The van der Waerden complex

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Abstract

We introduce the van der Waerden complex $\mathrm{vdW}(n, k)$ defined as the simplicial complex whose facets correspond to arithmetic progressions of length k in the vertex set $\{1, 2, \ldots, n\}$. We show the van der Waerden complex $\mathrm{vdW}(n, k)$ is homotopy equivalent to a CW-complex whose cells asymptotically have dimension at most $\log k / \log \log k$. Furthermore, we give bounds on n and k which imply that the van der Waerden complex is contractible.

Keywords: van der Waerden complex; discrete Morse theory; arithmetic progressions

1 Introduction

Recall that van der Waerden's Theorem from Ramsey theory states that given positive integers k and r, there is an integer $M = M(k, r)$ so that when the integers 1 through n with $n \geq M$ are colored with r colors, there is a monochromatic arithmetic progression of length k [8]. There is an upper bound for $M(k, r)$ due to Gowers coming out of his proof of Szemeredi's theorem [2].

Motivated by van der Waerden's theorem, we define the *van der Waerden complex* $\mathrm{vdW}(n, k)$ to be the simplicial complex on the vertex set $\{1, 2, \ldots, n\}$ whose facets correspond to all arithmetic progressions of length k, that is, the facets have the form $\{x, x + d, x + 2 \cdot d, \ldots, x + k \cdot d\}$, where d is a positive integer and $1 \leq x < x + k \cdot d \leq n$.

Observe that the van der Waerden $\mathrm{vdW}(n, k)$ is a pure simplicial complex of dimension k. Furthermore, when $k = 1$, the complex $\mathrm{vdW}(n, 1)$ is the complete graph K_n, which is homotopy equivalent to a wedge of $\binom{n-1}{2}$ circles.

This paper is concerned with understanding the topology of the van der Waerden complex. By constructing a discrete Morse matching, we show that the dimension of the homotopy type of the van der Waerden complex is bounded above by the maximum of the number of distinct primes factors of all positive integers less than or equal to k. See Theorem 3.8. This bound is asymptotically described as $\log k / \log \log k$. See Theorem 3.11. In Section 4 we give bounds under which the van der Waerden complex is contractible. We then look at the implications of our results when studying the topology of the family of van der Waerden complexes $\mathrm{vdW}(5k, k)$ where k is any positive integer. We end with open questions in the concluding remarks.

2 Preliminaries

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $[i, j]$ denote the interval $\{i, i + 1, \ldots, j\}$.

Definition 2.1. *The* van der Waerden complex $\mathrm{vdW}(n,k)$ *is the simplicial complex on the vertex set* $[n]$ *whose facets correspond to all arithmetic progressions of length* k *in* $[n]$*, that is, the facets have the form*

$$\{x, x+d, x+2\cdot d, \ldots, x+k\cdot d\},$$

where d *is a positive integer and* $1 \leq x < x+k\cdot d \leq n$*.*

We remark that the van der Waerden complex is not monotone in the variable k. For instance, the set $\{1,4,7\}$ is a facet in $\mathrm{vdW}(7,2)$, but is not a face of $\mathrm{vdW}(7,3)$.

Let P be a finite partially ordered set (poset) with partial order relation $\prec$. For further information on posets, see [7, Chapter 3]. A *matching* M on P is a collection of disjoint pairs of elements of P such that $x \prec y$ for each pair $(x,y) \in M$. Given a matching M, we define the two partial functions u and d on P as $u(x) = y$ and $d(y) = x$ when $(x,y) \in M$.

Definition 2.2. *A matching* M *on a poset* P *is* acyclic *if there does not exist elements* $x_1, x_2, \ldots, x_j$ *such that*

$$x_1 \prec u(x_1) \succ x_2 \prec u(x_2) \succ \cdots \succ x_j \prec u(x_j) \succ x_1.$$

One can describe this definition in more concrete terms. In the Hasse diagram of the poset P, direct all of the edges upward except for those in the matching M. These are directed downward. The matching M is acyclic if and only if the resulting directed graph has no directed cycles.

One way to construct an acyclic matching is by taking the union of acyclic matchings on subposets as detailed below. For details, see [5, Theorem 11.10].

Theorem 2.3 (Patchwork Theorem)**.** *Let* $\varphi : P \longrightarrow Q$ *be an order-preserving map between the two posets* P *and* Q*. Suppose that an acyclic matching exists on each subposet* $\varphi^{-1}(q)$ *for every* $q \in Q$*. Then the union of these matchings is itself an acyclic matching on* P*.*

An element in the poset P that does not belong to matching M is called *critical*. We can now state the main theorem of discrete Morse theory [1, Theorem 2.5].

Theorem 2.4 (Forman)**.** *Let* Δ *be a simplicial complex and let* M *be an acyclic matching on the face poset of* $\Delta - \{\emptyset\}$*. Let* c_i *denote the number of critical elements of dimension* i*. Then the simplical complex* Δ *is homotopy equivalent to a CW-complex consisting of* c_i *cells of dimension* i *for* $i \geq 0$*.*

A CW-complex consisting of one 0-dimensional cell and c i-dimensional cell is homotopy equivalent to a wedge of c i-dimensional spheres. Combining this observation with Theorem 2.4, we have the following result.

