{\overline{S}},$$

where $\overline{S}$ denotes the complement of S with respect to $\{0, 1, \ldots, n-1\}$.

Posets and Eulerian posets will be introduced in Lecture 2.

1.3. **The ab-index and cd-index.**

We would like to encode the flag h-vector data in a more efficient manner. The **ab**-*index* of an n-polytope P is defined by

$$\Psi(P) = \sum_{S} h_S \cdot u_S,$$

where the sum is taken over all subsets $S \subseteq \{0, \ldots, n-1\}$ and $u_S = u_0 u_1 \ldots u_{n-1}$ is the non-commutative monomial encoding the subset S by

$$u_i = \begin{cases} \mathbf{a} & \text{if } i \notin S, \\ \mathbf{b} & \text{if } i \in S. \end{cases}$$

Observe the resulting **ab**-index is a noncommutative polynomial of degree n in the noncommutative variables **a** and **b**.

We now introduce another change of basis. Let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ be two noncommutative variables of degree 1 and 2, respectively. The following result was conjectured by J. Fine and proven by Bayer–Klapper for polytopes, and Stanley for Eulerian posets [4, 65].

Theorem 1.3.1 (Bayer–Klapper, Stanley)**.** *For the face lattice of a polytope, and more generally, an Eulerian poset, the* **ab**-*index* $\Psi(P)$ *can be written uniquely in terms of the noncommutative variables* $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$, *that is,* $\Psi(P) \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.

The resulting noncommutative polynomial is called the **cd**-*index*.

Bayer and Billera proved that the **cd**-index removes *all* of the linear redundancies holding among the flag vector entries [3]. Hence the **cd**-monomials form a natural basis for the vector space of **ab**-indexes of polytopes. These linear relations, known as the *generalized Dehn-Sommerville relations*, are given by

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}} = (1 - (-1)^{k-i-1}) \cdot f_S, \tag{1.5}$$

where $i \leq k-2$, the elements i and k are elements of $S \cup \{-1, n\}$, and the subset S contains no integer between i and k. These are all the linear relations holding among the flag f-vector entries. Observe that Euler-Poincaré follows if we take $S = \emptyset$, $i = -1$ and $k = n$.

The **cd**-index did not generate very much excitement in the mathematical community until Stanley's proof of the nonnegativity of its coefficients, which we state here.

Theorem 1.3.2 (Stanley)**.** *The* **cd**-*index of the face lattice of a polytope, more generally, the augmented face poset of any spherically-shellable regular CW-sphere, has nonnegative coefficients*

Stanley's result opened the door to the following question.

Open question 1.3.3. *Give a combinatorial interpretation of the coefficients of the* **cd***-index.*

One interpretation of the coefficients of the **cd**-index is due to Karu, who, for each **cd**-monomial, gave a sequence of operators on sheaves of vector spaces to show the non-negativity of the coefficients of the **cd**-index for Gorenstein* posets [45]. See Exercise 2.9.4 for Purtill's combinatorial interpretation of the **cd**-index coefficients for the n-simplex and the n-cube.

1.4. **Notes.**

For general references on polytopes, we refer the reader to the second edition of Grünbaum's treatise [39], Coxeter's book on regular polytopes [21] and Ziegler's text [73].

See [73] for more information on the Fourier–Motzkin algorithm.

Euler's formula that $v - e + f = 2$ was mentioned in a 1750 letter Euler wrote to Goldbach, and proved by Descartes about 100 years earlier [22]. The "scissor" proof is due to von Staudt in 1847 [71]. Poincaré's proof of the more general Euler–Poincaré–Schläfli formula for polytopes required him to develop homology groups and algebraic topology. This discussion can be found in H.S.M. Coxeter's Regular Polytopes, Chapter IX, pages 165–166 [21]. Sommerville's 1929 proof of Euler–Poincaré–Schläfli was not correct as he assumed polytopes could be built facet by facet in an inductive manner which controls the homotopy type of the cell complex at each stage. This problem was rectified in 1971 when Bruggesser and Mani proved that polytopes are *shellable*. The concept of shellability has proven to be very powerful as it allows controlled inductive arguments for results on polytopes. See Lecture 3 for further discussion.

For any finite polyhedral complex C with Betti numbers given by the reduced integer homology $\beta_i = \text{rank}(\widetilde{H}_i(C, \mathbb{Z}))$, the Euler–Poincaré formula is

$$f_0 - f_1 + f_2 - \cdots + (-1)^{d-1} f_{d-1} = 1 = \beta_0 - \beta_1 + \beta_2 - \cdots + (-1)^{d-1} \beta_{d-1}. \quad (1.6)$$

This holds for more general cell complexes. See [15].

Shortly after McMullen and Shephard's book [57] was published, the Upper and Lower Bound Theorems were proved [1, 55].

Theorem 1.4.1 (Upper Bound and Lower Bound Theorems)**.** *(a)* [McMullen] *For fixed nonnegative integers n and k, the maximum number of j-dimensional faces in an n-dimensional polytope P with k vertices is given by the cyclic polytope $C(n, k)$, that is,*

$$f_j(P) \leq f_j(C(n, k)), \quad \textit{for } 0 \leq j \leq n.$$

(b) [Barnette] *For an n-dimensional simplicial polytope P with $n \geq 4$,*

$$f_j(\mathrm{Stack}(n,k)) \leq f_j(P), \quad \text{for } 0 \leq j \leq n,$$

where Stack(n,k) *is any n-dimensional polytope on k vertices formed by repeatedly adding a pyramid over the facet of a simplicial n-polytope, beginning with the n-simplex.*

The g-theorem which characterizes f-vectors of simplicial polytopes involved a geometric construction of Billera and Lee [13] for the sufficiency proof, and tools from algebraic geometry for Stanley's necessity proof [64]. In particular, this required the Hard Lefschetz Theorem.

For convenience and those who are interested, we include Björner's reformulation of the g-theorem as stated in [39, section 10.6]:

Theorem 1.4.2 (The g-theorem)**.** (Billera–Lee; Stanley)
The vector $(1, f_0, \ldots, f_{d-1})$ is the f-vector of a simplicial d-polytope if and only if it is a vector of the form $\mathbf{g} \cdot M_d$, where M_d is the $([d/2]+1) \times (d+1)$ matrix with nonnegative entries given by

$$M_d = \left(\binom{d+1-j}{d+1-k} - \binom{j}{d+1-k} \right)_{0 \leq j \leq d, 0 \leq k \leq d}, \tag{1.7}$$

*and $\mathbf{g} = (g_0, \ldots, g_{[d/2]})$ is an M-*sequence*, that is, a nonnegative integer vector with $g_0 = 1$ and $g_{k-1} \geq \partial^k(g_k)$ for $0 < k \leq d/2$. The upper boundary operator ∂^k is given by*

$$\partial^k = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_i - 1}{i-2} \tag{1.8}$$

where the unique binomial expansion of n is

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_i}{i}, \tag{1.9}$$

with $a_k > a_{k-1} > \cdots a_i \geq i > 0$. For a given polytope P the vector $g = g(P)$ is determined by the f-vector, respectively h-vector, as $g_k = h_k - h_{k-1}$ for $0 < k \leq d/2$ with $g_0 = 1$.

The motivating question in Purtill's dissertation was to prove nonnegativity of the coefficients of the **cd**-index for convex polytopes. He did prove nonnegativity in the case of the n-cube and the n-simplex by giving a combinatorial interpretation of the coefficients using André and signed André permutations [60]. Stanley [65] proved nonnegativity for spherically-shellable posets, of which face lattice of polytopes are examples.

1.5. **Exercises.**

Exercise 1.5.1. Build Platonic solids and other polytopes from nets.

Exercise 1.5.2. Prove Steinitz' theorem.
Hint: Every face is at least a triangle, so what are the inequalities on the vectors?
Hint': Every vertex is at least incident to 3 edges, again inequalities?
This will be the foundation for the Kalai product in Lecture II
Hint": What happens to f_0 and f_2 after cutting off a simple vertex?

Exercise 1.5.3. Show that for 3-dimensional polytopes, the entries of the flag f-vector are determined by the values f_0 and f_2.

Exercise 1.5.4. We call a polytope *cubical* if all of its faces are combinatorial cubes. For example, the facets of 3-dimensional cubical polytopes are squares.
a. Show that a 3-dimensional cubical polytope satisfies $f_0 - f_2 = 2$ and $f_0 \geq 8$.
b. Show there is no 3-dimensional cubical polytope with $(f_0, f_2) = (9, 7)$.
c. Show that any other lattice point on the line $f_0 - f_2 = 2$ for $f_0 \geq 8$ comes from a cubical polytope.

Exercise 1.5.5. a. Starting with the cube, cut off a vertex and compute the **cd**-index.
b. Repeat part a. with the 4-cube.

Exercise 1.5.6. a. What is the **cd**-index of the n-gon?
b. What is the **cd**-index of the prism of the n-gon?
c. What is the **cd**-index of the pyramid of the n-gon?

2. Lecture II: Coalgebraic techniques and geometric operations

It will be useful to view the face structure of a polytope in terms of its *face lattice*, that is, the partially ordered set consisting of the faces of a polytope ordered by inclusion. We will see that geometric operations on polytopes correspond to poset operations on the face lattice, and hence, to operations on any "reasonable" poset. The resulting **ab**-index of the prism and pyramid operations strongly suggest an underlying coalgebraic structure, which we also introduce.

2.1. Posets, polytopes and geometric operations on them.

Recall a *partially ordered set* P, or *poset* for short, consists of a finite number of elements with a partial order $\leq$ which is

(1) reflexive: $x \leq x$ for all elements $x \in P$,
(2) antisymmetric: if $x \leq y$ and $y \leq x$ then $x = y$,
(3) transitive: $x \leq y$ and $y \leq z$ implies $x \leq z$.

Most of the posets we will work with will have a unique minimal and maximal elements, denoted by $\hat{0}$ and $\hat{1}$ respectively. Additionally, we say a poset P with unique minimal and maximal elements is *graded* if any saturated chain of elements from $\hat{0}$ to x, that is, $c = \{\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k = x\}$ has the same length for a fixed element $x \in P$. We call this length the *rank* of x, denoted $\rho(x)$ and the rank of a graded poset is $\rho(\hat{1})$. A poset is a *lattice* if every pair of elements has a unique least upper bound and unique greatest lower bound. For more information about posets, we refer the reader to [66, Chapter 3].

There are a number of operations of posets we will need. Given posets P and Q, the Cartesian product is

$$P \times Q = \{(p, q) : p \in P \text{ and } q \in Q\}$$

with the partial order $(p, q) \leq_{P\times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$.

Example 2.1.1. The Boolean algebra. The *Boolean algebra* B_n consisting of all subsets of $\{1, \ldots, n\}$ ordered by inclusion can be realized as the product $B_n \cong \underbrace{B_1 \times \cdots \times B_1}_{n}$. As a remark, the Boolean algebra B_{n+1} is isomorphic to the face lattice of the n-simplex.

Assuming P and Q are bounded posets, that is, each has a unique minimal and maximal element, the *diamond product* is

$$P \diamond Q = (P - \{\hat{0}_P\}) \times (Q - \{\hat{0}_Q\}) \cup \{\hat{0}\}$$

and the *Stanley product* $P * Q$ consists of the elements

$$P * Q = (P - \{\hat{1}_P\}) \cup (Q - \{\hat{0}_Q\})$$

with the order relation $x \leq_{P*Q} y$ if (i) $x, y \in P$ and $x \leq_P y$, (ii) $x, y \in Q$ and $x \leq_Q y$, or (iii) $x \in P$ and $y \in Q$. Finally, the *dual* of a poset P is the poset P^* where the order relation is $x \leq_{P^*} y$ if and only if $y \leq_P x$.

In important subclass of graded posets are the *Eulerian posets*. These satisfy the condition that $\mu(x, y) = (-1)^{\rho(x,y)}$, where $\rho(x, y) = \rho(y) - \rho(x)$ and the *Möbius function* is defined by $\mu(x, x) = 1$ and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$. Equivalently, in every non-trivial interval the number of elements of even rank equals the number of elements of odd rank. One important family of Eulerian posets are the face lattices of convex polytopes.

We can now discuss some geometric operations on polytopes. For P an n-polytope in $\mathbb{R}^n$, embed P in $\mathbb{R}^{n+1}$. (For instance, let the $(n+1)$st coordinate of each point in P be zero.) The *pyramid* of P is

$$\mathrm{Pyr}(P) = \mathrm{conv}(P \cup \{x\}),$$

where x is a point outside the affine hull of P. For example, $\Delta_n = \mathrm{Pyr}(\Delta_{n-1}) = \mathrm{Pyr}^n(\text{point})$. Note the dimension of $\mathrm{Pyr}(P)$ is one more than the dimension of P. The *bipyramid* of P is

$$\mathrm{Bipyr}(P) = \mathrm{conv}(P \cup \{x_+\} \cup \{x_-\}),$$

where x_+ and x_- are two points outside of the affine hull of P such that some point in the open interval (x_+, x_-) intersects the interior of P. Note that the n-dimensional cross-polytope is the repeated application of the bipyramid operation, starting with a point.

For $P \subseteq \mathbb{R}^p$ and $Q \subseteq \mathbb{R}^q$ two polytopes of dimension p and q, respectively, their *Cartesian product of polytopes* is

$$P \times Q = \{(x_1, \ldots, x_p, y_1, \ldots, y_q) = (\vec{x}, \vec{y}) : \vec{x} \in P, \vec{y} \in Q\}$$

Observe $\dim(P \times Q) = \dim(P) + \dim(Q)$. The *prism* of P is

$$\mathrm{Prism}(P) = P \times [0, 1],$$

where $[0, 1]$ is the line segment from 0 to 1. Thus the n-cube is $C_n = \mathrm{Prism}(C_{n-1})$.

The *free join* $P * Q$ of two polytopes P and Q is formed in the following manner. Embed P in the p-dimensional affine subspace of $\mathbb{R}^{p+q+1}$ as $\{(x_1, \ldots, x_p, 0, \ldots, 0) : \vec{x} \in P\}$. Embed Q in a q-dimensional affine subspace as $\{(0, \ldots, 0, y_1, \ldots, y_q, 1) : \vec{y} \in Q\}$. Then take the convex hull of these two embeddings. The resulting polytope has dimension $p+q+1$. Geometrically the free join corresponds to putting the two polytopes P and Q in orthogonal non-intersecting affine subspaces of $\mathbb{R}^{p+q+1}$ and then taking the convex hull.

It is natural to ask how geometric operations on a polytope, such as the prism and pyramid, change the **cd**-index of the original polytope. We first consider the change on the face lattice itself [44].

Proposition 2.1.2 (Kalai)**.** *For two convex polytopes P and Q we have*

$$\begin{aligned}\mathscr{L}(P * Q) &= \mathscr{L}(P) \times \mathscr{L}(Q),\\ \mathscr{L}(P \times Q) &= \mathscr{L}(P) \diamond \mathscr{L}(Q).\end{aligned}$$

Especially,

$$\mathscr{L}(\mathrm{Pyr}(P)) = \mathscr{L}(P) \times B_1 \text{ and } \mathscr{L}(\mathrm{Prism}(P)) = \mathscr{L}(P) \diamond B_2.$$

Definition 2.1.3. *For a graded poset P, define the pyramid and prism operations by* $\mathrm{Pyr}(P) = P \times B_1$ *and* $\mathrm{Prism}(P) = P \diamond B_2$.

An equivalent definition of the **ab**-index is as follows. Let P be a graded poset of rank $n+1$. Given a chain $c = \{\hat{0} < x_1 < \cdots < x_k < \hat{1}\}$ of P, we associate a weight $w(c) = z_1 \cdots z_n$, where

$$z_i = \begin{cases} \mathbf{a}-\mathbf{b} & \text{if } i \notin \{\rho(x_1), \ldots, \rho(x_k)\},\\ \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}.\end{cases}$$

Then the **ab**-*index* of a poset P is given by

$$\Psi(P) = \sum_c w(c),$$

where the sum is over all chains c in P. Observe that this is just a way to directly compute the flag h-vector from the flag f-vector, rather than having to compute the flag h-vector via certain alternating sums of the flag f-vector entries. For example, for the face lattice of a hexagon, we have $\Psi(\text{hexagon}) = 1(\mathbf{a}-\mathbf{b})^2+6\mathbf{b}\cdot(\mathbf{a}-\mathbf{b})+6(\mathbf{a}-\mathbf{b})\cdot\mathbf{b}+12\mathbf{bb} = \mathbf{a}+5\mathbf{ab}+5\mathbf{ba}+\mathbf{bb}$.

Proposition 2.1.4. [31] *Let P be a graded poset. Then*

$$\begin{aligned}\Psi(\mathrm{Pyr}(P)) &= \frac{1}{2}\left[\Psi(P)\cdot\mathbf{c} + \mathbf{c}\cdot\Psi(P) + \sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}} \Psi([\hat{0},x])\cdot\mathbf{d}\cdot\Psi([x,\hat{1}])\right],\\ \Psi(\mathrm{Prism}(P)) &= \Psi(P)\cdot\mathbf{c} + \sum_{\substack{x\in P\\ \hat{0}<x<\hat{1}}} \Psi([\hat{0},x])\cdot\mathbf{d}\cdot\Psi([x,\hat{1}]).\end{aligned}$$

Proof. The first identity follows by a careful chain argument. Consider a chain c in $P \times B_1$. We have

$$c = \{(\hat{0},\hat{0}) = (x_0,y_0) < (x_1,y_1) < \cdots < (x_k,y_k) = (\hat{1},\hat{1})\}.$$

Let i be the smallest index such that $y_i = \hat{1}$. Let $x = x_i$. This implies $y_0 = \cdots = y_{i-1} = \hat{0}$, $y_i = \cdots = y_k = \hat{1}$ and $x_{i-1} \leq x_i$. We also have the two chains $c_1 = \{\hat{0} = x_0 < x_1 < \cdots < x_{i-1} \leq x\}$ in $[\hat{0},x]$ and $c_2 = \{x < x_{i+1} < \cdots < x_k = \hat{1}\}$ in $[x,\hat{1}]$.

Three cases occur:

(1) $\hat{0} < x < \hat{1}$. Then the element $(x, \hat{0})$ may or may not be in the chain c. Let c' denote the chain $c - \{(x, \hat{0})\}$, that is, the chain without the element $(x, \hat{0})$. Similarly, let c'' denote the chain $c \cup \{(x, \hat{0})\}$, that is, the chain with the element $(x, \hat{0})$. Observe that the element $(x, \hat{1})$ belongs to both the chains c' and c'', so the weight of these chains at rank $\rho(x) + 1$ is $\mathbf{b}$. Hence we have

$$\begin{aligned} w(c') &= w_{[\hat{0},x]}(c_1) \cdot (\mathbf{a} - \mathbf{b}) \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c'') &= w_{[\hat{0},x]}(c_1) \cdot \mathbf{b} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2), \\ w(c') + w(c'') &= w_{[\hat{0},x]}(c_1) \cdot \mathbf{a} \cdot \mathbf{b} \cdot w_{[x,\hat{1}]}(c_2). \end{aligned}$$

(2) $x = \hat{1}$. Then the element $(\hat{1}, \hat{0})$ may or may not be in the chain c. Let c' be the chain $c - \{(\hat{1}, \hat{0})\}$ and let c'' be the chain $c \cup \{(\hat{1}, \hat{0})\}$. Then

$$\begin{aligned} w(c') &= w_P(c_1) \cdot (\mathbf{a} - \mathbf{b}), \\ w(c'') &= w_P(c_1) \cdot \mathbf{b}, \\ w(c') + w(c'') &= w_P(c_1) \cdot \mathbf{a}. \end{aligned}$$

(3) $x = \hat{0}$. Then the element $(\hat{0}, \hat{1})$ lies in the chain c, and the weight of the chain c is

$$w(c) = \mathbf{b} \cdot w_P(c_2).$$

Summing over all chains c in $P \times B_1$, we obtain

$$\Psi(P \times B_1) = \mathbf{b} \cdot \Psi(P) + \Psi(P) \cdot \mathbf{a} + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \cdot \mathbf{a} \cdot \mathbf{b} \cdot \Psi([x, \hat{1}]). \quad (2.1)$$

Applying equation (2.1) to the dual poset P^* and applying the involution * to obtain

$$\Psi(P \times B_1) = \Psi(P) \cdot \mathbf{b} + \mathbf{a} \cdot \Psi(P) + \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \cdot \mathbf{b} \cdot \mathbf{a} \cdot \Psi([x, \hat{1}]). \quad (2.2)$$

Adding equations (2.1) and (2.2) gives the desired result.

The proof of the second identity, which we omit, is similar. □

2.2. **The Newtonian coalgebra of ab-polynomials.**

Proposition 2.1.4 is very suggestive that a coalgebraic structure is occurring here. We introduce these ideas in this section.

Let $\mathcal{A} = \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ be the polynomial algebra in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$ with the usual multiplication. Define the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by

$$\Delta(v_1 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n,$$

and extend by linearity. For example $\Delta(\mathbf{abba}) = 1 \otimes \mathbf{bba} + \mathbf{a} \otimes \mathbf{ba} + \mathbf{ab} \otimes \mathbf{a} + \mathbf{abb} \otimes 1$.