Proposition 2.5. *Let* Δ *be a simplicial complex and let* M *be an acyclic matching on* $\Delta - \{\emptyset\}$*. If there only are* c *critical cells of dimension* $i > 0$ *and one critical cell of dimension* 0 *then* Δ *is homotopy equivalent to a wedge of* c *spheres each of dimension* i*.*

For a detailed treatment of the theory of CW-complexes, we direct the reader to [3].

3 A bound on the dimension of the homotopy type

We begin by studying a family of sets that we use as building blocks in order to understand the van der Waerden complex.

Definition 3.1. *Define* $\Gamma(k)$ *to be the family of subsets given by*

$$\Gamma(k) = \{F \ : \ \{0,k\} \subseteq F \subseteq [0,k], \ \gcd(F) = 1\},$$

where $[0,k] = \{0,1,\ldots,k\}$ *and* $\gcd(F)$ *denotes the greatest common divisor of the elements of* F*.*

In other words, $\Gamma(k)$ is the family of all subsets containing the pair $\{0,k\}$ and contained in the interval $[0,k]$ such that the only arithmetic sequence from 0 to k containing this family is the entire interval $[0,k]$. Even though $\Gamma(k)$ is not a simplicial complex, we refer to the sets in the collection $\Gamma(k)$ as faces.

Lemma 3.2. *Let* k *be a positive integer which is not squarefree. Then the family* $\Gamma(k)$ *has an acyclic matching with no critical cells.*

Proof. Let s denote the product of all of the distinct primes dividing k. Since k is not squarefree, we have that $s < k$.

Let F be a face in $\Gamma(k)$ such that $s \notin F$. Note that $\gcd(F \cup \{s\}) = \gcd(1, s) = 1$. Thus the union $F \cup \{s\}$ also belongs to the family $\Gamma(k)$. Similarly, if we have a face F in $\Gamma(k)$ such that $s \in F$, we claim that $F - \{s\}$ is also a face of $\Gamma(k)$. Assume that the prime p divides $\gcd(F - \{s\})$. It follows from $k \in F - \{s\}$ that $p|k$. Since s is the squarefree part of k, the prime p also divides s. This yields the contradiction that p divides $\gcd(\gcd(F - \{s\}), s) = \gcd(F) = 1$. Hence we conclude that the set difference $F - \{s\}$ is also in the family $\Gamma(k)$, proving the claim.

Given a face F, match it with the symmetric difference $F \triangle \{s\}$. This defines an acyclic matching on $\Gamma(k)$ with no critical cells, as claimed. □

Proposition 3.3. *Let* k *be a squarefree positive integer. There exists an acyclic matching on the family* $\Gamma(k)$ *which has only one critical cell given by*

$$\{0,k\} \cup \{k/q \ : \ q|k \text{ and } q \text{ prime}\}.$$

Proof. The proof is by induction on the number of prime factors of k. If there are no prime factors, that is, $k = 1$, then $\Gamma(1)$ only consists of the set $\{0,1\}$ which is then the only critical cell.

Assume now that k has at least one prime factor p. We begin to create the matching as follows. Let F be a set such that $k/p \notin F$ and $\gcd(F) = 1$. Note that $\gcd(F) = 1$ implies that $\gcd(F \cup \{k/p\}) = \gcd(\gcd(F), k/p) = 1$. Hence the two sets F and $F \cup \{k/p\}$ are faces of $\Gamma(k)$. Match these two faces. Note that among the elements that have been matched so far, there is no directed cycle.

The remaining unmatched faces are of the form $F \cup \{k/p\}$ where $k/p \notin F$, $\gcd(F) \neq 1$ and $\gcd(F \cup \{k/p)\}) = 1$. Observe that $\gcd(\gcd(F), k/p) = 1$. Since $\gcd(F)$ divides k, the only option for $\gcd(F)$ is the prime p. The unmatched faces are described by $F \cup \{k/p\}$ where $k/p \notin F$ and $\gcd(F) = p$. Note these sets F are in bijection with faces in the family $\Gamma(k/p)$ by sending F to the set $F/p = \{i/p \, : \, i \in F\}$, that is, the faces of $\Gamma(k)$ which have not yet been matched are the image of the map $\varphi(G) = p \cdot G \cup \{k/p\}$ applied to the family $\Gamma(k/p)$.

By the induction hypothesis we can find a Morse matching for $\Gamma(k/p)$ where the only critical cell is given by

$$C = \{0, k/p\} \cup \{k/pq \; : \; q|k/p \text{ and } q \text{ prime}\}.$$

By applying the map φ to the Morse matching of $\Gamma(k/p)$ we obtain a matching on the unmatched cells of $\Gamma(k)$. The only remaining unmatched cell is

$$\varphi(C) = \{0, k\} \cup \{k/q \; : \; q|k/p \text{ and } q \text{ prime}\} \cup \{k/p\}.$$

This is the claimed unmatched face. This completes the induction hypothesis.

By induction there are no directed cycles in the image $\varphi(\Gamma(k/p))$. Furthermore, since the image $\varphi(\Gamma(k/p))$ lies in ranks which are less than that of the elements which were first matched, there can be no directed cycles between these two parts. Hence we conclude the matching constructed is a Morse matching. □

As an application, the matching constructed in Lemma 3.2 and Proposition 3.3 yields a combinatorial identity for the number theoretic Möbius function. Recall for a positive integer k that the Möbius function is defined by

$$\mu(k) = \begin{cases} (-1)^r & \text{if } k \text{ is square-free and has } r \text{ prime factors,} \\ 0 & \text{if } k \text{ is not square-free.} \end{cases}$$

Corollary 3.4. *For k a positive integer Möbius function $\mu(k)$ is given by*

$$\mu(k) = \sum_{F \in \Gamma(k)} (-1)^{|F|}.$$

Proof. The matching of Lemma 3.2 and Proposition 3.3 yields a sign-reversing involution on the set $\Gamma(k)$. The only unmatched face $F = \{0, k\} \cup \{k/q \; : \; q|k \text{ and } q \text{ prime}\}$ occurs when k is squarefree, and it satisfies

$$(-1)^{|F|} = (-1)^{\#\text{ prime factors of } k} = \mu(k).$$ □

For $1 \leq x < y \leq n$, define the set $D(n, k, x, y)$ by

$$D(n, k, x, y) = \{d \in \mathbb{P} \; : \; d|y - x \text{ and } \{x, x + d, \ldots, y\} \in \mathrm{vdW}(n, k)\}.$$

The following inclusion holds.

$$D(n, k, x, y) \subseteq \{d \in \mathbb{P} \; : \; d|y - x \text{ and } (y - x)/d \leq k\}. \tag{3.1}$$

In general this inclusion is not an equality since the right-hand side does not depend on n. The following example illustrates this phenomenon.

Example 3.5. Consider the case $n = 7$ and $k = 3$. We have that $\{1, 7\}$ is an edge in $\mathrm{vdW}(7, 3)$ since it is contained in the facet $\{1, 3, 5, 7\}$. In fact, $D(7, 3, 1, 7) = \{2\}$, whereas the right-hand side of equation (3.1) is given by the set $\{2, 3, 6\}$.

The following notion will be useful in what follows.

Definition 3.6. *For a finite non-empty set $S = \{s_1 < s_2 < \cdots < s_j\}$ of positive integers, define the* greatest common divisor after translation *as*

$$\mathrm{gcdtr}(S) = \gcd(s_2 - s_1, s_3 - s_1, \ldots, s_j - s_1)$$

Observe that $\mathrm{gcdtr}(S)$ is the largest divisor d of the difference $\max(S)-\min(S)$ for which $\{\min(S), \min(S)+d, \ldots, \max(S) - d, \max(S)\}$ is an arithmetic progression that contains S.

Proposition 3.7. *The van der Waerden complex* $\mathrm{vdW}(n, k)$ *can be expressed as the disjoint union*

$$\mathrm{vdW}(n,k) \;=\; \{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\} \;\dot\cup\; \dot{\bigcup_{\substack{1\leq x<y\leq n\\ d\in D(n,k,x,y)}}} (d \cdot \Gamma((y-x)/d) + x)$$

where $d\cdot\Gamma(j)+x$ denotes the family $\{d\cdot F+x : F \in \Gamma(j)\}$ and $d\cdot F+x$ denotes the set $\{d\cdot i+x : i \in F\}$.

Proof. Let F be a face in $\mathrm{vdW}(n, k)$. If the face has cardinality at most 1, then it belongs to the family $\{\emptyset, \{1\}, \{2\}, \ldots, \{n\}\}$. Otherwise, it has cardinality at least 2. For such a face, let $x = \min(F)$, $y = \max(F)$ and $d = \mathrm{gcdtr}(F)$. Note that the face F is a subset of $d \cdot \Gamma((y - x)/d) + x$. This yields the decomposition. $\square$

We now use the aforementioned acyclic matching to obtain bounds for the dimension of the cells a CW-complex which is homotopy equivalent to the complex $\mathrm{vdW}(n, k)$. Our bounds are stated in terms of the *primorial function* $\Pi(x)$, which is defined as the product

$$\Pi(x) = \prod_{p\leq x} p$$

where p ranges over all primes less than or equal to x. Let p_i denote the ith prime. The product of the r first primes is then given by $\Pi(p_r)$.