Formally, we write the coproduct of an element x as

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}.$$

This should be thought of as the sum over all ways of breaking the element x into the pairs $x_{(1)}$ and $x_{(2)}$. The terms $x_{(1)}$ and $x_{(2)}$ are referred to as "x Sweedler 1" and "x Sweedler 2".

A *Newtonian coalgebra* is a coalgebra with respect to the coproduct Δ and an algebra with respect to the product μ where the *Newtonian condition* holds:

$$\Delta(u \cdot v) = \Delta(u) \cdot v + u \cdot \Delta(v).$$

Equivalently, using Sweedler notation, we have

$$\Delta(x \cdot y) = \sum_x x_{(1)} \otimes x_{(2)} y + \sum_y x y_{(1)} \otimes y_{(2)}.$$

It is straightforward to check

Lemma 2.2.1. *$\mathcal{A} = \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ is a Newtonian coalgebra with a unit, but no counit.*

The Newtonian coalgebra of **ab**-polynomials has a natural grading with $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$, where $\mathcal{A}_n$ is spanned by the **ab**-monomials of degree n. We also have $\dim(\mathcal{A}_n) = 2^n$, and

$$\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j} \text{ and } \Delta(\mathcal{A}_n) \subseteq \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j.$$

Lemma 2.2.2. *Consider the coproduct $\Delta : \mathcal{A}_n \longrightarrow \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$ as a linear map. Then the kernel of Δ is one-dimensional and is spanned by the element $(\mathbf{a} - \mathbf{b})^n$.*

Note the linear map $\Delta : \mathcal{A}_n \longrightarrow \bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j$ is not surjective for $n \geq 2$, because $\dim(\Delta(\mathcal{A}_n)) = 2^n - 1$ and $\dim\left(\bigoplus_{i+j=n-1} \mathcal{A}_i \otimes \mathcal{A}_j\right) = n \cdot 2^{n-1} > 2^n - 1$.

2.3. The ab-index as a coalgebra homeomorphism.

We state the following fundamental result concerning the **ab**-index [31].

Theorem 2.3.1 (Ehrenborg–Readdy)**.** *The **ab**-index is a Newtonian coalgebra homeomorphism from the linear space $\mathcal{P}$ of all graded posets to the polynomial algebra $\mathcal{A} = \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$, that is, $\Psi : \mathcal{P} \to \mathcal{A}$ with $\Psi(B_1) = 1$,*

$$\Psi(P * Q) = \Psi(P) \cdot \Psi(Q),$$

and

$$\Delta(\Psi(P)) = \sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} \Psi([\hat{0}, x]) \otimes \Psi([x, \hat{1}]) \tag{2.3}$$

Equivalently,

$$\Psi \circ \mu = \mu \circ (\Psi \otimes \Psi) \text{ and } \Delta \circ \Psi = (\Psi \otimes \Psi) \circ \Delta.$$

Let us read what this theorem says on the **ab**-level. If one obtains an expression of the form

$$\sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} B(\Psi([\hat{0}, x]), \Psi([x, \hat{1}])), \tag{2.4}$$

where $B : \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle \times \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbf{k}\langle \mathbf{a}, \mathbf{b} \rangle$ is a bilinear form, then we can evaluate (2.4) in terms of the **ab**-index of the entire poset P. That is, we have

$$\sum_{\substack{x \in P \\ \hat{0} < x < \hat{1}}} B(\Psi([\hat{0}, x]), \Psi([x, \hat{1}])) = \sum_{w} B(w_{(1)}, w_{(2)}),$$

where $w = \Psi(P)$. Note this circumnavigates having to compute the **ab**-index of every subinterval in the original poset P.

Lemma 2.3.2. *The subalgebra $\mathcal{F} = \mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$ of $\mathcal{A}$ is closed under the coproduct Δ.*

Proof. This follows from $\Delta(\mathbf{c}) = \Delta(\mathbf{a} + \mathbf{b}) = 1 \otimes 1 + 1 \otimes 1 = 2 \cdot 1 \otimes 1$ and $\Delta(\mathbf{d}) = \Delta(\mathbf{ab} + \mathbf{ba}) = \mathbf{a} \otimes 1 + 1 \otimes \mathbf{b} + \mathbf{b} \otimes 1 + 1 \otimes \mathbf{a} = \mathbf{c} \otimes 1 + 1 \otimes \mathbf{c}$. □

This Newtonian coalgebra inherits the grading from $\mathcal{A}$ in the following manner: $\mathcal{F} = \oplus_{n \geq 0} \mathcal{F}_n$ with $\dim(\mathcal{F}_0) = \dim(\mathcal{F}_1) = 1$ and $\mathcal{F}_n = \mathbf{c} \cdot \mathcal{F}_{n-1} + \mathbf{d} \cdot \mathcal{F}_{n-2}$, implying $\dim(\mathcal{F}_n) = F_n$, the nth Fibonacci number ($F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

Since the subalgebra $\mathcal{F} = \mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$ is closed under the coproduct, we have the immediate corollary to Theorem 2.3.1.

Corollary 2.3.3. *The* **cd**-*index is a coalgebra homeomorphism from the linear space $\mathcal{E}$ of all graded Eulerian posets to the polynomial algebra $\mathcal{A} = \mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$.*

Define an involution on $\mathcal{A}$, denoted *, by reading the **ab**-monomials in reverse. That is, $(v_1 \cdot v_2 \cdots v_n)^* = v_n \cdots v_2 \cdot v_1$ and extend it linearly to all of $\mathcal{A}$. This is also an involution on the **cd**-monomials $\mathcal{F}$, since $\mathbf{c}^* = (\mathbf{a} + \mathbf{b})^* = \mathbf{a} + \mathbf{b} = \mathbf{c}$ and $\mathbf{d}^* = (\mathbf{ab} + \mathbf{ba})^* = \mathbf{ba} + \mathbf{ab} = \mathbf{d}$. Also observe that taking the dual of a poset extends to an involution on the linear space $\mathcal{P}$.

2.4. **Derivations.**

Define a linear operator $D : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D(w) = \sum_w w_{(1)} \cdot \mathbf{d} \cdot w_{(2)}.$$

Recall that the Newtonian condition implies that D is a derivation on $\mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$. We could have defined D directly as a derivation on $\mathcal{A}$ such that $D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab}+\mathbf{ba} = \mathbf{d}$. Note that D is also a derivation on $\mathcal{F}$ since $D(\mathbf{c}) = 2 \cdot \mathbf{d}$ and $D(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$.

Combining Proposition 2.1.4 with the fact that Ψ is a Newtonian coalgebra map, we obtain:

Theorem 2.4.1 (Ehrenborg–Readdy)**.** *Let P be a graded poset. Then*

$$\begin{aligned} \Psi(\mathrm{Pyr}(P)) &= \frac{1}{2}\left[\Psi(P) \cdot \mathbf{c} + \mathbf{c} \cdot \Psi(P) + D(\Psi(P))\right], \\ \Psi(\mathrm{Prism}(P)) &= \Psi(P) \cdot \mathbf{c} + D(\Psi(P)). \end{aligned}$$

In Theorem 2.5.2 we will improve the formula for the pyramid.

As a corollary, Theorem 2.4.1 gives a new recursion formula for the **cd**-index of the n-dimensional cube C_n.

Corollary 2.4.2. *The* **cd***-index of the n-dimensional cube C_n satisfies the recursion*

$$\Psi(C_{n+1}) = \Psi(C_n) \cdot \mathbf{c} + D(\Psi(C_n)),$$

for $n \geq 0$ with $\Psi(C_0) = 1$.

This differs from Purtill's recursion obtained in [60, Corollaries 5.8 and 5.12].

2.5. **The derivation G.**

Define on the algebra $\mathcal{A}$ two derivations G and G' by letting

$$\begin{aligned} G(\mathbf{a}) &= \mathbf{ba}, & G'(\mathbf{a}) &= \mathbf{ab}, \\ G(\mathbf{b}) &= \mathbf{ab}, & G'(\mathbf{b}) &= \mathbf{ba}, \end{aligned}$$

and extending G and G' to all **ab**-polynomials by linearity and the product rule of derivations. Since $D(\mathbf{a}) = G(\mathbf{a}) + G'(\mathbf{a})$ and $D(\mathbf{b}) = G(\mathbf{b}) + G'(\mathbf{b})$, we obtain that $D(w) = G(w) + G'(w)$ for all **ab**-polynomials w, that is, $D = G + G'$.

Observe that $G(\mathbf{c}) = G(\mathbf{a} + \mathbf{b}) = \mathbf{ba} + \mathbf{ab} = \mathbf{d}$ and $G(\mathbf{d}) = G(\mathbf{a}) \cdot \mathbf{b} + \mathbf{a} \cdot G(\mathbf{b}) + G(\mathbf{b}) \cdot \mathbf{a} + \mathbf{b} \cdot G(\mathbf{a}) = \mathbf{bab} + \mathbf{aab} + \mathbf{aba} + \mathbf{bba} = \mathbf{cd}$. A similar computation gives $G'(\mathbf{c}) = \mathbf{d}$ and $G'(\mathbf{d}) = \mathbf{dc}$. Hence G and G' restrict to be derivations on $\mathcal{F}$.

Lemma 2.5.1. *For all* **ab***-monomials* w*, the identity*

$$w \cdot \mathbf{c} + G(w) = \mathbf{c} \cdot w + G'(w)$$

holds.

We leave the proof as an exercise. See Exercise 2.9.6.

Theorem 2.5.2 (Ehrenborg–Readdy)**.** *Let* P *be a graded poset. Then*

$$\begin{aligned} \Psi(\mathrm{Pyr}(P)) &= \Psi(P) \cdot \mathbf{c} + G(\Psi(P)) \\ &= \mathbf{c} \cdot \Psi(P) + G'(\Psi(P)). \end{aligned}$$

Proof. By Theorem 2.4.1 and the fact the **ab**-index is a coalgebra homeomorphism we have

$$\begin{aligned} 2 \cdot \Psi(P \times B_1) &= \Psi(P) \cdot \mathbf{c} + \mathbf{c} \cdot \Psi(P) + D(\Psi(P)) \\ &= (\Psi(P) \cdot \mathbf{c} + G(\Psi(P))) + \left(\mathbf{c} \cdot \Psi(P) + G'(\Psi(P))\right). \end{aligned}$$

But by Lemma 2.5.1 the two terms are equal. Thus we have $\Psi(P \times B_1) = \Psi(P) \cdot \mathbf{c} + G(\Psi(P)) = \mathbf{c} \cdot \Psi(P) + G'(\Psi(P))$. □

2.6. **Polytopes span.**

The **cd**-monomials give a basis for the vector space of **cd**-indexes of polytopes. Conversely, can we find a spanning set of polytopes whose **cd**-indexes give all **cd**-words? The answer, due to Bayer and Billera, is yes [3].