Theorem 3.8. *Let n, k and r be positive integers such that $k < \Pi(p_r)$. Then the van der Waerden complex* $\mathrm{vdW}(n, k)$ *is homotopy equivalent to a CW-complex whose cells have dimension at most r.*

Proof. The case $k = 1$ follows directly from the fact that $\mathrm{vdW}(n, 1)$ is the complete graph K_n. Consider the case when $k > 1$. Define the poset Q by

$$Q = \{(x, y, d) \in \mathbb{N}^3 \;:\; 1 \leq x < y \leq n,\ y - x \geq 2,\ d \in D(n, k, x, y)\} \cup \{\widehat{0}\},$$

with partial order defined by $(x, y, d) \leq_Q (x', y', d')$ if $x' \leq x < y \leq y'$ and $d'|d$; and $\widehat{0} \leq_Q z$ for all $z \in Q$. We also define the map $\varphi : \mathrm{vdW}(n, k) - \{\emptyset\} \longrightarrow Q$ as

$$\varphi(F) = \begin{cases} (\min(F), \max(F), \mathrm{gcdtr}(F)) & \text{if } \max(F) - \min(F) \geq 2,\\ \widehat{0} & \text{otherwise.}\end{cases}$$

Note that φ is order-preserving. Also, for $(x, y, d) \in Q$, the fiber $\varphi^{-1}((x, y, d))$ is given by the family $d \cdot \Gamma((y - x)/d) + x$. Finally, the fiber $\varphi^{-1}(\widehat{0})$ consists of singletons and the pairs $\{i, i + 1\}$, where $1 \leq i < n$.

The fibers of the map φ yield the decomposition of the complex $\mathrm{vdW}(n,k)$ in Proposition 3.7. Furthermore, the fiber $\varphi^{-1}(\widehat{0})$ has an acyclic matching, where the singleton $\{i\}$ is matched with the pair $\{i, i+1\}$. The other fibers each have an acyclic matching by Lemma 3.2 and Proposition 3.3. Finally, we apply Theorem 2.3 to conclude that the union of the aforementioned acyclic matchings is itself an acyclic matching on all of $\mathrm{vdW}(n,k)$.

The singleton $\{n\}$ is the only critical cell in the fiber $\varphi^{-1}(\widehat{0})$. If $(y-x)/d$ is not squarefree then there are no critical cells in the fiber $\varphi^{-1}((x,y,d))$. On the other hand, if $(y-x)/d$ is squarefree then the fiber $\varphi^{-1}((x,y,d))$ has exactly one critical cell with cardinality at most 2 plus the number of distinct primes factors of $(y-x)/d$. Since the inequality $(y-x)/d \leq k < p_r\#$ holds, the number of prime factors of $(y-x)/d$ is at most $r-1$. Hence, the size of the critical cells in our acyclic matching is at most $r+1$, that is, their dimensions are at most r. $\square$

Corollary 3.9. *If $k < \Pi(p_r)$ then the ith reduced homology group $\widetilde{H}_i(\mathrm{vdW}(n,k))$ of the van der Waerden complex vanishes for $i \geq r+1$.*

We end this section by considering the asymptotic behavior of r as k tends to infinity. Recall

$$\lim_{r \longrightarrow \infty} \frac{\log \Pi(p_r)}{r \cdot \log\log \Pi(p_r)} = 1; \tag{3.2}$$

see the beginning of Section 22.10 in [4] just before Theorem 430.

Theorem 3.10. *Let $r(k)$ be the unique positive integer r such that the inequalities $\Pi(p_{r-1}) \leq k < \Pi(p_r)$ hold. Then the following asymptotic result holds:*

$$r(k) \sim \frac{\log k}{\log\log k}$$

as $k \longrightarrow \infty$.

Proof. Note that the function $\log x/\log\log x$ is increasing for $x > e^e$. Hence for sufficiently large k (take $k \geq 30$) we have the two inequalities

$$\frac{r-1}{r} \cdot \frac{\log \Pi(p_{r-1})}{(r-1) \cdot \log\log \Pi(p_{r-1})} \leq \frac{\log k}{r \cdot \log\log k} < \frac{\log \Pi(p_r)}{r \cdot \log\log \Pi(p_r)},$$

where $r = r(k)$. Note that $r(k) \longrightarrow \infty$ as $k \longrightarrow \infty$. Hence by equation (3.2) both the right-hand and left-hand sides of the above bound tend to 1 as $k \longrightarrow \infty$. Thus the asymptotic follows by the squeeze theorem. $\square$

We can now state an asymptotic version of our homotopy equivalence dimension bound.

Theorem 3.11. *The van der Waerden complex $\mathrm{vdW}(n,k)$ is homotopy equivalent to a CW-complex whose cells have dimension asymptotically at most $\log k/\log\log k$.*

4 Bounds for contractible complexes

Recall that for a prime p and integer s that $p^r || s$ means that r is the largest power dividing s, that is, $p^r | s$ but $p^{r+1} \not| s$.

Proposition 4.1. *For an integer $a > 1$, let $L(a) = \mathrm{lcm}(1, 2, \ldots, a)$. Let p be a prime such that $p^\alpha || L$ for $\alpha \geq 1$. Let k be an integer greater than or equal to $L(a)/p^{\alpha-1}$, and let n be an integer bounded by $a \cdot k < n \leq (a+1) \cdot k$. Then the van der Waerden complex $\mathrm{vdW}(n, k)$ is contractible.*

Proof. Define the poset Q as

$$Q = \{(x, y) \in \mathbb{N}^2 \ : \ 1 \leq x < y \leq n, \ y - x \geq 2\} \cup \{\widehat{0}\},$$

with the partial order relation given by $(x, y) \leq_Q (x', y')$ if $x' \leq x < y \leq y'$, and $\widehat{0} \leq_Q z$ for all $z \in Q$. We also define the map $\varphi : \mathrm{vdW}(n, k) - \{\emptyset\} \longrightarrow Q$ as

$$\varphi(F) = \begin{cases} (\min(F), \max(F)) & \text{if } \max(F) - \min(F) \geq 2, \\ \widehat{0} & \text{otherwise.} \end{cases}$$

Note that φ is order-preserving.

As in the proof of Theorem 3.8, the fiber $\varphi^{-1}(\widehat{0})$ consists of singletons and the pairs $\{i, i+1\}$ such that $1 \leq i < n$. Again, we match the singleton $\{i\}$ with the pair $\{i, i+1\}$, leaving us with the critical cell $\{n\}$.

We will now describe a matching on the fiber $\varphi^{-1}((x, y))$ for $(x, y) \in Q$. We have that $\{x, y\}$ is an edge of $\mathrm{vdW}(n, k)$, and that $y - x \geq 2$. Note that all facets of $\mathrm{vdW}(n, k)$ have steps at most a, since for any $b > a$ the difference between $z + b \cdot k$ and z is greater than $n - 1$.

Let S be the set of steps of the facets containing the edge $\{x, y\}$, that is,

$$S = \{d \ : \ \{x, y\} \subseteq \{z, z + d, \ldots, z + k \cdot d\} \subseteq [n]\}.$$

Note that S is a non-empty subset of $[a]$. Next let T be the set of minimal elements of S with respect to divisibility. We have that $d | y - x$ for all $d \in T$. Thus the least common multiple $\mathrm{lcm}(T)$ also divides $y - x$. Also note that $\mathrm{lcm}(T) \leq \mathrm{lcm}(S) \leq L(a) \leq p^{\alpha-1} \cdot k$.

We claim that $y - x \neq \mathrm{lcm}(T)$. To reach a contradiction, suppose that $y - x = \mathrm{lcm}(T)$. Note that $1 \notin T$, since if the element 1 did belong to T, the set T would be the singleton $\{1\}$, implying that $y - x = \mathrm{lcm}(T) = 1$, which contradicts $y - x \geq 2$. Factor $L(a)$ as $p^\alpha \cdot c$ where c is not divisible by p. Assume now that $y - x$ factors as $p^\beta \cdot c'$ such that p does not divide c'. Since $y - x = \mathrm{lcm}(T)$ divides $L(a)$, we have that $c' | c$ and $\beta \leq \alpha$. We have two cases to consider.

The first case is $\beta \leq 1$. We have the following string of inequalities:

$$y - x = p^\beta \cdot c' \leq p^\beta \cdot c \leq p \cdot c = \frac{L(a)}{p^{\alpha-1}} \leq k.$$

This shows that the edge $\{x, y\}$ is contained in an edge of step 1, yielding the contradiction $1 \in T$.

The second case is $\beta \geq 2$. We then have

$$y - x = p^\beta \cdot c' \leq p^\beta \cdot c = p^\beta \cdot \frac{L(a)}{p^\alpha} = p^{\beta-1} \cdot \frac{L(a)}{p^{\alpha-1}} \leq p^{\beta-1} \cdot k.$$

We now claim that there is a facet with step $p^{\beta-1}$ that contains the edge $\{x, y\}$. Since p^α divides $L(a)$, we have the inequality $p^\alpha \leq a$. We have the following string of inequalities.

$$\begin{aligned} (k+1) \cdot p^{\beta-1} = k \cdot p^{\beta-1} + p^{\beta-1} &\leq k \cdot p^{\beta-1} + k \cdot (p-1) \cdot p^{\beta-1} \\ &= k \cdot p^\beta \leq k \cdot p^\alpha \leq k \cdot a < n. \end{aligned}$$

Hence the arithmetic sequence $F = \{p^{\beta-1}, 2 \cdot p^{\beta-1}, \ldots, (k+1) \cdot p^{\beta-1}\}$, which has step size $p^{\beta-1}$ and length k, is contained in the vertex set $[n]$, that is, F is indeed a facet. By shifting this facet F, we obtain all the facets of step $p^{\beta-1}$:

$$F - (p^{\beta-1} - 1),\ F - (p^{\beta-1} - 2),\ \ldots,\ F,\ \ldots,\ F + (n - (k+1) \cdot p^{\beta-1}).$$

Since $p^{\beta-1}$ divides $y - x$ and $y - x \leq k \cdot p^{\beta-1}$, one of the above facets contains the edge $\{x, y\}$, proving the claim.