Theorem 2.6.1 (Bayer–Billera)**.** *Let* $\mathscr{B}$ *be the set of* n*-tuples* $(R_1, \ldots, R_n)$ *such that each* R_i *is either the pyramid operation* Pyr*, or the prism operation* Prism *satisfying*

(1) R_i *and* R_{i+1} *are not both the prism operation, and*
(2) R_1 *is the pyramid operation.*

Then the set

$$\{\Psi(R_n(\cdots R_1(\mathit{point}) \cdots)) \;:\; (R_1, \ldots, R_n) \in \mathscr{B}\}$$

is a basis for the **cd***-polynomials of degree* n*.*

Bayer and Billera's original notation was in terms of the pyramid and bipyramid operations. The proof we give here is due to Billera, Ehrenborg and Readdy [11].

Proof. Let $\mathcal{F}_i$ denote the vector space of **cd**-polynomials of degree i. We have $\mathcal{F}_0 = \langle 1 \rangle$ and thus $\mathrm{Pyr}(1) = 1 \cdot \mathbf{c} + G(1) = \mathbf{c}$, which generates $\mathcal{F}_1$, that is,

$\mathcal{F}_1 = \langle \mathbf{c} \rangle$. By the induction hypothesis, we have a spanning set of polytopes for $\mathcal{F}_n$. We will use the pyramid and prism operations to build $\mathcal{F}_{n+1}$. Since $\mathrm{Prism} - \mathrm{Pyr} = G'$, the derivation with $G'(\mathbf{c}) = \mathbf{d}$ and $G'(\mathbf{d}) = \mathbf{dc}$, it is enough to show

$$G'(\mathcal{F}_n) + \mathrm{Pyr}(\mathcal{F}_n) = \mathcal{F}_{n+1}.$$

Recall by Lemma 2.5.1 we have for any **cd**-word w the relation $\mathbf{c} \cdot w + G'(w) = w \cdot \mathbf{c} + G(w)$. Thus

$$\begin{aligned} \mathbf{c} \cdot w &= w \cdot \mathbf{c} + G(w) - G'(w) \\ &= \mathrm{Pyr}(w) - G'(w), \end{aligned}$$

implying words of the form $\mathbf{c} \cdot w$ are in the span for $w \in \mathcal{F}_n$.

Let $v \in \mathcal{F}_{n-1}$. Then $\mathbf{d} \cdot v = G'(\mathbf{c} \cdot v) - c \cdot G'(v)$. since $G'(\mathbf{c} \cdot v) \in G'(\mathcal{F}_n)$ and $\mathbf{c} \cdot G'(v)$ is in the span, we conclude that $\mathbf{d} \cdot v$ is also in the span. □

The *Minkowski sum* of two subsets X and Y of $\mathbb{R}^n$ is defined as

$$X + Y = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^n : \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

The Minkowski sum of two convex polytopes is another convex polytope. For a vector $\mathbf{x}$ denote the set $\{\lambda \cdot \mathbf{x} : 0 \leq \lambda \leq 1\}$ by $[\mathbf{0}, \mathbf{x}]$. A *zonotope* is defined to be the Minkowski sum of line segments. Observe the prism operation can be realized as the Minkowski sum with a line segment.

Theorem 2.6.2 (Billera–Ehrenborg–Readdy)**.** [11] *The* **cd**-*indexes of* n-*dimensional zonotopes linearly span the space* $\mathcal{F}_n$ *of* **cd**-*polynomials of degree* n.

Open question 2.6.3. *Find a basis of zonotopes which span the space* $\mathcal{F}_n$ *of* **cd**-*polynomials of degree* n.

One such basis is conjectured by Liu [51] consisting of all BP-words of length n ending in P and having no two consecutive B's, where for a zonotope Z, the two operations $PZ = \mathrm{Prism}(Z)$ and $BZ = M(\mathrm{Prism}(Z))$. Here M is the Minkowski sum with a line segment taken in general direction in the same dimension.

2.7. **Inequalities: a first look.**

The f-vector for 3-dimensional polytopes determines the flag vector (see Exercise 1.5.3), by Steinitz' theorem all of the inequalities for flag vectors of 3-dimensional polytopes have been determined. The best-known linear inequalities for 4-dimensional polytopes are due to Bayer [2]:

Theorem 2.7.1 (Bayer)**.** *The flag* f-*vector of a* 4-*polytope satisfies*

(1) $f_{02} - 3f_2 \geq 0$
(2) $f_{02} - 3f_1 \geq 0$

(3) $f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0$
(4) $6f_1 - 6f_0 - f_{02} \geq 0$
(5) $f_0 - 5 \geq 0$
(6) $f_2 - f_1 + f_0 - 5 \geq 0$

Observe that (1) and (2) are dual, and (5) and (6) are dual, whereas (3) and (4) are self-dual.

2.8. **Notes.**

The theory of Hopf algebras is originally due to Sweedler [69]. The Newtonian coalgebra of posets is due to Ehrenborg and Hetyei (unpublished). Ehrenborg and Readdy discovered the inherent coalgebraic structure of flag vectors of polytopes and geometric operations on polytopes; see [31]

The first identity in 2.3.1 is due to Stanley [65, Lemma 1.1].

The generalized Dehn–Sommerville relations are also known as the *Bayer–Billera relations.*

The original proof of Theorem 2.6.1, due to Bayer and Billera, is long. Lou Billera described it as "Beating the beast until it was lying down".

Theorem 2.8.1 (Varchenko/Liu)**.** [11] *For an n-dimensional zonotope Z and $S = \{i_1, \ldots, i_k\}$ we have*

$$\frac{f_S(Z)}{f_{i_j}(Z)} < \binom{d - i_1}{i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k} \cdot 2^{i_k - i_1}.$$

Billera and Hetyei [12] determined the convex cone generated by all flag f-vectors of graded posets.

2.9. **Exercises.**

Exercise 2.9.1. Compute the **cd**-index of the n-dimensional simplex for $n = 1, \ldots, 5$.

Exercise 2.9.2. Compute the **cd**-index of the n-dimensional cube for $n = 1, \ldots, 5$.

Exercise 2.9.3. Compute the coefficient $\mathbf{c}^i\mathbf{d}\mathbf{c}^j$ in the **cd**-index of the Boolean algebra B_{i+j+3}.

Exercise 2.9.4. For $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, we say π has a *descent* at position j if $\pi_j > \pi_{j+1}$. Furthermore, a permutation π is an *André permutation* if π has no double descents, that is, no index j with $\pi_j > \pi_{j+1} > \pi_{j+2}$ and satisfies the more general "no double descent" condition: for all $1 < j < j' \leq n$ if $\pi_{j-1} = \max\{\pi_{j-1}, \pi_j, \pi_{j'-1}, \pi_{j'}\}$ and $\pi_{j'} = \min\{\pi_{j-1}, \pi_j, \pi_{j'-1}, \pi_{j'}\}$, then there exists a j'' with $j < j'' < j'$ such that $\pi_{j''} < \pi_{j'}$. Denote the set of

André permutations in $\mathfrak{S}_n$ by $\mathscr{A}_n$.

a. Determine the set of André permutations $\mathscr{A}_n$ for $n = 1, \ldots, 5$.
b. The noncommutative André polynomial of Foata and Schützenberger is $\sum_\pi \Omega(u_\pi)$ where the sum is over all André permutations $\pi \in \mathscr{A}_n$, u_π is the descent word of the permutation π and Ω is the map which replaces each occurrence of **ba** with a **d** and then each remaining **a** with a **c**. Compute the noncommutative André polynomials for $n = 1, \ldots, 5$.

Exercise 2.9.5. Define $\kappa : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \rightarrow \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle$ to be the algebra map such that $\kappa(\mathbf{a}) = \mathbf{a} - \mathbf{b}$ and $\kappa(\mathbf{b}) = 0$.
a. Prove

$$w = \kappa(w) + \sum_{w} \kappa(w_{(1)}) \cdot \mathbf{b} \cdot w_{(2)}.$$

b. What does this say about the **ab**-index of a poset?

Exercise 2.9.6. Prove Lemma 2.5.1.

Exercise 2.9.7. Let v be a vertex of a polytope P. Describe the **cd**-index of the resulting polytope when cutting off the vertex v in terms of $\Psi(P)$ and $\Psi(P/v)$. Here P/v is the vertex figure which has face lattice $[v, \hat{1}]$.

Exercise 2.9.8. Describe the **cd**-index of the bipyramid of a polytope P in terms of the **cd**-index of the original polytope P.

3. Lecture III: Hyperplane arrangements & zonotopes; Inequalities: a first look

3.1. **Zonotopes.**

Recall that a zonotope Z can be described as the Minkowski sum of line segments:

$$Z = [\mathbf{0}, \mathbf{v_1}] + \cdots + [\mathbf{0}, \mathbf{v_k}].$$

Associated with a zonotope is a hyperplane arrangement given by the set of k hyperplanes $H_1, \ldots, H_k$, where the hyperplane H_i is the subspace orthogonal to the normal vector $\mathbf{v_i}$, for $i = 1, \ldots, k$.

Given a hyperplane arrangement $\mathscr{H} = \{H_1, \ldots, H_k\}$ in $\mathbb{R}^n$, there are two associated lattices. The *intersection lattice* L consists of all the intersections of the hyperplanes in $\mathscr{H}$ ordered with respect to reverse inclusion. Thus the minimal element is the empty intersection $\mathbb{R}^n$ and the maximal element of L is the intersection of all the hyperplanes, that is, the zero vector. The second lattice is the more complicated *lattice of regions* T. It is formed by intersecting the arrangement $\mathscr{H}$ with an n-dimensional sphere. This gives a decomposition of the n-sphere and the open cells can be ordered by inclusion to form T.

The important function we will work with is the omega map ω, which we now describe.

Definition 3.1.1. *Define a linear function $\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b}\rangle \longrightarrow \mathbb{Z}\langle \mathbf{c}, 2\mathbf{d}\rangle$ as follows: For an* **ab***-monomial v we compute $\omega(v)$ by replacing each occurrence of* **ab** *in the monomial v with* $2\mathbf{d}$*, and then replacing the remaining letters with* **c***'s. Extend this definition by linearity to* **ab***-polynomials.*

The function ω takes an **ab**-polynomial of degree n into a **c**-2**d**-polynomial of degree n. As an example $\omega(\mathbf{bbaababba}) = \mathbf{c}^3 \cdot 2\mathbf{d} \cdot 2\mathbf{d} \cdot \mathbf{c}^2$.

We can now link the **ab**-index of the intersection lattice with the **cd**-index of the corresponding zonotope. Here an *essential* hyperplane arrangement $\mathscr{H}$ is one where there is no non-zero vector orthogonal to all of the hyperplanes in $\mathscr{H}$.