Hence the element $p^{\beta-1}$ belongs to S. Either $p^{\beta-1}$ is a minimal element of S or it is not. Let p^γ be an element of T that is less than or equal to $p^{\beta-1}$ in the divisor order, that is, $\gamma \leq \beta - 1$. Consider another element t in T, that is, $t \neq p^\gamma$, we have that $p^\delta \| t$ implies that $\delta < \gamma$. Hence p^γ is the largest p-power that divides $\mathrm{lcm}(T)$. Since $\gamma \leq \beta - 1 < \beta$, this contradicts the fact that p^β is the smallest power of p dividing $\mathrm{lcm}(T) = y - x$.

This proves our claim. Thus, we conclude that $\mathrm{lcm}(T) < y - x$.

We now continue to construct our matching on the fiber $\varphi^{-1}((x, y))$ as follows. For a face $F \in \mathrm{vdW}(n, k)$ such that $\varphi(F) = (x, y)$, we match F with the symmetric difference $F \triangle \{x + \mathrm{lcm}(T)\}$. This is well-defined because if the sum $x + \mathrm{lcm}(T)$ does not belong to the face F, then $F \cup \{x + \mathrm{lcm}(T)\}$ is also a face of the complex. Note that this matching on the fiber $\varphi^{-1}((x, y))$ is acyclic and has no critical cells.

Finally, we apply Theorem 2.3 to conclude that the union of the aforementioned acyclic matchings is an acyclic matching on $\mathrm{vdW}(n, k)$ with the unique critical cell $\{n\}$. It follows that the complex $\mathrm{vdW}(n, k)$ is contractible. □

Directly we have the following result.

Proposition 4.2. *Let $a > 1$ be an integer, and define $L(a)$ as in Proposition 4.1. Let $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_m^{\alpha_m}$ be the prime factorization of $L(a)$. Let $M(a)$ be the maximum of the set*

$$\left\{p_1^{\alpha_1 - 1}, p_2^{\alpha_2 - 1}, \ldots, p_m^{\alpha_m - 1}\right\}.$$

Let k be an integer greater than or equal to $L(a)/M(a)$, and let n be an integer bounded by $a \cdot k < n \leq (a+1) \cdot k$. Then the van der Waerden complex $\mathrm{vdW}(n, k)$ is contractible.

Remark. The prime p in $M(a) = p^{\alpha-1}$ must be 2 or 3. To see this, suppose that $p \geq 5$. Since $2^2 < p$, we have that $p^{\alpha-1} < 2^{\beta-1} < 2^\beta < p^\alpha$, so there are at least two 2-powers between $p^{\alpha-1}$ and p^α. Note that $2^\beta < p^\alpha < a$. Hence, the maximum $M(a)$ cannot be given by $p^{\alpha-1}$.

Lemma 4.3. *View L and M in the previous proposition as functions in the variable a. Then the quotient L/M is a weakly increasing function in a.*

Proof. First consider when a is not a prime power. We have $L(a-1) = L(a)$ and hence $M(a-1) = M(a)$. Thus the two quotients $L(a-1)/M(a-1)$ and $L(a)/M(a)$ are equal.

Next assume that a is the prime power p^α. Then $L(a)$ increases by a factor of p, that is, $L(a) = p \cdot L(a-1)$. If $M(a-1) = M(a)$ then $L(a)/M(a) = p \cdot L(a-1)/M(a-1) \geq L(a-1)/M(a-1)$. On the other hand, if $M(a) \neq M(a-1)$ then we have that $M(a) = p^{\alpha-1}$. Note now that $M(a-1) \geq p^{\alpha-2}$. We thus have

$$\frac{L(a-1)}{M(a-1)} = \frac{p \cdot L(a-1)}{p \cdot M(a-1)} \leq \frac{L(a)}{p \cdot p^{\alpha-2}} = \frac{L(a)}{M(a)}.$$

□

Using Lemma 4.3, we now remove the lower bound $a \cdot k$ on n.

Theorem 4.4. *For an integer $a > 1$, let $L(a)$ and $M(a)$ be as in Proposition 4.2. Let k be an integer greater than or equal to the quotient $L(a)/M(a)$, and let n be a positive integer bounded by $n \leq (a+1) \cdot k$. Then the van der Waerden complex* $\mathrm{vdW}(n, k)$ *is contractible.*

Proof. Let $b \leq a$ be a positive integer such that $b \cdot k < n \leq (b+1) \cdot k$. By Lemma 4.3 we have that $L(b)/M(b) \leq L(a)/M(a) \leq k$. Hence, applying Proposition 4.2 with the parameter b implies that $\mathrm{vdW}(n, k)$ is contractible. □

Remark. If $a+1$ is not a prime power, then the statement of Theorem 4.4 can be sharpened by increasing a by 1, because we have that $L(a) = \mathrm{lcm}(1, 2, \ldots, a) = \mathrm{lcm}(1, 2, \ldots, a+1)$. Hence in this case we have that the complex $\mathrm{vdW}(n, k)$ is contractible for $k \geq L(a)/M(a)$ and $n \leq (a+2)k$.

We have the following lower and upper bounds on $M(a)$.

Lemma 4.5. *For an integer $a > 1$, let $M(a)$ be defined as in Proposition 4.2. Then we have that*

$$a/4 < M(a) \leq a/2.$$

Proof. Assume that $M(a) = p^{\alpha-1}$ where p is a prime and $\alpha > 0$. We have that $p^{\alpha} \leq a$ and hence $M(a) = p^{\alpha-1} \leq p^{\alpha}/2 \leq a/2$. As for the lower bound, let $\beta \geq 0$ satisfy $2^{\beta} \leq a < 2^{\beta+1}$. Then we have $a/4 < 2^{\beta-1} \leq M(a)$. □

5 Examples

For the sake of illustration, in this section we discuss the topology of the family of van der Waerden complexes $\mathrm{vdW}(5k, k)$ for $k \geq 1$.

Let $L(a)$ and $M(a)$ be defined as in Proposition 4.2. Then $L(4) = 12 = 2^2 \cdot 3$ and accordingly $M(4) = 2$. Applying Theorem 4.4 for $a = 4$ we conclude the complex $\mathrm{vdW}(5k, k)$ is contractible for all $k \geq L(4)/M(4) = 6$. In fact, when $k \geq 6$ the van der Waerden complex $\mathrm{vdW}(n, k)$ is contractible for all positive integers $n \leq 5k$.

The remaining van der Waerden complexes in the family, namely $\mathrm{vdW}(5, 1)$ through $\mathrm{vdW}(25, 5)$, are not contractible. For instance, the complex $\mathrm{vdW}(5, 1)$ is the complete graph K_5 which is homotopy equivalent to a wedge of $\binom{4}{2} = 6$ circles. To determine the homotopy equivalence of the other four van der Waerden complexes, we construct on each complex an acyclic matching with a minimal number of critical cells.

Example 5.1. The complex $\mathrm{vdW}(10, 2)$ has a discrete Morse matching with the critical cells $\{10\}$ and $\{x, x+3\}$ for $1 \leq x \leq 7$. Thus, this complex is homotopy equivalent to a wedge of seven circles.

Proof. On the face poset of $\mathrm{vdW}(10, 2) - \{\emptyset\}$, match the singleton $\{x\}$ with $\{x, x+1\}$ for $1 \leq x \leq 9$. Furthermore, match $\{x, x+2d\}$ with $\{x, x+d, x+2d\}$ for $1 \leq x < x+2d \leq 10$. It is straightforward to see that this matching is acyclic and the critical cells have the form $\{x, x+3\}$ for $1 \leq x \leq 7$. The homotopy result follows from Proposition 2.5. □

$\mathrm{vdW}(5k, k)$	homotopy type
$\mathrm{vdW}(5,1)$	$(S^1)^{\vee 6}$
$\mathrm{vdW}(10,2)$	$(S^1)^{\vee 7}$
$\mathrm{vdW}(15,3)$	$(S^2)^{\vee 9}$
$\mathrm{vdW}(20,4)$	$(S^2)^{\vee 22}$
$\mathrm{vdW}(25,5)$	$(S^2)^{\vee 32}$

Table 1: The homotopy types for $\mathrm{vdW}(5k, k)$ such that $1 \leq k \leq 5$. By Theorem 4.4 the complex $\mathrm{vdW}(5k, k)$ is contractible for $k \geq 6$.

Let Q and $\varphi : \mathrm{vdW}(n, k) - \{\emptyset\} \longrightarrow Q$ be the poset and the order-preserving map which appear in the beginning of the proof of Proposition 4.1. We will use them in Examples 5.2 through 5.4.

Example 5.2. The complex $\mathrm{vdW}(15, 3)$ has a discrete Morse matching with the critical cells $\{15\}$ and $\{x, x + 3, x + 6\}$ for $1 \leq x \leq 9$. Hence, this complex is homotopy equivalent to a wedge of nine 2-dimensional spheres.

Proof. For a non-empty face $F \in \mathrm{vdW}(15, 3)$, the difference $\max(F)-\min(F)$ takes one of the following values: 0, 1, 2, 3, 4, 6, 8, 9 and 12. Consider the following matching.

1. If the difference $\max(F) - \min(F) \in \{0, 1, 2, 4, 8\}$, we match the face F in the same manner as in Example 5.1. This leaves $\{15\}$ as a critical cell.
2. If $1 \leq x < x + 3d \leq n$, we match the face F in the fiber $\varphi^{-1}((x, x + 3d))$ with the symmetric difference $F\triangle\{x+d\}$. Note that when the difference is 6, this leaves the critical cell $\{x, x+3, x+6\}$ for $1 \leq x \leq 9 = n - 6$.