Theorem 3.1.2 (Billera–Ehrenborg–Readdy)**.** *Let $\mathscr{H}$ be an essential hyperplane arrangement and let L be the intersection lattice of $\mathscr{H}$. Let Z be the zonotope corresponding to $\mathscr{H}$. Then the* **c**-2**d***-index of the zonotope Z is given by*

$$\Psi(Z) = \omega(\mathbf{a} \cdot \Psi(L)).$$

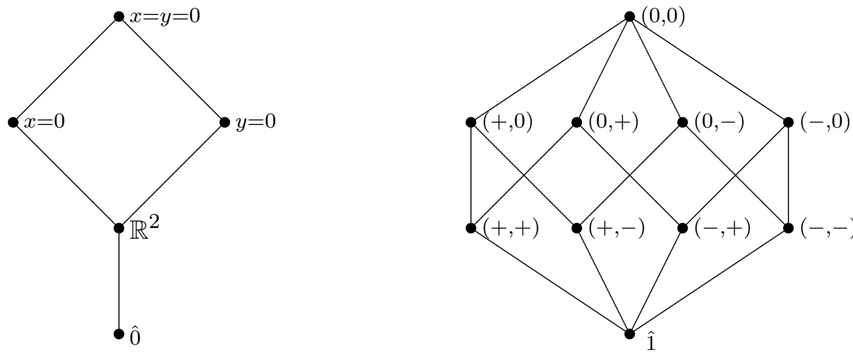

FIGURE 1. The lattice $\widehat{L}$ and the lattice T^*, the dual of the lattice of regions.

For example, the intersection lattice corresponding to the hexagonal prism has **ab**-index $\Psi(L) = \mathbf{aa} + 3 \cdot \mathbf{ba} + 3 \cdot \mathbf{ab} + 2 \cdot \mathbf{bb}$, so

$$\begin{aligned}\Psi(Z) &= \omega(\mathbf{a} \cdot (\mathbf{aa} + 3 \cdot \mathbf{ba} + 3 \cdot \mathbf{ab} + 2 \cdot \mathbf{bb})) \\ &= \omega(\mathbf{aaa} + 3 \cdot \mathbf{aba} + 3 \cdot \mathbf{aab} + 2 \cdot \mathbf{abb}) \\ &= \mathbf{c}^3 + 3 \cdot 2\mathbf{dc} + 3 \cdot \mathbf{c} \cdot 2\mathbf{d} + 2 \cdot 2\mathbf{d} \cdot \mathbf{c}) \\ &= \mathbf{c}^3 + 10 \cdot \mathbf{cd} + 6 \cdot \mathbf{dc}\end{aligned}$$

Theorem 3.1.2 was originally stated in terms of oriented matroids. For further details, see [10].

Theorem 3.1.3. *[Billera–Ehrenborg–Readdy] Let $\mathscr{M}$ be an oriented matroid, T the lattice of regions of $\mathscr{M}$ and L the lattice of flats of $\mathscr{M}$. Then the* **c**-2**d**-*index of the lattice of regions T is given by*

$$\Psi(T) = \omega(\mathbf{a} \cdot \Psi(L))^*.$$

The proof of Theorem 3.1.3 involves three ingredients. First, one orients the hyperplane arrangement so that each region has an associated sign vector. There is a map z, called the zero map, from the dual of the lattice of regions to the lattice $L \cup \{\hat{0}\}$ which sends a sign vector with zero coordinates $I = \{i_1, \ldots, i_k\}$ to the intersection $H_{i_1} \cap \cdots \cap H_{i_k}$. See Figure 1. Secondly, a result of Bayer and Sturmfels gives the cardinality of the inverse image of a chain in $L \cup \{\hat{0}\}$ [5].

Theorem 3.1.4 (Bayer–Sturmfels)**.** *For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in $L \cup \{\hat{0}\}$, the cardinality of its inverse image is given by*

$$|z^{-1}(c)| = \prod_{i=1}^{k-1} \sum_{x_i \leq y \leq x_{i+1}} (-1)^{\rho(x_i,y)} \cdot \mu(x_i, y).$$

Finally, coalgebraic techniques from [31] allow one to translate this into a straightforward-to-compute expression.

3.2. **Application: R-labelings.**

Let P be a graded poset with $\hat{0}$ and $\hat{1}$. We say $\lambda : E(P) \rightarrow \mathbb{Z}$ is an *R-labeling* if for every interval $[x, y]$ of P, there exists a unique saturated chain that is rising, that is, $c : x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ with

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k).$$

The classical R-labeling on the Boolean algebra is to label an edge $S \prec T$ by the unique element $T - S$. The $n!$ maximal chains in the Boolean algebra B_n then correspond to the $n!$ permutation in $\mathfrak{S}_n$. See [66, Chapter 3].

Theorem 3.2.1 (Björner; Stanley)**.** *Let P be a poset with R-labeling λ. Then*

$$h_S = \text{ \# maximal chains from } \hat{0} \text{ to } \hat{1} \text{ in } P \text{ with descent set } S.$$

As a first application, the theory of R-labelings gives the **ab**-index of the Boolean algebra as

$$\Psi(B_n) = \sum_{\pi \in \mathfrak{S}_n} u_{D(\pi)},$$

where $D(\pi)$ is the descent word of the permutation π. Since this corresponds to the **ab**-index of the hyperplane arrangement consisting of the n coordinate hyperplanes in $\mathbb{R}^n$, we can apply Theorem 3.1.2 to obtain the **cd**-index of the zonotope, that is, the cubical lattice.

Theorem 3.2.2 (Billera–Ehrenborg–Readdy)**.** *The* **c**-2**d***-index of the n-dimensional cube C_n is given by*

$$\Psi(C_n) = \sum_{\pi \in \mathfrak{S}_n} \omega(\mathbf{a} \cdot u_{D(\pi)}).$$

As a second application, Stanley conjectured that the **cd**-index of any convex polytope, and more generally, any Gorenstein* lattice, is coefficient-wise greater than or equal to the **cd**-index of the simplex of the same dimension, i.e., the Boolean algebra of the same rank. We obtain a zonotopal analogue of this conjecture.

Corollary 3.2.3 (Billera–Ehrenborg–Readdy)**.** *Among all zonotopes of dimension n, the n-dimensional cube has the smallest* **c**-2**d***-index.*

3.3. **Kalai convolution and 4-polytope inequalities.**

Knowing inequalities for the **cd**-index implies inequalities for the flag h-vector and the flag f-vector. This follows from expanding the **cd**-index back into the **ab**-index ($\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ are each non-negative linear combinations of monomials in **a** and **b**), then expanding the **ab**-index back into the flag f-vector via equation (1.4) (another non-negative linear combination).

Before examining inequalities for the **cd**-index, we begin with a critique of the known linear inequalities for 4-dimensional polytopes. See Theorem 2.7.1.

Kalai's convolution is a method to lift known inequalities on flag vectors of m and n-dimensional polytopes to an inequality which holds for $(m+n+1)$-dimensional polytopes [44]. We follow [26].

Definition 3.3.1. *The* Kalai convolution *is*

$$f_S^m * f_T^n = f^{m+n+1}_{S\cup\{m\}\cup(T+m+1)},$$

where $S \subseteq \{0,\ldots,m-1\}$ and $T \subseteq \{0,\ldots,n-1\}$. The superscripts indicate the dimension of the polytope that the flag vector is from, and $T+m+1$ denotes shifting all the elements of the subset of T by $m+1$.

The Kalai product implies that for linear operators M and N defined on m-, respectively n-, dimensional polytopes yields a linear functional on $(m+n+1)$-dimensional polytopes P by

$$(M^m * N^n)(P) = \sum_{\substack{x \\ \dim(x)=m}} M^m([\hat{0},x]) \cdot N^n([x,\hat{1}]).$$

Corollary 3.3.2. *If M and N are two linear functionals that are nonnegative on polytopes, then so is their Kalai convolution $M * N$.*

Example 3.3.3. Since every 2-dimensional face has at least 3 vertices, we have

$$0 \leq (f_0^2 - 3f_\emptyset^2) * f_\emptyset^1 = f_{0,2}^4 - 3f_2^4.$$

This is (1) of Theorem 2.7.1. The dual is

$$0 \leq f_\emptyset^1 * (f_0^2 - 3f_\emptyset^2) = f_{1,2}^4 - 3f_1^4 = \frac{1}{2}f_{012}^4 - 3f_1^4 = f_{02}^4 - 3f_1^4,$$

which is (2) of Theorem 2.7.1. This inequality states that each edge of a 4-dimensional polytope is surrounded by three 2-faces.

We continue discussing the inequalities of Theorem 2.7.1. Inequality (3) of is the toric g-vector inequality $g_2 \geq 0$; see [43, 44].

Inequality (4) comes from the following computation:

$$0 \leq f_\emptyset^0 * (f_0^2 - 3f_\emptyset^2) * f_\emptyset^0 = f_{013}^4 - 3f_{03}^4 = 6f_1^4 - 6f_0^4 - f_{02}^4, \qquad (3.1)$$

where the last equality is Exercise 3.7.2.

Finally, inequalities (5) and (6) state that every 4-dimensional polytope has at least five vertices and at least five 3-dimensional faces.

3.4. Shelling and cd-index inequalities.

Let us return to inequalities for the **cd**-index. Recall that Stanley proved the nonnegativity of the **cd**-index for polytopes, and more generally, for spherically-shellable regular CW-spheres. See Theorem 1.3.2. Stanley conjectured that for n-dimensional polytopes, more generally, Gorenstein* lattices, the **cd**-index was minimized on the simplex of the same dimension, respectively Boolean algebra of the same rank. Both of these conjectures were shown to be true. See [9, 28].

Theorem 3.4.1 (Billera–Ehrenborg)**.** *The* **cd***-index of a convex n-polytope is coefficient-wise greater than or equal to the* **cd***-index of the n-simplex.*

Theorem 3.4.2 (Ehrenborg–Karu)**.** *The* **cd***-index of a Gorenstein* lattice of rank n is coefficient-wise greater than or equal to the* **cd***-index of the Boolean algebra B_n.*

Define an inner product on $\mathbf{k}\langle \mathbf{c}, \mathbf{d}\rangle$ by

$$\langle u|v\rangle = \delta_{u,v}$$

where u and v are **cd**-monomials and extend by linearity. We can use this notation to encode inequalities easily. For example,

$$\left\langle \mathbf{d} - \mathbf{c}^2 | \Psi(P) \right\rangle \geq 0$$

says the for a 2-dimensional polytope the coefficient of $\mathbf{d}$ minus the coefficient of $\mathbf{c}^2$ is nonnegative. (True, as $(n-2)-1 \geq 0$ for $n \geq 3$.) We can now state Ehrenborg's lifting technique [26, Theorem 3.1].

Theorem 3.4.3 (Ehrenborg)**.** *Let u and v be two* **cd***-monomials. Suppose u does not end in* **c** *and v does not begin with* **c***. Then the inequality*

$$\langle H|\Psi(P)\rangle \geq 0 \text{ implies } \langle u \cdot H \cdot v|\Psi(P)\rangle \geq 0.$$

where H is a **cd***-polynomial such that the inequality $\langle H|\Psi(P)\rangle \geq 0$ holds for all polytopes P.*

Corollary 3.4.4. *For two* **cd***-monomials u and v the following inequality holds for all polytopes P:*

$$\langle u \cdot \mathbf{d} \cdot v|\Psi(P)\rangle \geq \left\langle u \cdot \mathbf{c}^2 \cdot v|\Psi(P)\right\rangle.$$

This corollary says the coefficient of a **cd**-monomial increases when replacing a $\mathbf{c}^2$ with a $\mathbf{d}$.

3.5. A word about shellings.

A pure n-dimensional polytopal complex is *shellable* if there is an ordering of its facets $F_1, \ldots, F_s$, called a *shelling order*, such that (i) ∂F_1 is shellable, (ii) for all $1 \leq k \leq s$, the intersection of $F_k \cap \cup_{i=1}^{k-1} F_i$ is shellable of dimension $n-1$. If a polytopal complex is of dimension 0, then any order of its vertices is declared to be a valid shelling order.

In Section 1.4, it was pointed out that many proofs for results about polytopes were incomplete as they assumed all polytopes (that is, the complex formed by the boundary of the polytope) are shellable. Shellability of polytopes was settled in 1971 by Bruggesser and Mani [20].

Theorem 3.5.1 (Bruggesser–Mani)**.** *Polytopes are shellable.*

Proof. The idea of the proof is to treat the boundary of a polytope as a planet and to send a space rocket off from the planet. Unlike NASA, your rocket travels in a straight line. As you are taking off, you should write down the order of the new facets you are seeing, starting with the first facet you took off from. Eventually you will see all the facets on one side of the polytope. The rocket goes off to infinity, then returns from the other direction along the same straight line. You begin to descend on the other side of the polytope. Now record the facets which begin to disappear as you approach your landing spot. The order of the facets you recorded is a shelling order. □

The shelling order in Bruggesser–Mani is called a *line shelling*.

Given a shelling order for a polytope, observe this builds the polytope one facet at a time a polytope one facet at a time so that at each shelling step the polytopal complex is topologically a ball except the last step when it becomes a sphere.

The notion of spherical shellability is closely related to shellability. The **cd**-index is only defined for regular decompositions of a sphere?? In order for Stanley to proof of the nonnegativity of the **cd**-index, he had to work with spherical objects. At each shelling step of a polytope, he attached an artificial facet to close off the complex $F_1 \cup \cdots \cup F_i$ into a sphere. He was then able to show at each shelling step that the coefficients of the **cd**-index were weakly increasing and hence nonnegative.

Proposition 3.5.2. *[Stanley] Let $F_1, \ldots, F_s$ be a spherical-shelling of a regular cellular sphere Ω. Then*

$$0 \leq \Psi(F_1') \leq \Psi((F_1 \cup F_2)') \leq \cdots \leq \Psi((F_1 \cup \cdots \cup F_{n-1})') = \Psi(\Omega), \quad (3.2)$$

where the notation Γ' indicates attaching a cell to the boundary $\partial\Gamma$ of the complex Γ so that is topologically a sphere.

The inequalities in Proposition 3.5.2 were essential in the proof of Theorem 3.4.1, that is, that the n-simplex minimizes the **cd**-index coefficient-wise for all n-polytopes. The proof also required using coalgebra techniques to derive a number of identities, and combining the inequalities into the desired inequality. The proof of Theorem 3.4.3 also used shellings. However, the inequality relations in Proposition 3.5.2 were replaced with a different type of inequality.

3.6. **Notes.**

The labels in an R-labeling do not necessarily have to be the integers, but instead elements from some poset. There are other notions of edge labelings, including EL-labelings (edge-lexicographic labelings), CL-labelings (chain-lexicograpic labelings), and analogues for nonpure complexes. See [18] and the references therein.

In [26] Ehrenborg has determined the best linear inequalities for polytopes of dimension up to dimension 8. There has been some work on finding *quadratic* inequalities for flag vectors of polytopes due to Ling [50]. Bayer's 1987 paper also includes some quadratic inequalities [2].

3.7. **Exercises.**

Exercise 3.7.1. The 3-dimensional permutahedron, depicted on the WAM poster, is a zonotope. Describe the associated hyperplane arrangement, intersection lattice and compute the **cd**-index using Theorem 3.1.2.

Exercise 3.7.2. Finish the computation in (3.1).

Exercise 3.7.3. Use line shellings to prove the Euler–Poincaré formula.

Exercise 3.7.4. Prove Corollary 3.4.4.

4. Lecture IV: New Horizons

In this lecture we describe recent developments regarding chain enumeration and the **cd**-index which involve algebra, graph theory and topology. The first is a non-homogeneous **cd**-index for Bruhat graphs due to Billera and Brenti [8]. One motivation for studying the **cd**-index of Bruhat graphs is that the **cd**-index of the interval $[u, v]$ determines the Kazhdan–Lusztig polynomial $P_{u,v}(q)$; see [8, Section 3]. These polynomials arise out of Kazhdan and Lusztig's study of the Springer representations of the Hecke algebra of a Coxeter group [48, 49]. The Kazhdan–Lusztig polynomials have many applications, including to Verma modules and to the algebraic geometry and topology of Schubert varieties. See Section 4.1 for a further discussion.

The second recent development is the theory of balanced graphs, due to Ehrenborg and Readdy [35]. This theory relaxes the graded, poset and Eulerian requirements for chain enumeration in graded posets. Bruhat graphs are a special case of balanced graphs, and the theory simplifies the proof techniques from using quasi-symmetric theory to edge labelings in the graphs. In the case a balanced graph has a *linear edge labeling*, the authors conjecture the **cd**-index has nonnegative coefficients.

The third development is both a topological and poset theoretic generalization of flag enumeration. Ehrenborg, Goresky and Readdy have extended the theory of face incidence enumeration of polytopes, and more generally, chain enumeration in graded Eulerian posets, to that of Whitney stratified spaces and quasi-graded posets [27]. It is important to point out that, unlike the case of polytopes, the coefficients of the **cd**-index of Whitney stratied manifolds can be negative. It is hoped that by applying topological techniques to stratified manifolds, a tractable interpretation of the coefficients of the **cd**-index will emerge. This may ultimately explain Stanley's nonnegativity results for spherically shellable posets [65] and Karu's results for Gorenstein* posets [45], and settle the conjecture that non-negativity holds for regular cell complexes

4.1. **Bruhat graphs.**

Another family of Eulerian posets is formed by taking the (strong) Bruhat order on a Coxeter group [70]. Hence any interval has a **cd**-index which is homogeneous of degree one more than the length of the interval. By removing the adjacent rank assumption on the cover relation of the Bruhat order, a directed graph known as the Bruhat graph is obtained which in effect allows algebraic "short cuts" between elements.

More formally, let (W, S) be a Coxeter system, where W denotes a (finite or infinite) Coxeter group with generators S and $\ell(u)$ denotes the length of a group element u. Let T be the set of reflections, that is, $T = \{w \cdot s \cdot w^{-1}\ :$

$s \in S, w \in W\}$. The *Bruhat graph* has the group W as its vertex set and its set of labels Λ is the set of reflections T. The edges and their labeling are defined as follows. There is a directed edge from u to v labeled t if $u \cdot t = v$ and $\ell(u) < \ell(v)$. The underlying poset of the Bruhat graph is called the *(strong) Bruhat order*. It is important to note that every interval of the Bruhat order is Eulerian, that is, every interval $[x, y]$ has Möbius function given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$, where ρ denotes the rank function. For a more complete description of Coxeter systems, see Björner and Brenti's text [17].

Using the fact that the generalized Dehn–Sommerville relations hold for coefficients of polynomials arising in Kazhdan–Lusztig polynomials [19, Theorem 8.4] and quasisymmetric functions, Billera and Brenti show that the Bruhat graph has a non-homogeneous **cd**-index [8].

Theorem 4.1.1 (Billera–Brenti)**.** *For an interval $[u, v]$ in the Bruhat order, where $u < v$, the following three conditions hold:*

(i) *The interval $[u, v]$ in the Bruhat graph has a* **cd***-index $\Psi([u, v])$.*
(ii) *Restricting the* **cd***-index $\Psi([u, v])$ to those terms of degree $\ell(v) - \ell(u) - 1$ equals the* **cd***-index of the graded poset $[u, v]$.*
(iii) *The degree of a term in the* **cd***-index $\Psi([u, v])$ is less than or equal to $\ell(v) - \ell(u) - 1$ and has the same parity as $\ell(v) - \ell(u) - 1$.*

For an alternate proof using labelings of balanced graphs, see [35].

4.2. **Bruhat and balanced graphs.**

The notion of a labeled acyclic digraph was introduced in [35] in order to model poset structure in this more general setting.

Let $G = (V, E)$ be a directed, acyclic and locally finite graph with multiple edges allowed. Recall that an *acyclic graph* does not have any directed cycles and the property of a graph being *locally finite* requires that there are a finite number of paths between any two vertices. Each directed edge e has a tail and a head, denoted respectively by $\text{tail}(e)$ and $\text{head}(e)$. View each directed edge as an arrow from its tail to its head. A directed path p of length k from a vertex x to a vertex y is a list of k directed edges $(e_1, e_2, \ldots, e_k)$ such that $\text{tail}(e_1) = x$, $\text{head}(e_k) = y$ and $\text{head}(e_i) = \text{tail}(e_{i+1})$ for $i = 1, \ldots, k-1$. We denote the length of a path p by $\ell(p)$.

Since the graph is acyclic, it does not have any loops. Furthermore, the acyclicity condition implies there is a natural partial order on the vertices of G by defining the order relation $x \leq y$ if there is a directed path from the vertex x to the vertex y. It is straightforward to verify that this relation is reflexive, antisymmetric and transitive. It allows us to define the *interval* $[x, y]$ to be the set of all vertices z in $V(G)$ such that there is a directed path from x to z and a directed path from z to y. We view the interval $[x, y]$ as the vertex-induced subgraph of the digraph G, where the edges have the same labels as in the digraph G. The locally finite condition is now equivalent to that every interval $[x, y]$ in the graph has finite cardinality.