Observe that all of these matchings stay inside the corresponding fiber of the poset map φ and are acyclic in each fiber. Hence by Theorem 2.3 their union is an acyclic matching . The critical cells are $\{15\}$ and $\{x, x + 3, x + 6\}$ for $1 \leq x \leq 9$. It follows from Proposition 2.5 that $\mathrm{vdW}(15, 3)$ is homotopy equivalent to a wedge of nine spheres. $\square$

Example 5.3. The complex $\mathrm{vdW}(20, 4)$ has a discrete Morse matching with the critical cells $\{20\}$, $\{x, x+3, x+6\}$ for $1 \leq x \leq 14$ and $\{x, x+4, x+12\}$ for $1 \leq x \leq 8$. Hence, this complex is homotopy equivalent to a wedge of 22 2-dimensional spheres.

Proof. Now we have that for a non-empty face $F \in \mathrm{vdW}(20, 4)$, the difference $\max(F) - \min(F) \in \{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\}$.

1. When $\max(F) - \min(F) \in \{0, 1, 2, 3, 6, 9\}$, we match in the same manner as in Example 5.2. above. This leaves one zero-dimensional critical cell $\{20\}$ and 14 two-dimensional critical cells of the form $\{x, x + 3, x + 6\}$ for $1 \leq x \leq 14$.
2. For $1 \leq x \leq 16$, match each face F in the fiber $\varphi^{-1}((x, x + 4))$ with the symmetric difference $F\triangle\{x + 2\}$.

3. For $1 \le x \le 12$, match each face F in the fiber $\varphi^{-1}((x, x+8))$ with the symmetric difference $F \triangle \{x+4\}$.

4. For $1 \le x \le 8$ and $F \in \varphi^{-1}((x, x+12))$, there are three cases to consider. If $F \triangle \{x+6\} \in \mathrm{vdW}(20, 4)$, then we match F with the symmetric difference $F \triangle \{x+6\}$. The remaining cases are when F is one of the following three faces: $\{x, x+4, x+12\}$, $\{x, x+8, x+12\}$ or $\{x, x+4, x+8, x+12\}$. We proceed by matching the last two faces and leaving $\{x, x+4, x+12\}$ as a critical face, for $1 \le x \le 8$.

5. For $1 \le x \le 4$, we match each face $F \in \varphi^{-1}((x, x+16))$ with the symmetric difference $F \triangle \{x+8\}$.

The result follows by the same reasoning as the end of Example 5.2. □

Example 5.4. The complex $\mathrm{vdW}(25, 5)$ has a discrete Morse matching with the critical cells $\{25\}$, $\{x, x+3, x+6\}$ for $1 \le x \le 19$ and $\{x, x+4, x+12\}$ for $1 \le x \le 13$. Hence, this complex is homotopy equivalent to a wedge of 32 2-dimensional spheres.

Proof. Note that for a non-empty face F we have that the difference $\max(F) - \min(F)$ belongs to the set $\{0, 1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 20\}$. Consider the following matching:

1. When the difference $\max(F) - \min(F) \in \{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\}$, we match as in Example 5.3. This leaves the critical cells $\{25\}$, $\{x, x+3, x+6\}$ for $1 \le x \le 19$ and $\{x, x+4, x+12\}$ for $1 \le x \le 13$.

2. For $1 \le x < x + 5d \le 25$ (which implies $1 \le d \le 4$), match the face $F \in \varphi^{-1}((x, x+5d))$ with the symmetric difference $F \triangle \{x+d\}$.

The same reasoning as in the previous examples yields the result. □

Note that since 3 and 5 are prime, it was easier to describe the matchings for the complexes $\mathrm{vdW}(15, 3)$ and $\mathrm{vdW}(25, 5)$.

6 Concluding remarks

Some natural questions arise concerning the topology of the van der Waerden complex. If $\mathrm{vdW}(n, k)$ is noncontractible, is it always homotopy equivalent to a wedge of spheres? There is some evidence for this in the examples described in Section 5. One possible way to prove such a result would be to construct an acyclic matching that satisfies Kozlov's alternating-path condition [6].

When the complex $\mathrm{vdW}(n, k)$ is noncontractible, can one predict its Betti numbers or say something nontrivial about their behavior with respect to n or k? Moreover, do sequences of these Betti numbers have an underlying connection to other mathematical structures, including number-theoretic ones?

What is the error term in the asymptotic bound $\log k / \log \log k$ for the dimension of the homotopy type in Theorem 3.11? Given that the Prime Number Theorem is fundamentally encoded into this asymptotic via the primorial function, it is conceivable that a successful analysis in this direction may yield a statement about the homotopy dimension bound of $\mathrm{vdW}(n, k)$ that is equivalent to the Riemann Hypothesis.

Are Theorems 3.8 and 4.4 provably the tightest possible general bounds for the dimension (with respect to k) and for contractibility (with respect to n/k). Can one generalize Theorem 4.4 to a bound on the dimension of the homotopy type of $\mathrm{vdW}(n, k)$ with respect to both n and k, as opposed to just k, as in Theorem 3.8?

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