We next relax the notions of R-labeling and the **ab**-index of a poset. Let Λ be a set with a relation $\sim$, that is, there is a subset $R \subseteq \Lambda \times \Lambda$ such that for $i, j \in \Lambda$ we have $i \sim j$ if and only if $(i, j) \in R$. A *labeling of* G is a function λ from the set of edges of G to the set Λ. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commutative variables each of degree one. For a path $p = (e_1, \ldots, e_k)$ of length k, where $k \geq 1$, we define the *descent word* $u(p)$ to be the **ab**-monomial $u(p) = u_1 u_2 \cdots u_{k-1}$, where

$$u_i = \begin{cases} \mathbf{a} & \text{if } \lambda(e_i) \sim \lambda(e_{i+1}), \\ \mathbf{b} & \text{if } \lambda(e_i) \not\sim \lambda(e_{i+1}). \end{cases}$$

Observe that the descent word $u(p)$ has degree $k - 1$, that is, one less than the length of the path p. The **ab**-*index* of an interval $[x, y]$ is defined to be

$$\Psi([x, y]) = \sum_{p} u(p), \tag{4.1}$$

where the sum is over all directed paths p from x to y.

An analogue of the coalgebraic groundwork for for flag enumeration in posets holds for labeled acyclic digraphs. More specifically, the **ab**-index of a labeled acyclic digraph is a coalgebra homeomorphism from the linear span of bounded labeled acyclic digraphs to the polynomial ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$.

The following result gives three equivalent statements which imply the (non-homogeneous) **ab**-index of an acyclic digraph can be written as a (non-homogeneous) **cd**-index [35].

Theorem 4.2.1 (Ehrenborg–Readdy)**.** *For a labeled acyclic digraph G, the following three statements are equivalent:*

(i) *For every interval $[x, y]$ in the digraph G and for every non-negative integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(ii) *For every interval $[x, y]$ in the digraph G and for every* even *positive integer k, the number of rising paths from x to y of length k is equal to the number of falling paths from x to y of length k.*
(iii) *The* **ab**-*index of every interval $[x, y]$ in the digraph G, where $x < y$, is a polynomial in $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$.*

Definition 4.2.2. *A labeled acyclic digraph G is said to be* balanced *if it satisfies condition (i) in Theorem 4.2.1. Such a labeling is called a* balanced labeling *or B-*labeling *for short.*

An edge labeling *linear* if the underlying relation $(\Lambda, \sim)$ is that of a linear order.

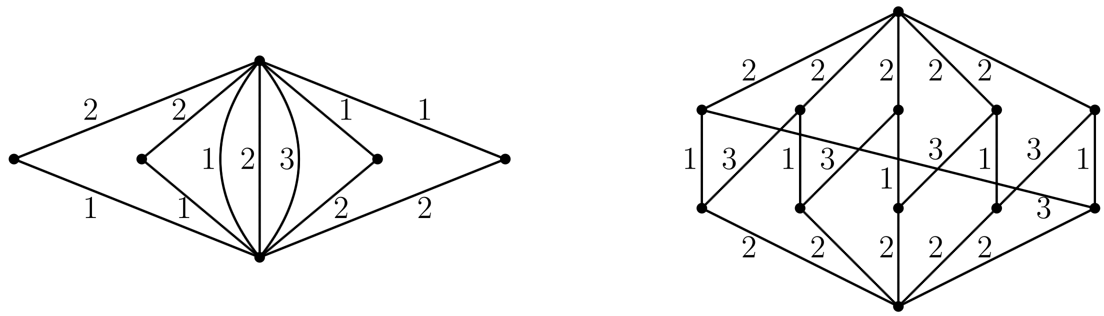

FIGURE 2. Two balanced directed graphs where the relation on the labeled set $\Lambda = \{1, 2, 3\}$ is the natural linear order. Their respective **cd**-indexes are $2 \cdot \mathbf{c} + 3$ and $5 \cdot \mathbf{d}$. These two examples show that the **cd**-index of a graph is not necessarily homogeneous and that the coefficient of the **c**-power term is not necessarily 1.

Theorem 4.2.3 (Ehrenborg–Readdy)**.** *Let u be a non-zero* **cd***-polynomial with non-negative coefficients. Then there exists a bounded balanced labeled acyclic digraph G where the relation on the set of labels is a linear order and which satisfies $\Psi(G) = w$.*

Theorem 4.2.3 motivates the following conjecture.

Conjecture 4.2.4 (Ehrenborg–Readdy)**.** *The* **cd***-index of a bounded labeled acyclic digraph G with a balanced linear edge labeling is non-negative.*

4.3. **Euler flag enumeration of Whitney stratified spaces.**

We begin with a modest example.

Example 4.3.1. Consider the non-regular CW-complex Ω consisting of one vertex v, one edge e and one 2-dimensional cell c such that the boundary of c is the union $v \cup e$, that is, boundary of the complex Ω is a one-gon. Its face poset is the four element chain $\mathscr{F}(\Omega) = \{\hat{0} < v < e < c\}$. This is not an Eulerian poset. The **ab**-index of Ω is $\mathbf{a}^2$. Note that $\mathbf{a}^2$ cannot be written in terms of **c** and **d**.

Observe that the edge e is attached to the vertex v twice. Hence it is natural to change the value of f_{01} to 2. This changes h_{01} to be 1. The **ab**-index becomes $\Psi(\Omega) = \mathbf{a}^2 + \mathbf{b}^2$ and hence its **cd**-index is $\Psi(\Omega) = \mathbf{c}^2 - \mathbf{d}$.

The *Euler characteristic* of an n-dimensional polytopal complex Δ is defined as the alternating sum of its face numbers, that is,

$$\chi(\Delta) = f_0(\Delta) - f_1(\Delta) + f_2(\Delta) - \cdots + (-1)^n \cdot f_n(\Delta).$$

This is a topological invariant, that is, any two complexes that are homotopy equivalent have the same Euler characteristic. Especially, any contractible space has Euler characteristic 1.

The motivation for the value 2 in Example 4.3.1 is best expressed in terms of the Euler characteristic of the link. The link of the vertex v in the edge e is two points whose Euler characteristic is 2. In order to view this example in the right topological setting, we review the notion of a Whitney stratification. For more details, see [23, 37, 38, 53].

A subset S of a topological space M is *locally closed* if S is a relatively open subset of its closure $\overline{S}$. Equivalently, for any point $x \in S$ there exists a neighborhood $U_x \subseteq S$ such that the closure $\overline{U_x} \subseteq S$ is closed in M. Another way to phrase this is a subset $S \subset M$ is locally closed if and only if it is the intersection of an open subset and a closed subset of M.

Definition 4.3.2. *Let W be a closed subset of a smooth manifold M which has been decomposed into a finite union of locally closed subsets*

$$W = \bigcup_{X \in \mathcal{P}} X.$$

Furthermore suppose this decomposition satisfies the condition of the frontier*:*

$$X \cap \overline{Y} \neq \emptyset \Longleftrightarrow X \subseteq \overline{Y}.$$

This implies the closure of each stratum is a union of strata, and it provides the index set $\mathcal{P}$ with the partial ordering:

$$X \subseteq \overline{Y} \Longleftrightarrow X \leq_{\mathcal{P}} Y.$$

This decomposition of W is a Whitney stratification *if*

1. *Each $X \in \mathcal{P}$ is a (locally closed, not necessarily connected) smooth submanifold of M.*
2. *If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i} Y$ converge to some limiting plane τ. Then the following inclusions hold:*

$$\text{(A)}\ \ T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B)}\ \ \ell \subseteq \tau.$$

Remark 4.3.3. For convenience we will henceforth also assume that W is pure dimensional, meaning that if $\dim(W) = n$ then the union of the n-dimensional strata of W forms a dense subset of W. Strata of dimension less than n are referred to as *singular strata*.

Whitney's conditions A and B ensure there is no fractal behavior and no infinite wiggling. A crucial result is that the links are well-defined in a Whitney stratification. See [27].

Recall the *incidence algebra* of a poset P is the set of all functions $f : I(P) \to \mathbb{C}$ where $I(P)$ denotes the set of intervals in the poset. The multiplication is given by $(f * g)(x, y) = \sum_{x \le z \le y} f(x, z) \cdot g(z, y)$ and the identity is given by the delta function $\delta(x, y) = \delta_{x,y}$, where the second delta is the usual Kronecker delta function $\delta_{x,y} = 1$ if $x = y$ and zero otherwise. The *zeta function* ζ is defined by $\zeta(x, y) = 1$ if $x \le y$ in the poset P and 0 otherwise. The *Möbius function* μ is the inverse of the zeta function in the incidence algebra, that is, $\mu * \zeta = \zeta * \mu = \delta$.

Recall a poset is said to be *ranked* if every maximal chain in the poset has the same length. This common length is called the *rank* of the poset. A poset is said to be *graded* if it is ranked and has a minimal element $\hat{0}$ and a maximal element $\hat{1}$. For other poset terminology, we refer the reader to Stanley's text [66].

We introduce the notion of a quasi-graded poset. This extends the notion of a ranked poset.

Definition 4.3.4. *A quasi-graded poset* $(P, \rho, \bar{\zeta})$ *consists of*

(i) *a finite poset* P *(not necessarily ranked),*
(ii) *a strictly order-preserving function* ρ *from* P *to* $\mathbb{N}$*, that is,* $x < y$ *implies* $\rho(x) < \rho(y)$ *and*
(iii) *a function* $\bar{\zeta}$ *in the incidence algebra* $I(P)$ *of the poset* P*, called the* weighted zeta function*, such that* $\bar{\zeta}(x, x) = 1$ *for all elements* x *in the poset* P*.*

Observe that we do not require the poset to have a minimal element or a maximal element. Since $\bar{\zeta}(x, x) \neq 0$ for all $x \in P$, the function $\bar{\zeta}$ is invertible in the incidence algebra $I(P)$ and we denote its inverse by $\bar{\mu}$.

For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the face poset of a Whitney stratified space, define

$$\bar{\zeta}(c) = \chi(x_1) \cdot \chi(\mathrm{link}_{x_2}(x_1)) \cdots \chi(\mathrm{link}_{x_{k-1}}(x_k)),$$

where χ denotes the Euler characteristic.

The usual **ab**-index for polytopes and Eulerian posets is via the flag f- and flag h-vectors. We extend this route by introducing the flag $\bar{f}$- and flag $\bar{h}$-vectors. Let $(P, \rho, \bar{\zeta})$ be a quasi-graded poset of rank $n+1$ having a $\hat{0}$ and $\hat{1}$ such that $\rho(\hat{0}) = 0$. For $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset of $\{1, \ldots, n\}$, define the *flag* $\bar{f}$*-vector* by

$$\bar{f}_S = \sum_{c} \bar{\zeta}(c), \tag{4.2}$$

where the sum is over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}\}$ in P such that $\rho(x_i) = s_i$ for all $1 \le i \le k$. The *flag* $\bar{h}$*-vector* is defined by the relation (and by inclusion–exclusion, we also display its inverse relation)

$$\bar{h}_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \bar{f}_T \quad \text{and} \quad \bar{f}_S = \sum_{T \subseteq S} \bar{h}_T. \tag{4.3}$$

For a subset $S \subseteq \{1, \ldots, n\}$ define the **ab**-monomial $u_S = u_1 u_2 \cdots u_n$ by $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The **ab**-index of the quasi-graded poset $(P, \rho, \bar{\zeta})$ is then given by

$$\Psi(P, \rho, \bar{\zeta}) = \sum_S \bar{h}_S \cdot u_S,$$

where the sum ranges over all subsets S. Again, in the case when we take the weighted zeta function to be the usual zeta function ζ, the flag $\bar{f}$ and flag $\bar{h}$-vectors correspond to the usual flag f- and flag h-vectors.

Definition 4.3.5. *A quasi-graded poset is said to be* Eulerian *if for all pairs of elements $x \leq z$ we have that*

$$\sum_{x \leq y \leq z} (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y) \cdot \bar{\zeta}(y, z) = \delta_{x,z}. \tag{4.4}$$

In other words, the function $\bar{\mu}(x, y) = (-1)^{\rho(x,y)} \cdot \bar{\zeta}(x, y)$ is the inverse of $\bar{\zeta}(x, y)$ in the incidence algebra. In the case $\bar{\zeta}(x, y) = \zeta(x, y)$, we refer to relation (4.4) *as the* classical Eulerian relation*.*

Generalizing the classical result of Bayer and Klapper for graded Eulerian posets, we have the analogue for quasi-graded posets.

Theorem 4.3.6 (Ehrenborg–Goresky–Readdy)**.** *For an Eulerian quasi-graded poset $(P, \rho, \bar{\zeta})$ its **ab**-index $\Psi(P, \rho, \bar{\zeta})$ can be written uniquely as a polynomial in the non-commutative variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$.*

Theorem 4.3.7 (Ehrenborg–Goresky–Readdy)**.** *Let M be a manifold with a Whitney stratified boundary. Then the face poset is quasi-graded and Eulerian, with*

$$\rho(x) = \dim(x) + 1$$

and

$$\bar{\zeta}(x, y) = \chi(\text{link}_y(x)).$$

We now give a few examples of Whitney stratifications beginning with the classical polygon.

Example 4.3.8. Consider a two dimensional cell c with its boundary subdivided into n vertices $v_1, \ldots, v_n$ and n edges $e_1, \ldots, e_n$. There are three ways to view this as a Whitney stratification.

(1) Declare each of the $2n + 1$ cells to be individual strata. This is the classical view of an n-gon. Here the weighted zeta function is the classical zeta function, that is, always equal to 1 (assuming $n \geq 2$).
(2) Declare each of the n edges to be one stratum $e = \cup_{i=1}^n e_i$, that is, we have the $n + 2$ strata $v_1, \ldots, v_n, e, c$. Here the non-one values of the weighted zeta function are given by $\bar{\zeta}(\hat{0}, e) = n$ and $\bar{\zeta}(v_i, e) = 2$.

S	$\bar{f}_S$	$\bar{h}_S$	$\mathbf{c}^3$	$-\mathbf{cd}$
$\emptyset$	1	1	1	0
$\{0\}$	2	1	1	0
$\{1\}$	1	0	1	-1
$\{2\}$	1	0	1	-1
$\{0,1\}$	2	0	1	-1
$\{0,2\}$	2	0	1	-1
$\{1,2\}$	2	1	1	0
$\{0,1,2\}$	4	1	1	0

Table 2. The flag $\bar{f}$- and flag $\bar{h}$-vectors, **ab**-index and **cd**-index of the sphere with an edge on it. The sum of the last two columns equals the flag h column, showing the **cd**-index is $\mathbf{aaa} + \mathbf{baa} + \mathbf{abb} + \mathbf{bbb} = \mathbf{c}^3 - \mathbf{cd}$.

(3) Lastly, we can have the three strata $v = \cup_{i=1}^{n} v_i$, $e = \cup_{i=1}^{n} e_i$ and c. Now non-one values of the weighted zeta function are given by $\bar{\zeta}(\hat{0}, v) = \bar{\zeta}(\hat{0}, e) = n$ and $\bar{\zeta}(v, e) = 2$.

In contrast, we cannot have $v, e_1, \ldots, e_n, c$ as a stratification, since the link of a point p in e_i depends on the point p in v chosen.

The **cd**-index of each of the three Whitney stratifications in Example 4.3.8 is the same, that is, $\mathbf{c}^2 + (n-2) \cdot \mathbf{d}$. Hence we have the immediate corollary.

Corollary 4.3.9. *The* **cd***-index of an* n*-gon is given by* $\mathbf{c}^2 + (n-2) \cdot \mathbf{d}$ *for* $n \geq 1$.

The last stratification in the previous example can be be extended to any simple polytope.

Example 4.3.10. Let P be an n-dimensional simple polytope. Recall that the simple condition that implies that every interval $[x, y]$, where $\hat{0} < x \leq y$, is a Boolean algebra. We obtain a different stratification of the ball by joining all the facets together to one strata. We note that the **cd**-index does not change, since the information is carried in the weighted zeta function. We continue by joining all the subfacets together to one strata. Again the **cd**-index remains unchanged. In the end we obtain a stratification where the union of all the i-dimensional faces forms the ith strata. The face poset of this stratification is the $(n+2)$-element chain $C = \{\hat{0} = x_0 < x_1 < \cdots < x_{n+1} = \hat{0}\}$, with the rank function $\rho(x_i) = i$ and weighted zeta function $\bar{\zeta}(\hat{0}, x_i) = f_{i-1}(P)$ and $\bar{\zeta}(x_i, x_j) = \binom{n+1-i}{n+1-j}$. We have $\Psi(C, \rho, \bar{\zeta}) = \Psi(P)$.

A similar stratification can be obtained for any regular polytope.

Example 4.3.11. Consider the 2-sphere with an edge with two incident vertices on it. See Table 2 for the **cd**-index computation.

Example 4.3.12. Consider the stratification of an n-dimensional manifold with boundary, denoted $(M, \partial M)$, into its boundary ∂M and its interior M°. The face poset is $\{\hat{0} < \partial M < M^\circ\}$ with the elements having ranks 0, n and $n+1$, respectively. The weighted zeta function is given by $\bar{\zeta}(\hat{0}, \partial M) = \chi(\partial M)$, $\bar{\zeta}(\hat{0}, M^\circ) = \chi(M)$ and $\bar{\zeta}(\partial M, M^\circ) = 1$. If n is even then ∂M is an odd-dimensional manifold without boundary and hence its Euler characteristic is 0. In this case the **ab**-index is $\Psi(M) = \chi(M) \cdot (\mathbf{a}-\mathbf{b})^n$. If n is odd then we have the relation $\chi(\partial M) = 2 \cdot \chi(M)$ and hence the **ab**-index is given by $\Psi(M) = \chi(M) \cdot (\mathbf{a}-\mathbf{b})^n + 2 \cdot \chi(M) \cdot (\mathbf{a}-\mathbf{b})^{n-1} \cdot \mathbf{b}$. Passing to the **cd**-index we conclude

$$\Psi(M) = \begin{cases} \chi(M) \cdot (\mathbf{c}^2 - 2\mathbf{d})^{n/2} & \text{if } n \text{ is even,} \\ \chi(M) \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c} & \text{if } n \text{ is odd.} \end{cases}$$

The next example is a higher dimensional analogue of the one-gon in Example 4.3.1.

Example 4.3.13. Consider the subdivision Ω_n of the n-dimensional ball $\mathbb{B}^n$ consisting of a point p, an $(n-1)$-dimensional cell c and the interior b of the ball. If $n \geq 2$, the face poset is $\{\hat{0} < p < c < b\}$ with the elements having ranks 0, 1, n and $n+1$, respectively. In the case $n = 1$, the two elements p and c are incomparable. The weighted zeta function is given by $\bar{\zeta}(\hat{0}, p) = \bar{\zeta}(\hat{0}, c) = \bar{\zeta}(\hat{0}, b) = 1$, $\bar{\zeta}(p, c) = 1 + (-1)^n$, and $\bar{\zeta}(p, b) = \bar{\zeta}(c, b) = 1$. Thus the **ab**-index is

$$\Psi(\Omega_n) = (\mathbf{a}-\mathbf{b})^n + \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-1} + (\mathbf{a}-\mathbf{b})^{n-1} \cdot \mathbf{b} + (1+(-1)^n) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{b}. \tag{4.5}$$

When n is even the expression (4.5) simplifies to

$$\begin{aligned} \Psi(\Omega_n) &= \mathbf{a} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{a} + \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{b} \\ &= \frac{1}{2} \cdot \left[(\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}-\mathbf{b}) + (\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}+\mathbf{b})\right] \\ &= \frac{1}{2} \cdot \left[(\mathbf{c}^2 - 2\mathbf{d})^{n/2} + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-2)/2} \cdot \mathbf{c}\right]. \end{aligned} \tag{4.6}$$

When n is odd the expression (4.5) simplifies to

$$\begin{aligned} \Psi(\Omega_n) &= \mathbf{a} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{a} - \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot \mathbf{b} \\ &= \frac{1}{2} \cdot \left[(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}-\mathbf{b}) + (\mathbf{a}-\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b})^{n-2} \cdot (\mathbf{a}+\mathbf{b})\right] \\ &= \frac{1}{2} \cdot \left[\mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} + (\mathbf{c}^2 - 2\mathbf{d})^{(n-1)/2} \cdot \mathbf{c}\right]. \end{aligned} \tag{4.7}$$

As a remark, these **cd**-polynomials played an important role in proving that the **cd**-index of a polytope is coefficient-wise minimized on the simplex, namely, $\Psi(\Omega_n) = (-1)^{n-1} \cdot \alpha_n$, where α_n are defined in [9].

Open question 4.3.14. *Find the linear inequalities that hold among the entries of the* **cd***-index of a Whitney stratified manifold.*

This expands the program of determining linear inequalities for flag vectors of polytopes. Since the coefficients may be negative, one must ask what should the new minimization inequalities be. Observe that Kalai's convolution [44] still holds. More precisely, let M and N be two linear functionals defined on the **cd**-coefficients of any m-dimensional, respectively, n-dimensional manifold. If both M and N are non-negative then their convolution is non-negative on any $(m + n + 1)$-dimensional manifold.

Other inequality questions are:

Open question 4.3.15. *Can Ehrenborg's lifting technique* [26] *be extended to stratified manifolds?*

Open question 4.3.16. *What non-linear inequalities hold among the* **cd***-coefficients?*

One interpretation of the coefficients of the **cd**-index is due to Karu [45] who, for each **cd**-monomial, gave a sequence of operators on sheaves of vector spaces to show the non-negativity of the coefficients of the **cd**-index for Gorenstein* posets [45].

Open question 4.3.17. *Is there a signed analogue of Karu's construction to explain the negative coefficients occurring in the* **cd***-index of quasi-graded posets?*

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